an extension (T, ψ) of Q such that T contains E as local subgroup and $\psi | E = \phi | E$. We call the pair (Y, X) *elementary* with respect to Q if every extension of X is extensible over Q from Y.

We can now state a group-theoretic equivalent of the condition $p_2(Y, X) = 0$.

THEOREM 3. *Let X, Y be local subgroups of a group Q such that (1) Y is a local subgroup of X; (2) Q is simply connected relative to Y; (3) X contains no elements of order 2. A necessary and sufficient condition that (Y, X) be elementary with respect to Q is that $p_2(Y, X) = 0$*

With the aid of Theorem 3, it is possible to give a proof of the existence of Lie groups in the large based on the vanishing of the second homotopy group rather than on the theorem of E. Levi.[3] The details will appear elsewhere.

[1] For definition see Reidemeister, *Einfuhrung in die kombinatorische Topologie*, p. 27.

[2] Although this seems not to be explicitly stated in the literature, it is implied in E. Cartan, *La topologie des groupes de Lie*, p. 13 and pp. 18–23.

[3] See Pontrjagin, *Topological Groups*, p. 269 (theorem 78).

INEQUALITIES BETWEEN THE TWO KINDS OF EIGENVALUES OF A LINEAR TRANSFORMATION

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With a linear transformation A in an n-dimensional vector space (matrix consisting of $n \times n$ complex numbers $a_{ii'}$) there are connected two kinds of eigenvalues: the roots $z = \alpha_1, \ldots, \alpha_n$ of the characteristic polynomial $|zE - A|$ of A (E = unit matrix) and the roots $z = \kappa_1, \ldots, \kappa_n$ of $|zE - K|$ where K is the Hermitian matrix A^*A composed of A and its Hermitian conjugate A^*. The κ_i are non-negative, and one would naturally compare the $\lambda_i = |\alpha_i|^2$ with the κ_i. If A is normal, $A^*A = AA^*$, they coincide; in general, however, they do not. Arrange the κ as well as the λ in descending order,

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n, \qquad \kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_n.$$

I shall prove the following

THEOREM. *Let $\varphi(\lambda)$ be an increasing function of the positive argument λ, $\varphi(\lambda) \geq \varphi(\lambda')$ for $\lambda \geq \lambda' > 0$, such that $\varphi(e^\xi)$ is a convex function of ξ and $\varphi(0) = \lim_{\lambda \to 0} \varphi(\lambda) = 0$. Then the eigenvalues λ_i and κ_i in descending order satisfy the inequalities*

$$\varphi(\lambda_1) + \ldots + \varphi(\lambda_m) \leq \varphi(\kappa_1) + \ldots + \varphi(\kappa_m) \qquad (m = 1, 2, \ldots, n), \quad (1)$$

in particular

$$\lambda_1^s + \ldots + \lambda_m^s \leq \kappa_1^s + \ldots + \kappa_m^s \qquad (m = 1, 2, \ldots, n) \quad (2)$$

for any real exponent $s > 0$.

According to a familiar argument[1]

$$\lambda_1 \leq \kappa_1. \quad (3)$$

Indeed the equation $Ax = \alpha_1 x$ has a vector solution $x = a \neq 0$: $Aa = \alpha_1 a$, $a^*A^* = \bar{\alpha}_1 a^*$, hence $a^*A^*Aa = \bar{\alpha}_1\alpha_1(a^*a)$ or

$$a^*Ka = \lambda_1(a^*a), \qquad a^*a > 0.$$

Since every vector satisfies the inequality $x^*Kx \leq \kappa_1(x^*x)$, (3) follows.

