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Research article

An analysis on the approximate controllability of neutral impulsive stochastic integrodifferential inclusions via resolvent operators

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ABSTRACT

This article focuses on the approximate controllability of impulsive neutral stochastic integrodifferential inclusions in Hilbert spaces. We used resolvent operators, fixed point approaches, and semigroup theory to achieve the article's main results. First, we focus on the existence of approximate controllability, and we develop the existence results with nonlocal conditions. At last, an application is provided to illustrate the concept.

1. Introduction

Controllability is a fundamental approach in mathematical control theory and is used in many scientific and technological fields. In the academic world, it is generally agreed that nonlinear deterministic systems can be controlled. Moreover, exact controllability enables us to steer the system to an arbitrary final state, while approximate controllability means that the system can be steered to an arbitrary small neighborhood of the final state using the set of admissible controls. The controllability of nonlinear systems was studied in [1]. In [2], the authors established the approximate controllability of a second-order semilinear stochastic system. The researchers of [3] discussed the approximate controllability of second-order non-autonomous integrodifferential inclusions through resolvent operators. Refer to the publications for more information [4–7].

Nowadays, different areas of applied science extensively utilize stochastic differential equations. A common development of a deterministic model of a differential equation is the structure of a stochastic differential equation, where appropriate parameters are modeled for applicable stochastic processes. This is a result of stochastic systems rather than deterministic systems being the primary

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model for real-world problems. Therefore, the system of stochastic differential equations has attracted a lot of attention because it can be used to describe many dynamic systems in medicine, biology, and physical science. The researchers in [8] examined the existence of solutions for impulsive nonlocal stochastic functional integrodifferential systems. Likewise, the authors of [9] studied the approximate controllability of second-order neutral stochastic non-autonomous integrodifferential systems to be solved by resolvent operators. Very recently, the author of [10] discussed the approximate controllability of stochastic degenerate systems through the decomposition of Hilbert space. Consult the monographs [11,12] and publications [13–19] for more details. The controllability of linear and nonlinear deterministic systems has significantly improved in the last few years, and nonlocal initial conditions generally execute the applications much more than formal initial conditions (see publications [20–22]). Therefore, in this article, the nonlocal condition has been handled separately.

Additionally, the resolvent condition provides a more useful approach to solving integrodifferential equations. So, integrodifferential equations are used in a wide range of scientific fields, including medicine, control theory, biology, and many others. The researcher in [23] examined the approximate controllability results for analytic resolvent integrodifferential systems. In [24], the authors utilized the resolvent operator method and an approximating approach to demonstrate the approximate controllability of impulsive neutral integrodifferential systems with nonlocal conditions. We suggest the monographs [25–28] and the publications [29–34] and the references therein.

Furthermore, impulsive differential systems explain the dynamics of the process wherein sudden variations and discontinuous jumps appear at particular moments, such as during earthquakes and natural disasters. The authors of [35] established the approximate controllability of impulsive differential equations with nonlocal conditions. In the manuscript [36], the authors discussed the approximate controllability of neutral stochastic integrodifferential systems with impulsive effects. For more details, refer to [37–40].

The fixed point approach is particularly relevant and practical for the controllability of nonlinear systems. Most of the time, the fixed point technique works well for many control theory problems. So, motivated by the above works, we conclude to examine the approximate controllability of impulsive nonlocal neutral stochastic integrodifferential system through the resolvent operator. To solve our main results, we use Bohnenblust-Karlin's fixed point approach.

In this article, we examined the approximate controllability of impulsive neutral stochastic integrodifferential inclusions of the form

$$\begin{cases} d\left(y(\tau)+\int_0^\tau \mathbb{Q}(\tau-\vartheta)y(\vartheta)d\vartheta\right) \in \left[Ay(\tau)+\int_0^\tau g(\tau-\vartheta)y(\vartheta)d\vartheta+Bu(\tau)\right]d\tau \\ \qquad\qquad\qquad\qquad +\mathcal{G}(\tau,y(\tau))dw(\tau),\ \tau\in\mathcal{J}=[0,c],\ \tau\neq\tau_{\mathfrak{r}}, \\ \Delta y|_{\tau=\tau_{\mathfrak{r}}}=\mathcal{I}_{\mathfrak{r}}(y(\tau_{\mathfrak{r}})),\ \mathfrak{r}=1,2,\dots,\hat{n}, \\ y(0)=y_0, \end{cases} \tag{1}$$

where $A: D(A)\subseteq\mathbb{H}\to\mathbb{H}$ makes an analytic semigroup $\{\mathcal{T}(\tau)\}_{\tau\geq 0}$ of bounded linear operators and $y(\cdot)$ takes the values in a separable Hilbert space $\mathbb{H}$ with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$; $\{\mathbb{Q}(\tau)\}_{\tau\geq 0}$ is a set of continuous linear operators on $\mathbb{H}$; $B:\mathcal{Z}\to\mathbb{H}$ is a bounded linear operator; $g(\tau)$ is a closed linear operator on $D(A)$ for each $\tau\geq 0$; the control function $u(\cdot)$ in a Hilbert space $\mathscr{L}^2(\mathcal{J},\mathcal{Z})$; $\mathcal{G}:\mathcal{J}\times\mathbb{H}\to 2^{\mathbb{H}}\backslash\{\emptyset\}$ is nonempty, closed, bounded and convex multivalued map; suppose $\{w(\tau);\tau\geq 0\}$ is a given $\mathbb{W}$-valued Wiener process with a finite trace nuclear operator $Q\geq 0$; $0=\tau_0<\tau_1<\dots<\tau_{\hat{n}}<\tau_{\hat{n}+1}=c$, $\mathcal{I}_{\mathfrak{r}}:\mathbb{H}\to\mathbb{H}$, $\mathfrak{r}=1,2,\dots,\hat{n}$, $\Delta y(\tau_{\mathfrak{r}})$ denote the jump of y at $\tau_{\mathfrak{r}}$ indicate by $\Delta y(\tau_{\mathfrak{r}})=y(\tau_{\mathfrak{r}}^+)-y(\tau_{\mathfrak{r}}^-)$. Here, $(\tau_{\mathfrak{r}}^+)$ and $(\tau_{\mathfrak{r}}^-)$ expressed the right and left limits of $y(\tau)$ at $\tau=\tau_{\mathfrak{r}}$.

This article is split into consecutive sections: In Section 2, provide the preliminaries of this article. In Section 3, we examined the solution for system (1), which is based on the resolvent operator, fixed point approach, and semigroup theory. In Section 4, we introduce the approximate controllability results for nonlocal conditions. In Section 5, an example is presented to verify the concept and results.

2. Preliminaries

We now provide some fundamental theories, lemmas, and facts to discuss our manuscript's principal results.

We denote $(\mathbb{H},\|\cdot\|_{\mathbb{H}})$ and $(\mathbb{W},\|\cdot\|_{\mathbb{W}})$ two real separable Hilbert spaces. We just use $\|\cdot\|$ for the norms and $\langle\cdot,\cdot\rangle$ for the inner product in $\mathbb{H},\mathbb{W}$ to avoid confusion. Consider $\mathscr{L}(\mathbb{W},\mathbb{H}):\mathbb{W}\to\mathbb{H}$ stands for the space of bounded linear operator. Let $(\Omega,\mathscr{F},\{\mathscr{F}_\tau\}_{\tau\geq 0},\mathscr{P})$ be a complete filtered probability space fulfilling the standard conditions, the filtration $\{\mathscr{F}_\tau\}_{\tau\geq 0}$ is a right continuous increasing family and $\mathscr{F}_0$ contains all $\mathscr{P}$ null sets. Suppose that $\{w(\tau):\tau\geq 0\}$ is a cylindrical $\mathbb{W}$-valued Wiener process including a finite trace nuclear covariance operator $Q\geq 0$, such that $Tr(Q)<\infty$. Moreover, we consider that there exists a complete orthonormal basis $\{e_\eta\}_{\eta\geq 1}$ in $\mathbb{W}$, and a bounded sequence of $\Bbbk_\eta>0$ such that $Qe_\eta=\Bbbk_\eta e_\eta$, $\eta\in\mathbb{N}$, and the sequence of independent Wiener process. We assume that $\mathscr{F}_\tau=\sigma\{w(\vartheta):0\leq\vartheta\leq\tau\}$ is the σ-algebra generated through w and $\mathscr{F}_c=\mathscr{F}$. For $\tilde{\Theta}\in\mathscr{L}(\mathbb{W},\mathbb{H})$, represents

$$\|\tilde{\Theta}\|_Q^2=Tr(\tilde{\Theta}Q\tilde{\Theta}^*)=\sum_{\eta=1}^{\infty}\|\sqrt{\Bbbk_\eta}\tilde{\Theta}e_\eta\|^2.$$

If $\|\tilde{\Theta}\|_Q^2=Tr(\tilde{\Theta}Q\tilde{\Theta}^*)<\infty$, then $\tilde{\Theta}$ is said to be a Q-Hilbert Schmidt operator. Consider $\mathscr{L}_Q(\mathbb{W},\mathbb{H})$ stands for the space of all Q-Hilbert Schmidt operators $\tilde{\Theta}:\mathbb{W}\to\mathbb{H}$. The completion $\mathscr{L}_Q(\mathbb{W},\mathbb{H})$ of $\mathscr{L}(\mathbb{W},\mathbb{H})$ with respect to the topology induced through $\|\cdot\|_Q$, where $\|\tilde{\Theta}\|_Q^2=\langle\tilde{\Theta},\tilde{\Theta}\rangle$ is a Hilbert space including the above norm topology. The set of all square integrable, strongly measurable, $\mathbb{H}$-valued random variables stand for $\mathscr{L}^2(\Omega,\mathbb{H})$ is a Banach space equipped with the norm

$$\| \cdot \|_{\mathscr{L}_2} = \left(E\|y(\cdot, w)\|^2_{\mathbb{H}} \right)^{\frac{1}{2}}.$$

In the above expectation, E is represented through $Ey = \int_\Omega y(w)d\mathscr{P}$.

