[5] Cf. Bochner, S., "Group Invariance of Cauchy's Formula in Several Variables," *Ann. Math.*, **45**, 686 (1944).

[6] Tamarkin, J. D., and Zygmund A., "Proof of a Theorem of Thorin, *Bull. Am. Math. Soc.*, **50**, 279 (1944).

ON THE ASYMPTOTIC DISTRIBUTION OF THE SUM OF A RANDOM NUMBER OF RANDOM VARIABLES

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Communicated by Marston Morse, January 9, 1948

We shall state without proof some results on the asymptotic distribution as $\lambda \to \infty$ of the sum

$$Y = X_1 + \ldots + X_N$$

of a random number of independent random variables, where the X_j have the same fixed distribution function $F(x) = P[X_j \leqslant x]$ and where N is a non-negative integer-valued random variable independent of the X_j, whose distribution function depends on a parameter λ. We shall use the notation

$$a = E(X_j), \qquad c^2 = Var(X_j)(0 < c^2 < \infty),$$
$$\alpha = E(N), \qquad \gamma^2 = Var(N)(0 \leqslant \gamma^2 < \infty), \qquad M = (N - \alpha)/\gamma,$$
$$\theta(t) = E(e^{itM}), \qquad \sigma^2 = \alpha c^2 + \gamma^2 a^2, \qquad \delta = (\gamma a)/\sigma,$$
$$Z = (Y - E(Y))/\sqrt{Var(Y)} = (Y - \alpha a)/\sigma, \qquad \varphi(t) = E(e^{itZ}).$$

THEOREM 1. *If as* $\lambda \to \infty$

$$\sigma^2 \to \infty, \qquad \gamma = o(\sigma^2), \tag{1}$$

then for any t, as $\lambda \to \infty$

$$\varphi(t) = \theta(\delta t)e^{-1/2t^2(1-\delta^2)} + o(1). \tag{2}$$

COROLLARY 1. *If (1) holds and if as* $\lambda \to \infty$

$$a^2\gamma^2 = o(\alpha), \tag{3}$$

then for any t,

$$\lim_{\lambda \to \infty} \varphi(t) = e^{-1/2t^2}, \tag{4}$$

so that Y is asymptotically normal.

COROLLARY 2. *If (1) holds and if M has a non-normal limiting distribution function G(x), so that*

$$\lim_{\lambda \to \infty} \theta(t) = g(t) \not\equiv e^{-1/2t^2}, \tag{5}$$

and if

$$\lim_{\lambda \to \infty} \frac{\alpha c^2}{\gamma^2 a^2} = s(0 \leqslant s < \infty), \tag{6}$$

then

$$\lim_{\lambda \to \infty} \varphi(t) = g\left(\frac{t}{\sqrt{1+s}}\right) \cdot e^{-1/2t^2(s/1+s)} \tag{7}$$

so that Z has a non-normal limiting distribution function.

It is easy to show that if M has a limiting distribution function $G(x)$ such that $G(x) > 0$ for every x, then as $\lambda \to \infty$, $\alpha \to \infty$ and $\gamma = o(\alpha)$, so that (1) holds.

COROLLARY 3. *If N is asymptotically normal then Y is asymptotically normal.*

An example in which (1) does not hold is the following: for any positive integer λ let N have the values λ, 2λ with $P[N = \lambda] = P[N = 2\lambda] = 1/2$, and let $a = 0$. Then $\sigma^2 = (3\lambda c^2)/2$, $\gamma = \lambda/2$, $\gamma \neq o(\sigma^2)$. Here we may show directly that

$$\lim_{\lambda \to \infty} \varphi(t) = 1/2\{e^{-1/3t^2} + e^{-2/3t^2}\},$$

which is the characteristic function of a mixture of two different normal distributions. This is a special case of the following theorem.

THEOREM 2. *If*

$$a = 0, \quad \lim_{\lambda \to \infty} \gamma/\alpha = r(0 < r < \infty), \tag{8}$$

and if M has a limiting distribution function G(x) (necessarily such that G(x) = 0 for some x), then

$$\lim_{\lambda \to \infty} \varphi(t) = \int_0^\infty e^{-1/2t^2x} dG_1(x), \tag{9}$$

where

$$G_1(x) = G\left(\frac{x-1}{r}\right). \tag{10}$$

It follows that Z has the limiting distribution function

$$H(x) = \int_0^\infty H_0\left(\frac{x}{\sqrt{y}}\right) dG_1(y), \tag{11}$$

where

$$H_0(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-1/2u^2} du \tag{12}$$

is the normal distribution function with means 0 and variance 1.

A full account of these results will be published elsewhere.