The linear vector transformation A induces certain linear transformations $A^{[1]}, A^{[2]}, A^{[3]}, \ldots, A^{[n]}$ for the space elements (skew-symmetric tensors) of rank 1, 2, 3, ..., n. For instance $A^{[3]} = \|a^{[3]}_{JJ'}\|$ is given by

$$a^{[3]}_{JJ'} = \begin{vmatrix} a_{ii'}, & a_{ik'}, & a_{il'} \\ a_{ki'}, & a_{kk'}, & a_{kl'} \\ a_{li'}, & a_{lk'}, & a_{ll'} \end{vmatrix}$$

where J and J' range over the triples (i, k, l) and (i', k', l') with the restrictions $i < k < l$, $i' < k' < l'$, respectively. Application of the inequality (3) to these matrices $A^{[1]}, A^{[2]}, \ldots$ yields the relations

$$\lambda_1 \leq \kappa_1, \quad \lambda_1\lambda_2 \leq \kappa_1\kappa_2, \quad \ldots, \quad \lambda_1 \ldots \lambda_n \leq \kappa_1 \ldots \kappa_n \quad (4)$$

(with the equality sign prevailing in the last of them). Everything will be settled as soon as I prove the following

LEMMA: *Let* κ_i, λ_i $(i = 1, \ldots, m)$ *be non-negative numbers such that*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \quad (5)$$

and

$$\lambda_1 \leq \kappa_1, \quad \lambda_1\lambda_2 \leq \kappa_1\kappa_2, \quad \ldots, \quad \lambda_1 \ldots \lambda_m \leq \kappa_1 \ldots \kappa_m; \quad (6)$$

then

$$\textstyle\sum_i \varphi(\lambda_i) \leq \sum_i \varphi(\kappa_i) \qquad (i = 1, \ldots, m) \quad (7)$$

for any function φ *of the nature described in the Theorem.*

Of two real numbers α, β let max.(α, β) denote α if $\alpha \geq \beta$ and β if $\beta \geq \alpha$. With a variable argument $z \geq 0$ form the functions

$$f(z) = \prod_{i=1}^{m} \text{max.}(1, \kappa_i z) \text{ and } g(z) = \prod_{i=1}^{m} \text{max.}(1, \lambda_i z).$$

Then

$$g(z) \leq f(z) \text{ for } z \geq 0. \tag{8}$$

Indeed set

$$g_i(z) = 1 \text{ for } i = 0 \text{ and } g_i(z) = \lambda_1 \ldots \lambda_i z^i \quad \text{for} \quad i = 1, \ldots, m$$

and distinguish the intervals $\{0\}, \{1\}, \ldots, \{m-1\}, \{m\}$ as defined by

$$\lambda_1 z \leq 1,\ \lambda_1 z \geq 1 \geq \lambda_2 z,\ \ldots,\ \lambda_{m-1}z \geq 1 \geq \lambda_m z,\ \lambda_m z \geq 1.$$

Then $g(z) = g_i(z)$ for z in $\{i\}$. But, because of (6), $g_i(z) \leq f_i(z) \leq f(z)$, hence (8) holds in each of the $m + 1$ intervals.

With an increasing function $\psi(z)$ one can form the Stieltjes integral

$$\int_0^\infty \log g(z)\cdot d\psi(z) = \sum_i \varphi(\lambda_i), \tag{9}$$

provided $\int^\infty \log z . d\psi(z)$ converges. Here

$$\varphi(\lambda) = \int_0^\infty \log \max.\,(1, \lambda z)\cdot d\psi(z) = \int_{\lambda z \geq 1} \log (\lambda z)\cdot d\psi(z). \tag{10}$$

It is clear how (8) by means of (9) and the corresponding formula for $f(z)$ leads to (7).

Set $\lambda = e^\xi$. If $\varphi(\lambda) = G(\xi)$ is a given function satisfying the conditions of the Theorem, it can be expressed by means of a non-decreasing function $G'(t)$ in the form

$$G(\xi) = \int_{-\infty}^{\xi} G'(t)\cdot dt = -\int_{-\infty}^{\xi} (t - \xi)\cdot dG'(t). \tag{11}$$

(The integration per partes is justified since

$$-t.G'(t) \leq 2.\int_t^{t/2} G'(\tau)\cdot d\tau$$

converges to zero for $t \rightarrow -\infty$.) (10) goes over into (11) by the substitution $z = e^{-t}$, $\psi(z) = -G'(t)$.