Let $\mathcal{C}(\mathcal{J}, \mathscr{L}^2(\Omega,\mathbb{H}))$ be a Banach space of all continuous functions from $\mathcal{J}$ into $\mathscr{L}^2(\Omega,\mathbb{H})$, satisfying $\sup_{0\le\tau\le c} E\|y(\tau)\|^2_{\mathbb{H}} < \infty$, $\mathscr{L}^2_0(\Omega,\mathbb{H})$ denotes the family of all $\mathscr{F}_0$-measurable, $\mathbb{H}$-valued random variables.

We refer to the linear operator A and its resolvent family through $\rho(A)$. The concept is well known in [41], there exists a constant $\mathbb{Y} > 1$ and a real number ν such that $\|\mathcal{R}(\tau)\|^2 \le \mathbb{Y}e^{\nu\tau}$, $\tau \ge 0$, $\nu \ge 0$. Consider the Banach space $\mathcal{C}(\mathcal{J},\mathbb{H})$ of the continuous functions form $\mathcal{J} \to \mathbb{H}$ with the

$$\|y\| = \sup_{\tau\in\mathcal{J}} \|y(\tau)\|, \forall\ y \in \mathcal{C}(\mathcal{J},\mathbb{H}).$$

The function y from $\mathcal{J}$ to $\mathbb{H}$ is contained in $\mathcal{PC} = \mathcal{PC}(\mathcal{J},\mathbb{H})$ formed by all $\mathscr{F}_\tau$ adapted measurable, then $\mathbb{H}$ valued stochastic processes $\{y(\tau) : \tau \in \mathcal{J}\}$ such that $y(\tau)$ is continuous at $\tau \neq \tau_{\mathfrak{r}}$ and left continuous at $\tau = \tau_{\mathfrak{r}}$, with the right limit $y(\tau^+_{\mathfrak{r}})$ existing $\mathfrak{r} = 1,2,3,\dots,\hat{n}$. $\mathcal{PC}$ is definitely a Banach space, including the

$$\|y\|_{\mathcal{PC}} = \left(\sup_{\tau\in\mathcal{J}} E\|y(\tau)\|^2 \right)^{\frac{1}{2}}, \forall\ y \in \mathcal{PC}.$$

This is obvious that $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is a Banach space. According to our assumption, $\|\mathcal{R}(\tau)\|$ is uniformly bounded by $\mathbb{Y}$ and the analytic resolvent such that $0 \in \rho(A)$. Consider $\mathscr{B}(\mathbb{H})$ is a Banach space of bounded linear operator from $\mathbb{H}$ to $\mathbb{H}$ with operator norm. Now, we will describe the theory of resolvent operators as follows:

Definition 2.1. [42] A one parameter family $\{\mathcal{R}(\tau)\}_{\tau\ge0}$ in $\mathscr{B}(\mathbb{H})$ is said to be a resolvent operator for the abstract integrodifferential Cauchy problem

$$\begin{cases} \frac{d}{d\tau}\left(y(\tau) + \int_0^\tau \mathbb{Q}(\tau-\vartheta)y(\vartheta)d\vartheta\right) = Ay(\tau) + \int_0^\tau g(\tau-\vartheta)y(\vartheta)d\vartheta,\ \tau \in \mathcal{J}, \\ y(0) = y_0 \in \mathbb{H}. \end{cases} \tag{2}$$

If

(i) $\{\mathcal{R}(\tau)\} = \mathscr{I}$ (the identity operator on $\mathbb{H}$),
(ii) for all $y \in \mathbb{H}$, $\mathcal{R}(\tau)y$ is continuous for $\tau \in \mathcal{J}$,
(iii) since $y \in D(A)$, $\mathcal{R}(\cdot)y \in C([0,\infty), D(A)) \cap C^1((0,\infty),\mathbb{H})$, then

$$\frac{d}{d\tau}\left(\mathcal{R}(\tau)y + \int_0^\tau \mathbb{Q}(\tau-\vartheta)\mathcal{R}(\vartheta)yd\vartheta\right) = A\mathcal{R}(\tau)y + \int_0^\tau g(\tau-\vartheta)\mathcal{R}(\vartheta)yd\vartheta, \tag{3}$$

$$\frac{d}{d\tau}\left(\mathcal{R}(\tau)y + \int_0^\tau \mathcal{R}(\tau-\vartheta)\mathbb{Q}(\vartheta)yd\vartheta\right) = \mathcal{R}(\tau)Ay + \int_0^\tau \mathcal{R}(\tau-\vartheta)g(\vartheta)yd\vartheta,\ \tau \in \mathcal{J}. \tag{4}$$

The following assumptions will be using throughout this article:

($\mathbf{A_1}$) The operator $A{:}D(A) \subseteq \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of an analytic semigroup $\{\mathcal{T}(\tau)\}_{\tau\ge0}$ on $\mathbb{H}$ and $\rho(A) \supset \Sigma_\gamma = \{\ell \in \mathbb{C} : \ell \neq 0, |\arg(\ell)| < \gamma\}$ and $\|R(\ell,A)\| \le \mathbb{Y}_0 \mid \ell\mid^{-1}$ for $\mathbb{Y}_0 > 1, \gamma \in (\pi/2,\pi)$ for each $\ell \in \Sigma_\gamma$, where the resolvent of A is $R(\ell,A)$.

($\mathbf{A_2}$) The map $\mathbb{Q} : [0,\infty) \to \mathscr{B}(\mathbb{H})$ is strongly continuous. $\widehat{\mathbb{Q}}(\ell)\theta$ is absolutely convergent for any $\theta \in \mathbb{H}$ if $\mathbf{Re}(\ell) > 0$. There is an analytic extension of $\widehat{\mathbb{Q}}(\ell)$ (still expressed by $\widehat{\mathbb{Q}}(\ell)$) to Σ_γ such that $\|\widehat{\mathbb{Q}}(\ell)\theta\| \le \mathbb{Q}_1 \mid \ell \mid^{-1} \|\theta\|_1\ \forall\ \ell \in \Sigma_\gamma$ and $\theta \in D(A)$.

($\mathbf{A_3}$) The operator $g(\tau) : D(g(\tau)) \subseteq \mathbb{H} \to \mathbb{H}$ is linear and closed with $D(A) \subseteq D(g(\tau))$ for each $\tau \ge 0$. For any $\theta \in D(A)$, $g(\cdot)\theta$ is strongly measurable on $(0,\infty)$. There is a function $c(\cdot) \in \mathscr{L}^1_{loc}(\mathbb{R}^+) \ni \hat{c}(\ell)$ can be obtained for $\mathbf{Re}(\ell) > 0$ and $\|g(\tau)\theta\| \le c(\tau)\|\theta\|_1$ for each $\tau > 0$ and $\theta \in D(A)$. Additionally, $\hat{g} : \Sigma_{\pi/2} \to \mathscr{B}(D(A),\mathbb{H})$ has an analytical extension (still expressed by $\hat{g}$) to Σ_γ such that $\|\hat{g}(\ell)\theta\| \le \|\hat{g}(\ell)\|\|\theta\|_1$ for each $\theta \in D(A)$, then $\|\hat{g}(\ell)\| \to 0$ as $\mid \ell \mid \to \infty$.

($\mathbf{A_4}$) There is a subspace $\widehat{\mathbb{K}} \subseteq D(A)$ that is dense in $D(A)$ and constants $\widehat{\mathfrak{C}}_i > 0, i = 1,2$, such that $\hat{g}(\ell)(\widehat{\mathbb{K}}) \subseteq D(A)$, $\widehat{\mathbb{Q}}(\ell)(\widehat{\mathbb{K}}) \subseteq D(A)$, $\|A\hat{g}(\ell)\theta\| \le \widehat{\mathfrak{C}}_1\|\theta\|$ and $\|\widehat{\mathbb{Q}}(\ell)\theta\| \le \widehat{\mathfrak{C}}_2\|\ell\|^{-\chi}\|\theta\|_1$ for each $\theta \in \widehat{\mathbb{K}}$ and $\ell \in \Sigma_\gamma$.
In the continuation, for each $\hat{s} > 0$ and $\varpi \in (\pi/2,\gamma)$,

$$\Sigma_{\hat{s},\varpi} = \{\ell \in \mathbb{C} : \ell \neq 0 : \mid \ell \mid > \hat{s}, \mid \arg(\ell) \mid < \varpi\},$$

$\Gamma_{\hat{s},\varpi}$, $\Gamma^i_{\hat{s},\varpi}$, $i = 1,2,3$, are the paths $\Gamma^1_{\hat{s},\varpi} = \{\tau e^{i\varpi} : \tau \ge \hat{s}\}$, $\Gamma^2_{\hat{s},\varpi} = \{\hat{s}e^{i\pounds} : -\varpi \ge \pounds \ge \varpi\}$, $\Gamma^3_{\hat{s},\varpi} = \{\tau e^{-i\varpi} : \tau \ge \hat{s}\}$, and $\Gamma_{\hat{s},\varpi} = \cup^3_{i=1}\Gamma^i_{\hat{s},\varpi}$ oriented in positive sense.
Consider

$$\aleph(G) = \{\ell \in \mathbb{C} : G(\ell) = (\ell I + \ell\widehat{\mathbb{Q}}(\ell) - A - \hat{g}(\ell))^{-1} \in \mathscr{B}(\mathbb{H})\}.$$

Lemma 2.2. *[42] The constant $\hat{s}_1 > 0$ such that $\Sigma_{\hat{s}_1,\varpi} \subset \aleph(G)$ and $G : \Sigma_{\hat{s}_1,\varpi} \to \mathscr{B}(\mathbb{H})$ is analytic, and there exists $\mathbb{L}_1 > 0$ such that $\|\ell G(\ell)\| \leq \mathbb{L}_1$, $\ell \in \Sigma_{\hat{s}_1,\varpi}$.*

If $\mathcal{R}(\cdot)$ is a resolvent operator of (2), then the Laplace transform of (4) provides that

$$\widehat{\mathcal{R}}(\cdot)(\ell I + \ell \widehat{\mathbb{Q}}(\ell) - A - \widehat{g}(\ell))\varpi = \varpi, \text{ for all } \varpi \in D(A).$$

We conclude that $\mathcal{R}(\cdot)$ is the only resolvent operator of (2) to applying the Lemma 2.2 and the inverse Laplace transforms. We let $\hat{s} > \hat{s}_1$ in the remaining portion of the section. Now $\{\mathcal{R}(\tau)\}_{\tau\geq 0}$ is represented as

$$\mathcal{R}(\tau) = \begin{cases} \frac{1}{2i\pi} \int_{\Gamma_{\hat{s},\varpi}} e^{\ell\tau} G(\ell) d\ell, & \tau > 0, \\ I, & \tau = 0. \end{cases}$$

Lemma 2.3. *[42] If $\mathcal{R}(\ell_0, A)$ is compact for each $\ell_0 \in A$, then $\mathcal{R}(\tau)$ is compact for all $\tau > 0$.*

Lemma 2.4. *[42] The map $\mathcal{R} : (0, \infty) \to \mathscr{B}(\mathbb{H})$ has an analytic extension to $\Sigma_{\mathfrak{z}}$, $\mathfrak{z} = \min\{\gamma - \frac{\pi}{2}, \frac{\pi}{2} - \gamma\}$ and $\mathbb{Y} > 1$ such that $\sup_{\tau\in\mathcal{J}} E\|\mathcal{R}(\tau)\|^2 \leq \mathbb{Y}$.*

Theorem 2.5. *[28] Assume A is an infinitesimal generator of a C_0 semigroup $\mathcal{T}(\tau)$. Provided that $\mathcal{R}(\ell, A)$ is compact for all $\ell \in \rho(A)$ and $\mathcal{T}(\tau)$ is continuous in the uniform operator topology for $\tau > 0$, then the semigroup $\mathcal{T}(\tau)$ is compact.*

Lemma 2.6. *[27] A set $\widehat{\mathbb{K}} \subset \mathcal{PC}(\mathcal{J}, \mathbb{H})$ is relatively compact in $\mathcal{PC}(\mathcal{J}, \mathbb{H})$ iff the set $\widehat{\mathbb{K}}|_{[\tau_{\mathfrak{r}}, \tau_{\mathfrak{r}+1}]}$ is relatively compact in $C([\tau_{\mathfrak{r}}, \tau_{\mathfrak{r}+1}], \mathbb{H})$ for each $\mathfrak{r} = 0, 1, 2, \ldots, \hat{n}$.*

Further, we present a few fundamental results and explanations of multivalued maps. For additional information on multivalued maps, consult the monographs [43,44].

While $\widehat{\mathbb{K}}(y)$ is convex (closed), then the multivalued map $\widehat{\mathbb{K}} : \mathbb{H} \to 2^{\mathbb{H}} \backslash \{\emptyset\}$ is convex (closed) valued for all $y \in \mathbb{H}$. When $\widehat{\mathbb{K}}(y) = \bigcup_{y\in\mathscr{C}} \widehat{\mathbb{K}}(y)$ is bounded in $\mathbb{H}$ for all bounded set $\mathscr{C}$ of $\mathbb{H}$, then $\sup_{y\in\mathscr{C}}\{\sup\{\|z\| : z \in \widehat{\mathbb{K}}(y)\}\} < \infty$, indicates that $\widehat{\mathbb{K}}$ is bounded on bounded set.

Definition 2.7. [43] $\widehat{\mathbb{K}}$ is known as u.s.c. (upper semicontinuous for expansion) on $\mathbb{H}$, if for each $y_0 \in \mathbb{H}$, the set $\widehat{\mathbb{K}}(y_0)$ is a nonempty closed subset of $\mathbb{H}$ and if for each open set $\mathscr{C}$ of $\mathbb{H}$ containing $\widehat{\mathbb{K}}(y_0)$, there exists an open neighborhood $\mathcal{U}$ of y_0 such that $\widehat{\mathbb{K}}(\mathcal{U}) \subseteq \mathscr{C}$.

Definition 2.8. [43] $\widehat{\mathbb{K}}$ is known as completely continuous if $\widehat{\mathbb{K}}(\mathscr{C})$ is relatively compact for every bounded subset $\mathscr{C}$ of $\mathbb{H}$.

If the multivalued map $\widehat{\mathbb{K}}$ is completely continuous with nonempty values, then $\widehat{\mathbb{K}}$ is upper semicontinuous, iff $\widehat{\mathbb{K}}$ has a closed graph, i.e., $y_n \to y_*$, $z_n \to z_*$, $y_n \in \widehat{\mathbb{K}} y_n$ imply $y_* \in \widehat{\mathbb{K}} y_*$. $\widehat{\mathbb{K}}$ has a fixed point, provide that there is a $y \in \mathbb{H}$, such that $y \in \widehat{\mathbb{K}}(y)$.

In the following, $BCC(\mathbb{H})$ denotes the set of all nonempty, bounded, closed and convex subset of $\mathbb{H}$.

Definition 2.9. [43] A multivalued map $\widehat{\mathbb{K}} : \mathcal{J} \to BCC(\mathbb{H})$ is called measurable if for each $y \in \mathbb{H}$, the function $\tilde{v} : \mathcal{J} \to \mathbb{R}$, defined by

$$\tilde{v}(\tau) = d(y, \widehat{\mathbb{K}}(\tau)) = \inf\{\|y - z\| : z \in \widehat{\mathbb{K}}(\tau)\} \in \mathscr{L}^1(\mathcal{J}, \mathbb{R}).$$

Definition 2.10. [43] The multivalued map $\mathcal{G} : \mathcal{J} \times \mathbb{H} \to BCC(\mathbb{H})$ is said to be $\mathscr{L}^2$-Caratheodory if
(i) $\tau \to \mathcal{G}(\tau, y)$ is measurable for each $y \in \mathbb{H}$,
(ii) $y \to \mathcal{G}(\tau, y)$ is u.s.c. almost all $\tau \in \mathcal{J}$.
(iii) For each $q > 0$, there exists $\mathcal{L}_q \in \mathscr{L}^1(\mathcal{J}, \mathbb{R})$ such that

$$\|\mathcal{G}(\tau, y)\|^2 = \sup\left\{E\|\hat{h}\|^2 : \hat{h} \in \mathcal{G}(\tau, y)\right\} \leq \mathcal{L}_q(\tau),$$

for $\tau \in \mathcal{J}$ and all $\|y\|^2 \leq q$.

Definition 2.11. An $\mathscr{F}_\tau$-adapted stochastic process $y \in \mathcal{PC}\ (\mathcal{J}, \mathscr{L}^2(\Omega, \mathscr{F}, \mathscr{P}))$ is said to be a mild solution of (1), if $y(0) = y_0$, and the impulsive condition $\Delta y|_{\tau=\tau_{\mathfrak{r}}} = \mathcal{I}_{\mathfrak{r}}(y(\tau_{\mathfrak{r}}^-)), \mathfrak{r} = 1, 2, \ldots, \hat{n}$, then there exists $\hat{h} \in \mathscr{L}^2(\mathcal{J}, \mathscr{L}(\mathbb{W}, \mathbb{H}))$ such that $\hat{h}(\tau) \in \mathcal{G}(\tau, y(\tau))$ on $\tau \in \mathcal{J}$ and the integral equation

$$y(\tau) = \mathcal{R}(\tau)y_0 + \int_0^\tau \mathcal{R}(\tau - \vartheta)\hat{h}(\vartheta)dw(\vartheta) + \int_0^\tau \mathcal{R}(\tau - \vartheta)Bu(\vartheta)d\vartheta + \sum_{0<\tau_{\mathfrak{r}}<\tau} \mathcal{R}(\tau - \tau_{\mathfrak{r}})\mathcal{I}_{\mathfrak{r}}(y(\tau_{\mathfrak{r}})),$$

is satisfied.

It is realistic to define the operators here,

$$\Pi_0^c = \int_0^c \mathcal{R}(c-\vartheta)BB^*\mathcal{R}^*(c-\vartheta)d\vartheta : \mathbb{H} \to \mathbb{H},$$

$$\mathcal{S}(\alpha, \Pi_0^c) = (\alpha I + \Pi_0^c)^{-1} : \mathbb{H} \to \mathbb{H}.$$

In the above B^* and $\mathcal{R}^*(\tau)$ represents the adjoints of B and $\mathcal{R}(\tau)$. Clearly, Π_0^c is a bounded linear operator.

To examine the system of approximate controllability, we set the following assumption:

($\mathbf{H_0}$) $\alpha\mathcal{S}(\alpha, \Pi_0^c) \to 0$ as $\alpha \to 0^+$ the strong operator topology.

Observing in [5], ($\mathbf{H_0}$) holds iff the linear differential system

$$\begin{cases} y'(\tau) \in Ay(\tau) + \int_0^\tau g(\tau-\vartheta)y(\vartheta)d\vartheta, \ \tau \in \mathcal{J}, \\ y(0) = y_0, \end{cases} \tag{5}$$

is approximately controllable on $\mathcal{J}$.

Lemma 2.12. *[45] Assume that $\mathcal{J}$ is a compact real interval, the set of all nonempty, closed, bounded, and convex subsets based on $\mathbb{H}$ is known as $BCC(\mathbb{H})$, and $\mathcal{G}$ is a multivalued map fulfilling $\mathcal{G} : [0,c] \times \mathbb{H} \to BCC(\mathbb{H})$ is measurable to τ for each fixed $y \in \mathbb{H}$, u.s.c. to y for each $\tau \in \mathcal{J}$, and for every $y \in C(\mathcal{J}, \mathbb{H})$ the set*

$$\mathbb{S}_{\mathcal{G},y} = \left\{\hat{h} \in \mathscr{L}^2(\mathcal{J}, \mathscr{L}(\mathbb{W}, \mathbb{H})) : \hat{h}(\tau) \in \mathcal{G}(\tau, y(\tau)), \ \tau \in [0,c]\right\},$$

is nonempty. Consider Ξ as a linear continuous form $\mathscr{L}^2(\mathcal{J}, \mathbb{H}) \to C(\mathcal{J}, \mathbb{H})$, then the operator

$$\Xi \circ \mathbb{S}_{\mathcal{G}} : C(\mathcal{J}, \mathbb{H}) \to BCC(C(\mathcal{J}, \mathbb{H})), \ y \to (\Xi \circ \mathbb{S}_{\mathcal{G}})(y) = \Xi(\mathbb{S}_{\mathcal{G},y}),$$

is a closed graph operator in $C(\mathcal{J}, \mathbb{H}) \times C(\mathcal{J}, \mathbb{H})$.