Of the inequalities (2) thus proved, the most important is the last $m = n$, which is independent of any arrangement of the κ_i and λ_i,

$$\lambda_1^s + \ldots + \lambda_n^s \leq \kappa_1^s + \ldots + \kappa_n^s. \tag{2'}$$

Its application to $A^{[2]}, A^{[3]}, \ldots$ yields the further relations

$$\sum_{i_1 < i_2} \lambda_{i_1}^s \lambda_{i_2}^s \leq \sum_{i_1 < i_2} \kappa_{i_1}^s \kappa_{i_2}^s, \tag{2''}$$

$$\sum_{i_1 < i_2 < i_3} \lambda_{i_1}^s \lambda_{i_2}^s \lambda_{i_3}^s \leq \sum_{i_1 < i_2 < i_3} \kappa_{i_1}^s \kappa_{i_2}^s \kappa_{i_3}^s, \tag{2'''}$$

where all the indices $i_1, i_2, i_3, \ldots$ range from 1 to n. Together they state that the polynomial $Q_s(z) = \prod_{i=1}^{n} (1 + \lambda_i^s z)$ is majorized, coefficient for coefficient, by the polynomial $P_s(z) = \prod_{i=1}^{n} (1 + \kappa_i^s z)$. In the limit for $s \to \infty$ they lead back to the relations (4).

If A is non-singular, A^{-1} has the eigenvalues α_i^{-1}, and the eigenvalues of $A^{*-1}A^{-1}$ coincide with those of $A^*(A^{*-1}A^{-1})A^{*-1} = A^{-1}A^{*-1} = (A^*A)^{-1}$, i.e., with the κ_i^{-1}. Hence by application of (1) to A^{-1} corresponding inequalities

$$\sum_{i=m}^{n} \varphi(\lambda_i) \leq \sum_{i=m}^{n} \varphi(\kappa_i) \qquad (m = n, \ldots, 1)$$

will result for any *decreasing* function $\varphi(\lambda)$ for which $\varphi(\lambda) \to 0$ with $\lambda \to \infty$ and $\varphi(e^\xi)$ is convex; in particular for $\varphi(\lambda) = \lambda^s$ with a negative exponent s. This shows that for a non-singular A the inequalities (2′) and also (2″), (2‴), ... are valid even for $s \leq 0$.

Facts and proofs, except the last remarks which depend on the consideration of A^{-1}, carry over to completely continuous linear operators A in Hilbert space, especially to continuous kernels of integral equations.

Long ago I. Schur proved (2′) for $s = 1$.[2] Recently S. H. Chang showed in his thesis[3] that, in the case of integral equations, convergence of $\sum_i \kappa_i^s$ implies convergence of $\sum_i \lambda_i^s$. These two facts led me to conjecture the relation (2′), at least for $s \leq 1$. After having conceived the simple idea for the proof, I discussed the matter with C. L. Siegel and J. von Neumann; their remarks have contributed to the final form and generality in which the results are presented here.[4]

[1] For a generalization of this inequality see A. Loewy and R. Brauer, "Ueber einen Satz für unitäre Matrizen," *Tôhoku Math. Jour.*, **32**, 44–49 (1930), formula (13) on p. 48.

[2] Schur, I., "Ueber die charakteristischen Wurzeln einer linearen Substitution, mit einer Anwendung auf die Theorie der Integralgleichungen," *Math. Ann.*, **66**, 488–510 (1909).

[3] Chang, S. H., "Theory of Characteristic Values and Singular Values of Linear Integral Equations," Thesis, Cambridge, England, 1948; also, "On the Distribution of Characteristic Values and Singular Values of L^2 Kernels," *Trans. Am. Math. Soc.* (1949).

[4] While this note was in print a result due to J. Karamata, "Sur une inégalité relative aux fonctions convex," *Publ. Math. Univ. Belgrade*, **1**, 145–148 (1932), that comes very near to our lemma, was pointed out to me.