Lemma 2.13. *[46] Consider $\hat{\mathfrak{F}}$ as a nonempty subset of $\mathbb{H}$, which is bounded, closed and convex. Assume that $\widehat{\mathbb{K}} : \hat{\mathfrak{F}} \to 2^{\mathbb{H}} \setminus \{\emptyset\}$ is upper semicontinuous with closed, convex values, and such that $\widehat{\mathbb{K}}(\hat{\mathfrak{F}}) \subseteq \hat{\mathfrak{F}}$, and $\widehat{\mathbb{K}}(\hat{\mathfrak{F}})$ are compact. Therefore, $\widehat{\mathbb{K}}$ has a fixed point.*

3. Controllability results

In this section, we formulate and establish the approximate controllability results for the problem (1). We have the following assumptions to illustrate the main theorem:

($\mathbf{H_1}$) The operator $\mathcal{R}(\tau)$, $\tau > 0$ is compact.

($\mathbf{H_2}$) The multivalued map $\mathcal{G} : \mathcal{J} \times \mathbb{H} \to BCC(\mathbb{H})$ is an $\mathscr{L}^2$–Caratheodory function which fulfill the following assumption:
For each $\tau \in \mathcal{J}$, the function $\mathcal{G}(\tau, \cdot)$ is u.s.c., and for each $y \in \mathbb{H}$, the function $\mathcal{G}(\cdot, y)$ is measurable and for all $y \in \mathbb{H}$, the set

$$\mathbb{S}_{\mathcal{G},y} = \left\{\hat{h} \in \mathscr{L}^2(\mathcal{J}, \mathscr{L}(\mathbb{W}, \mathbb{H})) : \hat{h}(\tau) \in \mathcal{G}\big(\tau, y(\tau)\big), \ \tau \in \mathcal{J}\right\},$$

is nonempty.

($\mathbf{H_3}$) For $q > 0$ and $y \in \mathcal{PC}$ with $\|y\|^2_{\mathcal{PC}} \le q$ and $\mathcal{L}_{\hat{h},q}(\cdot) \in \mathscr{L}^1(\mathcal{J}, \mathbb{R}^+)$ such that

$$\sup\left\{E\|\hat{h}\|^2 : \hat{h}(\tau) \in \mathcal{G}(\tau, y(\tau))\right\} \le \mathcal{L}_{\hat{h},q}(\tau),$$

for a.e. $\tau \in \mathcal{J}$.

($\mathbf{H_4}$) The function $\vartheta \to \mathcal{L}_{\hat{h},q}(\vartheta) \in \mathscr{L}^1(\mathcal{J}, \mathbb{R}^+)$ and there exists $\mu > 0$ such that

$$\liminf_{q\to\infty} \frac{\int_0^\tau \mathcal{L}_{\hat{h},q}(\vartheta)d\vartheta}{q} = \mu < \infty.$$

($\mathbf{H_5}$) $\mathcal{I}_{\mathfrak{r}} \in C(\mathbb{H}, \mathbb{H})$ and there exists continuous non decreasing functions $\mathbb{Y}_{\mathfrak{r}}$ mapping from $[0, +\infty)$ into $(0, +\infty)$ such that

$$E\|\mathcal{I}_{\mathfrak{r}}(y)\|^2 \le \mathbb{Y}_{\mathfrak{r}}(\|y\|^2), \ \mathfrak{r} = 1, 2, \ldots, \hat{n}, \ y \in \mathbb{H},$$

and

$$\liminf_{q\to\infty} \frac{\mathbb{Y}_{\mathfrak{r}}(q)}{q} = \mathfrak{d}_{\mathfrak{r}} < \infty, \ \mathfrak{r} = 1, 2, \ldots, \hat{n}.$$

Lemma 3.1. *For any* $y_c \in \mathscr{L}^2(\mathscr{F}_c, \mathbb{H})$, *there exists* $\varphi \in \mathscr{L}^2_{\mathscr{F}}(\Omega, \mathscr{L}^2(\mathcal{J}, \mathscr{L}(\mathbb{W}, \mathbb{H})))$ *such that*

$$y_c = Ey_c + \int_0^c \varphi(\vartheta)dw(\vartheta).$$

As we establish (1) is approximately controllable, if for all $\alpha > 0$, $y_c \in \mathscr{L}^2(\mathscr{F}_c, \mathbb{H})$ and for $\hat{\hbar} \in \mathbb{S}_{\mathcal{G},y}$, then there exists a continuous function $y(\cdot)$ such that

$$y(\tau) = \mathcal{R}(\tau)y_0 + \int_0^\tau \mathcal{R}(\tau-\vartheta)\hat{\hbar}(\vartheta)dw(\vartheta) + \int_0^\tau \mathcal{R}(\tau-\vartheta)Bu(\vartheta,y)d\vartheta + \sum_{0<\tau_\mathfrak{r}<\tau} \mathcal{R}(\tau-\tau_\mathfrak{r})\mathcal{I}_\mathfrak{r}(y(\tau_\mathfrak{r})),\ \hat{\hbar} \in \mathbb{S}_{\mathcal{G},y}, \tag{6}$$

$$u(\tau, y) = B^*\mathcal{R}^*(c-\tau)\mathcal{S}(\alpha, \Pi_0^c)p(y(\cdot)), \tag{7}$$

where

$$p(y(\cdot)) = y_c - \mathcal{R}(c)y_0 - \int_0^c \mathcal{R}(c-\vartheta)\hat{\hbar}(\vartheta)dw(\vartheta) - \sum_{0<\tau_\mathfrak{r}<c} \mathcal{R}(c-\tau_\mathfrak{r})\mathcal{I}_\mathfrak{r}(y(\tau_\mathfrak{r})). \tag{8}$$

Theorem 3.2. *If the assumptions* $(\mathbf{H_0})$-$(\mathbf{H_5})$ *are fulfilled, then the system* (1) *has a mild solution on* $\mathcal{J}$ *given that*

$$4\mathbb{Y}^2\left(1 + \frac{4}{\alpha^2}\mathbb{Y}^4\mathbb{Y}_B^4c^2\right)\left[Tr(Q)\mu + \hat{n}\sum_{\mathfrak{r}=1}^{\hat{n}} \mathfrak{d}_\mathfrak{r}\right] < 1, \tag{9}$$

where $\mathbb{Y}_B = \|B\|$.

Proof. The primary intention of this theorem is to determine the conditions for (6) and (7) being solvable for $\alpha > 0$. By proving this, applying control $u(y,\tau)$ and the operator $\hat{\Lambda} : \mathcal{PC} \to 2^{\mathcal{PC}}$, defined by

$$\hat{\Lambda}(y) = \begin{cases} \mathbb{V} \in \mathcal{PC} : \mathbb{V}(\tau) = \mathcal{R}(\tau)y_0 + \int_0^\tau \mathcal{R}(\tau-\vartheta)\hat{\hbar}(\vartheta)dw(\vartheta) + \int_0^\tau \mathcal{R}(\tau-\vartheta)Bu(\vartheta,y)d\vartheta \\ \qquad + \sum_{0<\tau_\mathfrak{r}<\tau} \mathcal{R}(\tau-\tau_\mathfrak{r})\mathcal{I}_\mathfrak{r}(y(\tau_\mathfrak{r})),\ \tau \in \mathcal{J}, \end{cases} \tag{10}$$

has a fixed point y, which is a mild solution of (1). We now find that $\hat{\Lambda}$ fulfills the conditions of Lemma 2.13. In our convenient, we split the proof in to five steps.

Step 1: $\hat{\Lambda}$ is convex for each $y \in \mathcal{PC}$. In case, providing that $\mathbb{V}_1, \mathbb{V}_2 \in \hat{\Lambda}(y)$, there exists $\hat{\hbar}_1, \hat{\hbar}_2 \in \mathbb{S}_{\mathcal{G},y}$ such that for each $\tau \in \mathcal{J}$, we have

$$\begin{aligned}\mathbb{V}_i(\tau) &= \mathcal{R}(\tau)y_0 + \int_0^\tau \mathcal{R}(\tau-\vartheta)\hat{\hbar}_i(\vartheta)dw(\vartheta) + \int_0^\tau \mathcal{R}(\tau-\vartheta)BB^*\mathcal{R}^*(c-\tau)\mathcal{S}(\alpha,\Pi_0^c) \\ &\times \left[Ey_c + \int_0^c \varphi(\jmath)dw(\jmath) - \mathcal{R}(c)y_0 - \int_0^c \mathcal{R}(c-\jmath)\hat{\hbar}_i(\jmath)dw(\jmath) \right. \\ &\left. - \sum_{0<\tau_\mathfrak{r}<c} \mathcal{R}(c-\tau_\mathfrak{r})\mathcal{I}_\mathfrak{r}(y(\tau_\mathfrak{r}))\right]d\vartheta + \sum_{0<\tau_\mathfrak{r}<\tau} \mathcal{R}(\tau-\tau_\mathfrak{r})\mathcal{I}_\mathfrak{r}(y(\tau_\mathfrak{r})),\ i = 1,2.\end{aligned}$$

Let $\psi \in [0,1]$. Then $\forall\ \tau \in \mathcal{J}$, we get

$$\begin{aligned}\psi\mathbb{V}_1(\tau) + (1-\psi)\mathbb{V}_2(\tau) &= \mathcal{R}(\tau)y_0 + \int_0^\tau \mathcal{R}(\tau-\vartheta)[\psi\hat{\hbar}_1(\vartheta) + (1-\psi)\hat{\hbar}_2(\vartheta)]dw(\vartheta) \\ &+ \int_0^\tau \mathcal{R}(\tau-\vartheta)BB^*\mathcal{R}^*(c-\tau)\mathcal{S}(\alpha,\Pi_0^c) \times \left[Ey_c + \int_0^c \varphi(\jmath)dw(\jmath)\right. \\ &- \mathcal{R}(c)y_0 - \int_0^c \mathcal{R}(c-\jmath)[\psi\hat{\hbar}_1(\jmath) + (1-\psi)\hat{\hbar}_2(\jmath)]dw(\jmath) \\ &\left. - \sum_{0<\tau_\mathfrak{r}<c} \mathcal{R}(c-\tau_\mathfrak{r})\mathcal{I}_\mathfrak{r}(y(\tau_\mathfrak{r}))\right]d\vartheta + \sum_{0<\tau_\mathfrak{r}<\tau} \mathcal{R}(\tau-\tau_\mathfrak{r})\mathcal{I}_\mathfrak{r}(y(\tau_\mathfrak{r})).\end{aligned}$$

Since $\mathcal{G}$ has convex values, it is easy to observe that $\mathbb{S}_{\mathcal{G},y}$ is convex. So, $\psi\mathbb{V}_1 + (1-\psi)\mathbb{V}_2 \in \mathbb{S}_{\mathcal{G},y}$. Hence,

$$\psi\mathbb{V}_1 + (1-\psi)\mathbb{V}_2 \in \hat{\Lambda}(y).$$

Step 2: For $q > 0$, consider $\mathcal{B}_q = \{y \in \mathcal{PC} : \|y(\tau)\|^2_{\mathcal{PC}} \leq q \ \forall \ \tau \in \mathcal{J}\}$. Obviously, $\mathcal{B}_q$ is a closed, bounded and convex set of $\mathcal{PC}$. We state that there exists q such that $\hat{\Lambda}(\mathcal{B}_q) \subseteq (\mathcal{B}_q)$. Unless this is false, then for each $q > 0$, there exists $y^q \in \mathcal{B}_q$, but $\hat{\Lambda}(y^q) \notin \mathcal{B}_q$, that is

$$E\|\hat{\Lambda}(y^q)\|^2_{\mathcal{PC}} = \sup\{\|\mathbb{V}^q\|^2_{\mathcal{PC}} : \mathbb{V}^q \in \hat{\Lambda}(y^q)\} > q,$$

and

$$\begin{aligned}\mathbb{V}^q(\tau) =& \mathcal{R}(\tau)y_0 + \int_0^\tau \mathcal{R}(\tau-\vartheta)\hat{h}^q(\vartheta)dw(\vartheta) + \int_0^\tau \mathcal{R}(\tau-\vartheta)Bu^q(\vartheta,y)d\vartheta \\ &+ \sum_{0<\tau_\mathfrak{r}<\tau} \mathcal{R}(\tau-\tau_\mathfrak{r})\mathcal{I}_\mathfrak{r}(y(\tau_\mathfrak{r})),\end{aligned}$$

for some $\hat{h}^q \in \mathbb{S}_{\mathcal{G},y^q}$, applying $(\mathbf{H_0})-(\mathbf{H_5})$, we get

$$\begin{aligned} q &\leq E\|\hat{\Lambda}(y^q)(\tau)\|^2 \\ &\leq 4E\|\mathcal{R}(\tau)y_0\|^2 + 4E\|\int_0^\tau \mathcal{R}(\tau-\vartheta)\hat{h}^q(\vartheta)dw(\vartheta)\|^2 + 4E\|\int_0^\tau \mathcal{R}(\tau-\vartheta)Bu^q(\vartheta,y)d\vartheta\|^2 \\ &\quad + 4E\|\sum_{0<\tau_\mathfrak{r}<\tau}\mathcal{R}(\tau-\tau_\mathfrak{r})\mathcal{I}_\mathfrak{r}(y(\tau_\mathfrak{r}))\|^2 \\ &\leq 4\mathbb{Y}^2 E\|y_0\|^2 + 4\mathbb{Y}^2 Tr(Q)\int_0^\tau \mathcal{L}_{\hat{h},q}(\vartheta)d\vartheta + \frac{16}{\alpha^2}\mathbb{Y}^4\mathbb{Y}_B^4 c^2\Big[2E\|y_c\|^2 + 2E\|\int_0^c \varphi(\vartheta)dw(\vartheta)\|^2 \\ &\quad + \mathbb{Y}^2E\|y_0\|^2 + \mathbb{Y}^2 Tr(Q)\int_0^c \mathcal{L}_{\hat{h},q}(\vartheta)d\vartheta + \hat{n}\mathbb{Y}^2\sum_{\mathfrak{r}=1}^{\hat{n}}\mathbb{Y}_\mathfrak{r}(q)\Big] + 4\hat{n}\mathbb{Y}^2\sum_{\mathfrak{r}=1}^{\hat{n}}\mathbb{Y}_\mathfrak{r}(q).\end{aligned}$$

Dividing q on both sides and assuming limits as $q \to \infty$, applying $(\mathbf{H_3})-(\mathbf{H_5})$, we have

$$4\mathbb{Y}^2\left(1 + \frac{4}{\alpha^2}\ \mathbb{Y}^4\mathbb{Y}_B^4c^2\right)\left[Tr(Q)\mu + \hat{n}\sum_{\mathfrak{r}=1}^{\hat{n}}\mathfrak{d}_\mathfrak{r}\right] \geq 1.$$

This is contradiction to our assumptions (9). So, $q > 0$ and for all $\hat{h} \in \mathbb{S}_{\mathcal{G},y}$, $\hat{\Lambda}(\mathcal{B}_q) \subseteq \mathcal{B}_q$.
Step 3: $\hat{\Lambda}$ maps bounded sets into equicontinuous sets of $\mathcal{PC}$. For each $y \in \mathcal{B}_q, \mathbb{V} \in \hat{\Lambda}(y)$, there exists $\hat{h} \in \mathbb{S}_{\mathcal{G},y}$ such that

$$\mathbb{V}(\tau) = \mathcal{R}(\tau)y_0 + \int_0^\tau \mathcal{R}(\tau-\vartheta)\hat{h}(\vartheta)dw(\vartheta) + \int_0^\tau \mathcal{R}(\tau-\vartheta)Bu(\vartheta,y)d\vartheta + \sum_{0<\tau_\mathfrak{r}<\tau}\mathcal{R}(\tau-\tau_\mathfrak{r})\mathcal{I}_\mathfrak{r}(y(\tau_\mathfrak{r})).$$

Let $\varepsilon > 0$ and $0 < \tau_1 < \tau_2 \leq c$, then

$$\begin{aligned} E\|\mathbb{V}(\tau_1)-\mathbb{V}(\tau_2)\|^2 \leq\ & 9E\|\mathcal{R}(\tau_1)-\mathcal{R}(\tau_2)\|^2\|y_0\|^2 + 9E\left\|\int_0^{\tau_1-\varepsilon}[\mathcal{R}(\tau_1-\vartheta)-\mathcal{R}(\tau_2-\vartheta)]\hat{h}(\vartheta)dw(\vartheta)\right\|^2 \\ &+ 9E\left\|\int_{\tau_1-\varepsilon}^{\tau_1}[\mathcal{R}(\tau_1-\vartheta)-\mathcal{R}(\tau_2-\vartheta)]\hat{h}(\vartheta)dw(\vartheta)\right\|^2 \\ &+ 9E\left\|\int_{\tau_1}^{\tau_2}\mathcal{R}(\tau_2-\vartheta)\hat{h}(\vartheta)dw(\vartheta)\right\|^2 \\ &+ 9E\left\|\int_0^{\tau_1-\varepsilon}[\mathcal{R}(\tau_1-\jmath)-\mathcal{R}(\tau_2-\jmath)]Bu(\jmath,y)d\jmath\right\|^2 \\ &+ 9E\left\|\int_{\tau_1-\varepsilon}^{\tau_1}[\mathcal{R}(\tau_1-\jmath)-\mathcal{R}(\tau_2-\jmath)]Bu(\jmath,y)d\jmath\right\|^2 \\ &+ 9E\left\|\int_{\tau_1}^{\tau_2}[\mathcal{R}(\tau_2-\jmath)]Bu(\jmath,y)d\jmath\right\|^2 \end{aligned}$$

$$
\begin{aligned}
&+9E\left\|\sum_{0<\tau_{\mathfrak{r}}<\tau_1}[\mathcal{R}(\tau_1-\tau_{\mathfrak{r}})-\mathcal{R}(\tau_2-\tau_{\mathfrak{r}})]\mathcal{I}_{\mathfrak{r}}(y(\tau_{\mathfrak{r}}))\right\|^2\\
&+9E\left\|\sum_{\tau_1<\tau_{\mathfrak{r}}<\tau_2}[\mathcal{R}(\tau_2-\tau_{\mathfrak{r}})]\mathcal{I}_{\mathfrak{r}}(y(\tau_{\mathfrak{r}}))\right\|^2\\
\le\ &9E\|\mathcal{R}(\tau_1)-\mathcal{R}(\tau_2)\|^2\|y_0\|^2\\
&+9Tr(Q)\int_0^{\tau_1-\varepsilon}E\|[\mathcal{R}(\tau_1-\vartheta)-\mathcal{R}(\tau_2-\vartheta)]\|^2\mathcal{L}_{\hat{h},q}(\vartheta)d\vartheta\\
&+9Tr(Q)\int_{\tau_1-\varepsilon}^{\tau_1}E\|[\mathcal{R}(\tau_1-\vartheta)-\mathcal{R}(\tau_2-\vartheta)]\|^2\mathcal{L}_{\hat{h},q}(\vartheta)d\vartheta\\
&+9Tr(Q)\mathbb{Y}^2\int_{\tau_1}^{\tau_2}\mathcal{L}_{\hat{h},q}(\vartheta)d\vartheta\\
&+9\mathbb{Y}_B^2(\tau_1-\epsilon)\int_0^{\tau_1-\varepsilon}E\|\mathcal{R}(\tau_1-\jmath)-\mathcal{R}(\tau_2-\jmath)\|^2\|u(\jmath,y)\|^2d\jmath\\
&+9\mathbb{Y}_B^2(\epsilon)\int_{\tau_1-\varepsilon}^{\tau_1}E\|\mathcal{R}(\tau_1-\jmath)-\mathcal{R}(\tau_2-\jmath)\|^2\|u(\jmath,y)\|^2d\jmath\\
&+9\mathbb{Y}^2\mathbb{Y}_B^2(\tau_1-\tau_2)\int_{\tau_1}^{\tau_2}E\|u(\jmath,y)\|^2d\jmath\\
&+9\sum_{0<\tau_{\mathfrak{r}}<\tau_1}E\|[\mathcal{R}(\tau_1-\tau_{\mathfrak{r}})-\mathcal{R}(\tau_2-\tau_{\mathfrak{r}})]\mathcal{I}_{\mathfrak{r}}(y(\tau_{\mathfrak{r}}))\|^2\\
&+9\mathbb{Y}^2\sum_{\tau_1<\tau_{\mathfrak{r}}<\tau_2}\mathbb{Y}_{\mathfrak{r}}(q).
\end{aligned}
$$

Since $(\tau_1-\tau_2)\to 0$ and ε are sufficiently small, the R.H.S. of the previous inequality approaches zero independently of $y\in\mathcal{B}_q$, then, represents the compactness of $\mathcal{R}(\tau)$ requires the continuity in the uniform operator topology. As a result, $\hat{\Lambda}(y^q)$ expresses $\mathcal{B}_q$ into an equicontinuous set.

Step 4: The set $\Psi(\tau)=\{\mathbb{V}(\tau):\mathbb{V}\in\hat{\Lambda}(\mathcal{B}_q)\}$ is relatively compact in $\mathbb{H}$.

Consider $\tau\in(0,c]$ is fixed and ε a real number fulfilling $0<\varepsilon<\tau$. For $y\in\mathcal{B}_q$, we specify

$$
\mathbb{V}_\varepsilon(\tau)=\mathcal{R}(\tau)y_0+\int_0^{\tau-\varepsilon}\mathcal{R}(\tau-\vartheta)\hat{h}(\vartheta)dw(\vartheta)+\int_0^{\tau-\varepsilon}\mathcal{R}(\tau-\jmath)Bu(\jmath,y)d\jmath+\sum_{0<\tau_{\mathfrak{r}}<\tau-\varepsilon}\mathcal{R}(\tau-\tau_{\mathfrak{r}})\mathcal{I}_{\mathfrak{r}}(y(\tau_{\mathfrak{r}})).
$$

Since $\mathcal{R}(\tau)$ is a compact operator, the set $\Psi_\varepsilon(\tau)=\{\mathbb{V}_\varepsilon(\tau):\mathbb{V}_\varepsilon\in\hat{\Lambda}(\mathcal{B}_q)\}$ is relatively compact in $\mathbb{H}$ for all ε, $0<\varepsilon<\tau$. Further, for every $0<\varepsilon<\tau$, we get

$$
E\|\mathbb{V}(\tau)-\mathbb{V}_\varepsilon(\tau)\|^2\le 2\mathbb{Y}^2\int_{\tau-\varepsilon}^{\tau}\mathcal{L}_{\hat{h},q}(\vartheta)dw(\vartheta)+2\mathbb{Y}^2\mathbb{Y}_B^2\epsilon\int_{\tau-\varepsilon}^{\tau}E\|u(\jmath,y)\|^2d\jmath.
$$

Therefore,

$$
E\|\mathbb{V}(\tau)-\mathbb{V}_\varepsilon(\tau)\|^2\to 0 \text{ as } \varepsilon\to 0^+.
$$

Then, there exists relatively compact sets arbitrarily close to the set $\Psi(\tau)=\{\mathbb{V}(\tau):\mathbb{V}\in\hat{\Lambda}(\mathcal{B}_q)\}$, and the set $\widetilde{\Psi}(\tau)$ is relatively compact in $\mathbb{H}$ for all $\tau\in\mathcal{J}$. As a result, $\tau=0$, it is compact. Hence, $\hat{\Lambda}(\tau)$ is relatively compact in $\mathbb{H}$ for all $\tau\in\mathcal{J}$.

Step 5: $\hat{\Lambda}$ has a closed graph. Consider $y_n\to y_*$ as $n\to\infty$, $\mathbb{V}_n\in\hat{\Lambda}(y_n)$ and $\mathbb{V}_n\to\mathbb{V}_*$ as $n\to\infty$. As we explain $\mathbb{V}_*\in\hat{\Lambda}(y_*)$. Since $\mathbb{V}_n\in\hat{\Lambda}(y_n)$ there exists $\hat{h}_n\in\mathbb{S}_{\mathcal{G},y_n}$ such that

$$
\begin{aligned}
\mathbb{V}_n(\tau)=\ &\mathcal{R}(\tau)y_0+\int_0^{\tau}\mathcal{R}(\tau-\vartheta)\hat{h}_n(\vartheta)dw(\vartheta)+\int_0^{\tau}\mathcal{R}(\tau-\vartheta)BB^*\mathcal{R}^*(c-\tau)\mathcal{S}(\alpha,\Pi_0^c)\\
&\times\Big[Ey_c+\int_0^c\varphi(\jmath)dw(\jmath)-\mathcal{R}(c)y_0-\int_0^c\mathcal{R}(c-\jmath)\hat{h}_n(\jmath)dw(\jmath)
\end{aligned}
$$

$$- \sum_{0<\tau_{\mathfrak{r}}<c} \mathcal{R}(c-\tau_{\mathfrak{r}})\mathcal{I}_{\mathfrak{r}}(y_n(\tau_{\mathfrak{r}}))\big]d\vartheta + \sum_{0<\tau_{\mathfrak{r}}<\tau} \mathcal{R}(\tau-\tau_{\mathfrak{r}})\mathcal{I}_{\mathfrak{r}}(y_n(\tau_{\mathfrak{r}})),\ \tau \in \mathcal{J}.$$

To illustrate that there exists $\hat{h}_* \in \mathbb{S}_{\mathcal{G},y_*}$ such that

$$\mathbb{V}_*(\tau) = \mathcal{R}(\tau)y_0 + \int_0^\tau \mathcal{R}(\tau-\vartheta)\hat{h}_*(\vartheta)dw(\vartheta) + \int_0^\tau \mathcal{R}(\tau-\vartheta)BB^*\mathcal{R}^*(c-\tau)\mathcal{S}(\alpha,\Pi_0^c)$$
$$\times \big[Ey_c + \int_0^c \varphi(\jmath)dw(\jmath) - \mathcal{R}(c)y_0 - \int_0^c \mathcal{R}(c-\jmath)\hat{h}_*(\jmath)dw(\jmath)$$
$$- \sum_{0<\tau_{\mathfrak{r}}<c} \mathcal{R}(c-\tau_{\mathfrak{r}})\mathcal{I}_{\mathfrak{r}}(y_*(\tau_{\mathfrak{r}}))\big]d\vartheta + \sum_{0<\tau_{\mathfrak{r}}<\tau} \mathcal{R}(\tau-\tau_{\mathfrak{r}})\mathcal{I}_{\mathfrak{r}}(y_*(\tau_{\mathfrak{r}})),\ \tau \in \mathcal{J}.$$

Now, for each $\tau \in \mathcal{J}$, and clearly, we have

$$\Bigg\|\Bigg(\mathbb{V}_n - \mathcal{R}(\tau)y_0 - \int_0^\tau \mathcal{R}(\tau-\vartheta)BB^*\mathcal{R}^*(c-\tau)\mathcal{S}(\alpha,\Pi_0^c) \times \big[Ey_c + \int_0^c \varphi(\jmath)dw(\jmath)$$
$$- \mathcal{R}(c)y_0 - \sum_{0<\tau_{\mathfrak{r}}<c} \mathcal{R}(c-\tau_{\mathfrak{r}})\mathcal{I}_{\mathfrak{r}}(y_n(\tau_{\mathfrak{r}}))\big]d\vartheta - \sum_{0<\tau_{\mathfrak{r}}<\tau} \mathcal{R}(\tau-\tau_{\mathfrak{r}})\mathcal{I}_{\mathfrak{r}}(y_n(\tau_{\mathfrak{r}}))\Bigg)$$
$$- \Bigg(\mathbb{V}_* - \mathcal{R}(\tau)y_0 - \int_0^\tau \mathcal{R}(\tau-\vartheta)BB^*\mathcal{R}^*(c-\tau)\mathcal{S}(\alpha,\Pi_0^c) \times \big[Ey_c + \int_0^c \varphi(\jmath)dw(\jmath)$$
$$- \mathcal{R}(c)y_0 - \sum_{0<\tau_{\mathfrak{r}}<c} \mathcal{R}(c-\tau_{\mathfrak{r}})\mathcal{I}_{\mathfrak{r}}(y_*(\tau_{\mathfrak{r}}))\big]d\vartheta - \sum_{0<\tau_{\mathfrak{r}}<\tau} \mathcal{R}(\tau-\tau_{\mathfrak{r}})\mathcal{I}_{\mathfrak{r}}(y_*(\tau_{\mathfrak{r}}))\Bigg)\Bigg\|_{\mathcal{PC}}^2 \to 0 \text{ as } n \to \infty.$$

Consider the linear continuous operator $\mho : \mathscr{L}^2([0,c],\mathbb{H}) \to C(\mathcal{J},\mathbb{H})$,

$$(\mho\hat{h})(\tau) = \int_0^\tau \mathcal{R}(\tau-\vartheta)\Bigg[\hat{h}(\vartheta) - BB^*\mathcal{R}^*(c-\tau) \times \Bigg(\int_0^c \mathcal{R}(c-\jmath)\hat{h}(\jmath)d\jmath\Bigg)\Bigg]d\vartheta.$$

The operator $\mho$ is continuous and linear. For $\mho \circ \mathbb{S}_{\mathcal{G}}$ is a closed graph operator deriving once again from Lemma 2.13. Furthermore,

$$\Bigg(\mathbb{V}_n(\tau) - \mathcal{R}(\tau)y_0 - \int_0^\tau \mathcal{R}(\tau-\vartheta)BB^*\mathcal{R}^*(c-\tau)\mathcal{S}(\alpha,\Pi_0^c) \times \big[Ey_c + \int_0^c \varphi(\jmath)dw(\jmath)$$
$$- \mathcal{R}(c)y_0 - \sum_{0<\tau_{\mathfrak{r}}<c} \mathcal{R}(c-\tau_{\mathfrak{r}})\mathcal{I}_{\mathfrak{r}}(y_n(\tau_{\mathfrak{r}}))\big]d\vartheta - \sum_{0<\tau_{\mathfrak{r}}<\tau} \mathcal{R}(\tau-\tau_{\mathfrak{r}})\mathcal{I}_{\mathfrak{r}}(y_n(\tau_{\mathfrak{r}}))\Bigg) \in \mho(\mathbb{S}_{\mathcal{G},y_n}).$$

Then, $y_n \to y_*$ as $n \to \infty$, Lemma 2.13 again mentioned that

$$\Bigg(\mathbb{V}_*(\tau) - \mathcal{R}(\tau)y_0 - \int_0^\tau \mathcal{R}(\tau-\vartheta)BB^*\mathcal{R}^*(c-\tau)\mathcal{S}(\alpha,\Pi_0^c) \times \big[Ey_c + \int_0^c \varphi(\jmath)dw(\jmath)$$
$$- \mathcal{R}(c)y_0 - \sum_{0<\tau_{\mathfrak{r}}<c} \mathcal{R}(c-\tau_{\mathfrak{r}})\mathcal{I}_{\mathfrak{r}}(y_*(\tau_{\mathfrak{r}}))\big]d\vartheta - \sum_{0<\tau_{\mathfrak{r}}<\tau} \mathcal{R}(\tau-\tau_{\mathfrak{r}})\mathcal{I}_{\mathfrak{r}}(y_*(\tau_{\mathfrak{r}}))\Bigg) \in \mho(\mathbb{S}_{\mathcal{G},y_*}).$$

Therefore, $\hat{\Lambda}$ has a closed graph.

As a consequence of **Steps 1 – 5** together with the Arzela-Ascoli theorem, we conclude that $\hat{\Lambda}$ is a compact multivalued map, u.s.c. with convex closed values. As a consequence of Lemma 2.13, we can deduce that $\hat{\Lambda}$ has a fixed point y which is a mild solution of system (1).

Definition 3.3. The system (1) is said to be approximately controllable on $\mathcal{J}$, if $\overline{\mathscr{R}(c,y_0)} = \mathbb{H}$, then

$$\mathscr{R}(c,y_0) = \{y_c(y_0;u) : u(\cdot) \in \mathscr{L}^2(\mathcal{J},\mathcal{Z})\},$$

is known as the reachable set if (1) at terminal time c and its closure in $\mathbb{H}$ is denoted by $\overline{\mathscr{R}(c,y_0)}$; assume that $y_c(y_0,u)$ is the state value of (1) at terminal time c corresponding to the control u and the initial value $y_0 \in \mathbb{H}$.

In general, $y_0 \in \mathbb{H}$ is the result of a given initial point. The approximate controllability of the linear system (5) in the following theorem will be demonstrated to imply the approximate controllability of the nonlinear differential system (1) in specific cases.

Theorem 3.4. *Consider the assumptions* ($\mathbf{H_0}$)-($\mathbf{H_5}$) *are fulfilled, and the function* $\hat{h}$ *is uniformly bounded. Moreover,* $\mathcal{T}(\tau)$ *is compact, then the nonlinear stochastic differential system* (1) *is approximately controllable on* $\mathcal{J}$.

Proof. Consider $\hat{y}^{\alpha}(\cdot)$ is a fixed point of $\hat{\Lambda}$ in $\mathcal{B}_q$. By using stochastic Fubini theorem, clearly we observe that

$$\hat{y}^{\alpha}(c) = y_c - \alpha(\alpha I + \Pi_0^c)^{-1}\left[Ey_c + \int_0^c \varphi(\vartheta)dw(\vartheta)\right] - \alpha(\alpha I + \Pi_0^c)^{-1}\mathcal{R}(c)y_0 - \alpha(\alpha I + \Pi_0^c)^{-1}$$
$$\times \int_0^c \mathcal{R}(c-\vartheta)\hat{h}(\vartheta, \hat{y}^{\alpha}(\vartheta))dw(\vartheta) - \alpha(\alpha I + \Pi_0^c)^{-1} \sum_{0<\tau_{\mathfrak{r}}<c} \mathcal{R}(c-\tau_{\mathfrak{r}})\mathcal{I}_{\mathfrak{r}}(\hat{y}^{\alpha}(\tau_{\mathfrak{r}})). \tag{11}$$

Under the assumption $\hat{h}$ is uniformly bounded, then there exists $\mathbb{M} > 0$ such that

$$\|\hat{h}(\vartheta, \hat{y}^{\alpha}(\vartheta))\|^2 \le \mathbb{M},$$

in $\mathcal{J} \times \Omega$.

Then there is a subsequence represented by $\{\hat{h}(\vartheta, \hat{y}^{\alpha}(\vartheta))\}$ and $\{\mathcal{I}(\hat{y}^{\alpha}(\upsilon))\}$ are weakly convergent to say $\{\hat{h}(\vartheta)\}$ and $\{\mathcal{I}(\upsilon)\}$ in $\mathbb{H}\times\mathscr{L}_0^2$ and $\mathbb{H}\times\mathbb{H}$. Now, the compactness of $\mathcal{T}(\tau)$ implies that

$$\mathcal{R}(c-\vartheta)\hat{h}(\vartheta, \hat{y}^{\alpha}(\vartheta)) \to \mathcal{R}(c-\vartheta)\hat{h}(\vartheta) \text{ and } \mathcal{R}(c-\tau_{\mathfrak{r}})\mathcal{I}_{\mathfrak{r}}(\hat{y}^{\alpha}(\tau_{\mathfrak{r}})) \to \mathcal{R}(c-\tau_{\mathfrak{r}})\mathcal{I}_{\mathfrak{r}}(\upsilon).$$

$$\begin{aligned}
E\|\hat{y}^{\alpha}(c) - y_c\|^2 \le & 5E\|\alpha(\alpha I + \Pi_0^c)^{-1}y_c\|^2 \\
& + 5E\left(\|\alpha(\alpha I + \Pi_0^c)^{-1}\varphi(\vartheta)dw(\vartheta)\|\right)^2 \\
& + 5E\left(\|\alpha(\alpha I + \Pi_0^c)^{-1}\mathcal{R}(c)y_0 d\vartheta\|\right)^2 \\
& + 5E\left(\int_0^c \|\alpha(\alpha I + \Pi_0^c)^{-1}\mathcal{R}(c-\vartheta)\left[\hat{h}(\vartheta, \hat{y}^{\alpha}(\vartheta)) - \hat{h}(\vartheta)\right]d\vartheta\|\right)^2 \\
& + 5E\left(\|\alpha(\alpha I + \Pi_0^c)^{-1}\sum_{0<\tau_{\mathfrak{r}}<c}\mathcal{R}(c-\tau_{\mathfrak{r}})\mathcal{I}_{\mathfrak{r}}(\hat{y}^{\alpha}(\tau_{\mathfrak{r}}))\|\right)^2 \to 0 \text{ as } \alpha \to 0^+.
\end{aligned}$$

By referring the hypothesis ($\mathbf{H_0}$) and for all $0 \le \vartheta \le c$, the operator $\alpha(\alpha I + \Pi_0^c)^{-1}$ strongly as $\alpha \to 0^+$, and furthermore, $\|\alpha(\alpha I + \Pi_0^c)^{-1}\| \le 1$. Thus, by the Lebesgue-dominated convergence theorem, we obtain that $E\|\hat{y}^{\alpha}(c) - y_c\|^2 \to 0$ as $\alpha \to 0^+$. This is shown that the system (1) is approximate controllability.

4. Control systems with nonlocal conditions

The study of a system with nonlocal conditions is driven by physical problems. For example, inverse heat conduction situations are employed to determine unknown physical parameters [47]. To abstract Cauchy problems with the nonlocal condition was initially introduced by [48–50], their outcomes regard the existence and uniqueness of mild solutions. The researchers of [51] point out that describing physical processes is more useful for solving the nonlocal initial value problem. In the article [52], the authors established the existence of the mild solution for neutral stochastic integrodifferential systems with impulsive effects and nonlocal conditions. For further details, refer to [53–57].

We examine the approximate controllability of impulsive neutral stochastic integrodifferential systems with nonlocal conditions through resolvent operators of the form:

$$\begin{cases}
d\left(y(\tau) + \int_0^\tau \mathbb{Q}(\tau-\vartheta)y(\vartheta)d\vartheta\right) \in \left[Ay(\tau) + \int_0^\tau g(\tau-\vartheta)y(\vartheta)d\vartheta + Bu(\tau)\right]d\tau \\
\qquad\qquad +\mathcal{G}(\tau, y(\tau))dw(\tau), \ \mathcal{J} = [0, c], \ \tau \ne \tau_{\mathfrak{r}}, \\
\Delta y|_{\tau=\tau_{\mathfrak{r}}} = \mathcal{I}_{\mathfrak{r}}(y(\tau_{\mathfrak{r}})), \ \mathfrak{r} = 1, 2, \dots, \hat{n}, \\
y(0) = y_0 - \zeta(y).
\end{cases} \tag{12}$$

The system (12) satisfies the following assumption:

($\mathbf{H_6}$) $\zeta : \mathcal{PC}(\mathcal{J}, \mathbb{H}) \to \mathbb{H}$ is continuous and there exists a constant $\mathbb{L} > 0$ such that

$$E\|\zeta(y)\|^2 \le \mathbb{L}, \ y \in \mathcal{PC}(\mathcal{J}, \mathbb{H}).$$

The nonlocal term ζ has a better effect on the results and is also accurate for physical measurements than the classical condition $y(0) = y_0$ alone. Therefore, $\zeta(y)$ can be represented as

$$\zeta(y) = \sum_{j=1}^{\tilde{\eta}} l_j y(\tau_j),$$

where $l_j (j = 1, 2, \ldots, \tilde{\eta})$ are given constants and $0 < \tau_1 < \tau_2 < \ldots < \tau_{\tilde{\eta}} \le c$.

Definition 4.1. An $\mathscr{F}_\tau$-adapted stochastic process $y \in \mathcal{PC}\ (\mathcal{J}, \mathscr{L}^2(\Omega, \mathscr{F}, \mathscr{P}))$ is said to be a mild solution of (1), if $y(0) = y_0 - \zeta(y)$, and the impulsive condition $\Delta y|_{\tau=\tau_\mathfrak{r}} = \mathcal{I}_\mathfrak{r}(y(\tau_\mathfrak{r}^-)), \mathfrak{r} = 1, 2, \ldots, \hat{n}$, then there exists $\hat{h} \in \mathscr{L}^2(\mathcal{J}, \mathscr{L}(\mathbb{W}, \mathbb{H}))$ such that $\hat{h}(\tau) \in \mathcal{G}(\tau, y(\tau))$ on $\tau \in \mathcal{J}$ and the integral equation

$$y(\tau) = \mathcal{R}(\tau)[y_0 - \zeta(y)] + \int_0^\tau \mathcal{R}(\tau - \vartheta)\hat{h}(\vartheta) dw(\vartheta) + \int_0^\tau \mathcal{R}(\tau - \vartheta) Bu(\vartheta) d\vartheta + \sum_{0<\tau_\mathfrak{r}<\tau} \mathcal{R}(\tau - \tau_\mathfrak{r})\mathcal{I}_\mathfrak{r}(y(\tau_\mathfrak{r})),$$

is satisfied.

Theorem 4.2. *Assume the assumptions of Theorem 3.2 are fulfilled. Moreover, if assumption* ($\mathbf{H_6}$) *fulfilled, then the system* (12) *is approximately controllable on* $\mathcal{J}$ *given that*

$$4\mathbb{Y}^2\left(1 + \frac{4}{\alpha^2}\ \mathbb{Y}^4\mathbb{Y}_B^4 c^2\right)\left[Tr(Q)\mu + \hat{n}\sum_{\mathfrak{r}=1}^{\hat{n}} \mathfrak{d}_\mathfrak{r}\right] < 1,$$

where $\mathbb{Y}_B = \|B\|$.

Proof. For each $\alpha > 0$, we define the operator $\widehat{\Lambda}_\alpha$ at $\mathbb{H}$ through

$$(\widehat{\Lambda}_\alpha y) = x,$$

where

$$x(\tau) = \mathcal{R}(\tau)\big[y_0 - h(y)\big] + \int_0^\tau \mathcal{R}(\tau - \vartheta)\hat{h}(\vartheta) dw(\vartheta) + \int_0^\tau \mathcal{R}(\tau - \vartheta) B\xi(\vartheta, y) d\vartheta + \sum_{0<\tau_\mathfrak{r}<\tau} \mathcal{R}(\tau - \tau_\mathfrak{r})\mathcal{I}_\mathfrak{r}(y(\tau_\mathfrak{r})), \quad \hat{h} \in \mathbb{S}_{\mathcal{G},y},$$

$$\xi(\tau, y) = B^*\mathcal{R}^*(c - \tau)\mathcal{S}(\alpha, \Pi_0^c) p(y(\cdot)),$$

$$p(y(\cdot)) = Ey_c + \int_0^c \varphi(\vartheta) dw(\vartheta) - \mathcal{R}(c)\big[y_0 - h(y)\big] - \int_0^c \mathcal{R}(c - \vartheta)\hat{h}(\vartheta) dw(\vartheta) - \sum_{0<\tau_\mathfrak{r}<c} \mathcal{R}(c - \tau_\mathfrak{r})\mathcal{I}_\mathfrak{r}(y(\tau_\mathfrak{r})).$$

This is easily proved that the operator $\widehat{\Lambda}_\alpha$ has a fixed point if for all $\alpha > 0$ using the method from Theorem 3.2. The control system (12) is verified to be approximately controllable. This theorem's proof is already proved in Theorems 3.2 and 3.4, hence, it is not included here.

5. Example

We consider the nonlocal stochastic integrodifferential system with control of the form:

$$\begin{cases} d\left[y(\tau,\kappa) + \int_0^\tau (\tau-\vartheta)^\iota e^{-\lambda_1(\tau-\vartheta)} y(\vartheta,\kappa) d\vartheta\right] \in \left[\frac{\partial^2 y(\tau,\kappa)}{\partial\kappa^2} + \int_0^\tau e^{-\lambda_2(\tau-\vartheta)} \frac{\partial^2 y(\vartheta,\kappa)}{\partial\kappa^2} d\vartheta \right.\\ \qquad\qquad \left. + \wp(\tau,\kappa)\right] d\tau + \hat{h}(\tau,\kappa) dw(\tau),\ \tau \in [0,c],\ \kappa \in [0,\pi],\ \tau \neq \tau_\mathfrak{r}, \\ y(\tau,0) = y(\tau,\pi) = 0,\ \tau \in [0,c], \\ \big[y(\tau_\mathfrak{r}^+,\kappa) - y(\tau_\mathfrak{r}^-,\kappa)\big] = \mathcal{I}_\mathfrak{r}(y(\tau_\mathfrak{r})),\ \mathfrak{r} = 1,2,\ldots,\hat{n}, \\ y(0,\kappa) = y_0(\kappa) + \sum_{i=1}^{\tilde{\eta}} l_i y(\tau_i),\ 0 \le \kappa \le \pi, \end{cases} \tag{13}$$

where $w(\tau)$ denotes a standard cylindrical process in $\mathbb{H} = \mathbb{W} = \mathscr{L}^2([0,\pi],\mathbb{R})$ defined on a stochastic space $(\Omega, \mathscr{F}, \mathscr{P})$, $0 < \tau_1 < \tau_2 < \ldots < \tau_{\tilde{\eta}} < c$, $l_i (i = 1, 2, \ldots, \tilde{\eta})$ are real constants. To define the operator $A : D(A) \subseteq \mathbb{H} \to \mathbb{H}$, we construct $Ay = y''$ including $D(A) = \big\{y \in \mathbb{H} : y, y'$ are absolutely continuous, $y'' \in \mathbb{H},\ y(\pi) = 0 = y(0)\big\}$.

Clearly, the semigroup $\{\mathcal{T}(\tau), \tau \ge 0\}$ generated by A is analytic, compact, and self adjoint in $\mathbb{H}$. Further, the operator A is given by

$$Ay = -\sum_{j=1}^{\infty} j^2 \langle y, e_j \rangle e_j,\ y \in D(A),$$

and $\{\mathcal{T}(\tau)\}$ is represented by

$$\mathcal{T}(\tau)y = \sum_{\upsilon=1}^{\infty} e^{-j^2\tau}\langle y, e_j\rangle e_j,\ \ y \in \mathbb{H},$$

where $e_j(\kappa) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(j\kappa),\ j \in \mathbb{N}$. It is obvious that the set $\{e_j : j \in \mathbb{N}\}$ is an orthonormal basis for $\mathbb{H}$. Moreover, $(-A)^{\frac{1}{2}}$ is providing through

$$(-A)^{\frac{1}{2}} y = \sum_{j=1}^{\infty} j\langle y, e_j\rangle e_j,\ \ y \in D(-A)^{\frac{1}{2}},$$

where $D(-A)^{\frac{1}{2}} = \{y \in y : \sum_{j=1}^{\infty} j\langle y, e_j\rangle e_j \in \mathbb{H}\}$. Consider $B = \mathcal{J}$ and $\mathcal{Z} = D(-A)^{\frac{1}{2}}$ with $\|\cdot\|_{\frac{1}{2}} = \|(-A)^{\frac{1}{2}}\|$.
Directly, stands for the functions

$$y(\tau)(\kappa) = y(\tau, \kappa),$$

$$\hat{\hbar}(\tau, \varkappa)(\kappa) = \hat{\hbar}(\tau, \kappa),$$

$$Bu(\tau, \varkappa)(\kappa) = \wp(\tau, \kappa).$$

As well, we specify $g(\tau) : D(A) \subset \mathbb{H} \to \mathbb{H}$ and $\mathbb{Q}(\tau) : \mathbb{H} \to \mathbb{H}$ by

$$g(\tau)y = e^{-\lambda_2\tau} y \text{ for } y \in D(A),$$

$$\mathbb{Q}(\tau)y = \tau^{\iota} e^{-\lambda_1\tau} y \text{ for } y \in \mathbb{H}.$$

The system (13) can be abstracted from (1). Using the notations and conditions mentioned above. It is easy to find out that conditions $(\mathbf{A_1}) - (\mathbf{A_4})$ hold as $\widehat{\mathbb{Q}}(\ell) = \frac{\Gamma(\iota+1)}{(\ell+\lambda_1)^{\iota+1}} I$, $\widehat{g}(\ell) = \frac{1}{\ell+\lambda_2} A$ and $\widehat{\mathbb{K}} = C_0^{\infty}[(0,\pi)]$, if $C_0^{\infty}[(0,\pi)]$ stands for the set of infinitely differentiable functions disappear at $\kappa = 0$ and $\kappa = \pi$. The resolvent operator $\mathcal{R}(\cdot) : [0,\infty) \to \mathscr{B}(\mathbb{H})$ for the linear system of (13) is described by

$$\mathcal{R}(\tau) = \begin{cases} \frac{1}{2i\pi} \int_{\Gamma_{\hat{s},\varpi}} e^{\ell\tau} G(\ell) d\ell, & \tau > 0, \\ \mathscr{I}, & \tau = 0. \end{cases} \tag{14}$$

Obviously, the functions $\mathscr{I}_{\mathfrak{r}},\ \mathfrak{r} = 1,2,3$ are uniformly bounded and fulfill the hypothesis $(\mathbf{H_5})$. We achieve that $\mathcal{R}(\tau)$ is the resolvent operator and is compact for all $\tau \geq 0$.

Consider that functions fulfill the required hypotheses. We can convert (13) into an abstract form (1) by selecting the functions and evolution operator $A(\tau)$ from the list earlier and using $B = \mathcal{J}$. Theorem 3.4 states that all assumptions are fulfilled, and the system (13) is approximately controllable.

6. Conclusion

In this article, we examined the approximate controllability of nonlocal neutral stochastic integrodifferential inclusions with impulses via resolvent operators in Hilbert spaces. Our articles main results based on resolvent operators, stochastic integrodifferential evolution inclusions, nonlocal conditions, and the fixed point technique of Bohnenblust-Karlin's theorem. At last, we have provided an example of the presented theory.

In the future, we will focus on our study on approximate controllability of impulsive neutral stochastic integrodifferential systems with finite delay and nonlocal conditions via resolvent operators.

CRediT authorship contribution statement

Yong-Ki Ma: Writing – original draft. **J. Pradeesh:** Writing – original draft. **Anurag Shukla:** Writing – original draft. **V. Vijayakumar:** Writing – original draft. **K. Jothimani:** Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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