[10] Muller, H. J., *J. Genet.*, **22,** 299–334 (1930).

[11] No attempt was made to detect autosomal recessives.

[12] The F_2 progenies derived from XYY males and $C1B$ females were examined microscopically in order to eliminate the possibility that patroclinous sons resulting from secondary non-disjunction might mask a lethal.

[13] Breeding tests were conducted on all flies showing conspicuous, symmetrical alterations in phenotype, and such changes were scored as mutants even in those cases where inviability or sterility prevented actual progeny counts.

[14] It should be noted that it was not possible to compare the recessive lethal rates in the two kinds of sperm (X and XY) from XYY males, as was possible in the case of visibles.

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[21] X-raying was done at the University of Pennsylvania Hospital, through the courtesy of the Department of Radiology. A medium-voltage therapy machine was used with no filter; target distance, 17 cm.; average output, 450 r per minute; current maintained at 8 milliamperes and 135 kilovolts.

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*ON A THEOREM OF WEYL CONCERNING EIGENVALUES OF LINEAR TRANSFORMATIONS. I**

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Communicated by Hermann Weyl, September 22, 1949

H. Weyl[1] recently proved the following theorem:

THEOREM. *Let A be a linear transformation in the n-dimensional unitary space C_n. Let the eigenvalues of A and A^*A be denoted by λ_i and $\kappa_i (1 \leq i \leq n)$, respectively, which are so arranged that*

$$|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|, \qquad \kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_n. \tag{1}$$

For any non-decreasing function $\omega(t)$ on $t > 0$ such that $\omega(e^t)$ is a convex function of t and $\omega(0) = \lim_{t \to 0} \omega(t) = 0$, λ_i and κ_i satisfy the inequalities:

$$\sum_{i=1}^{q} \omega(|\lambda_i|^2) \leq \sum_{i=1}^{q} \omega(\kappa_i) \qquad (1 \leq q \leq n). \tag{2}$$

In the present note, we prove three related theorems. Theorem 1 gives an extremum property of the sum of the first q eigenvalues for Hermitian transformations. This property furnishes a recurrent characterization of successive eigenvalues without referring to any eigenvector. Theorem 2 gives a similar but stronger property for all normal transformations. For an arbitrary linear transformation A and for a positive integer s, we have in Theorem 3 inequalities comparing the eigenvalues of $(A^s)^*A^s$ with those of A^*A. Finally we shall see that Weyl's theorem in the most important case $\omega(t) = t^s$ $(s = 1, 2, 3, \ldots)$ can be derived from Theorems 1 and 3. The general case of Weyl's theorem will be discussed in a forthcoming note.[2] All linear transformations considered here are assumed to be in the n-dimensional unitary space C_n, but the results can be easily carried over to completely continuous linear operators in Hilbert space, especially to continuous kernels of linear integral equations.

THEOREM 1. *Let the eigenvalues λ_i of a Hermitian transformation H be so arranged that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. For any positive integer $q \leq n$, the sums $\sum_{i=1}^{q} \lambda_i$ and $\sum_{i=1}^{q} \lambda_{n+1-i}$ are, respectively, the maximum and minimum of $\sum_{j=1}^{q} (Hx_j, x_j)$, when q orthonormal vectors x_j $(1 \leq j \leq q)$ vary in the space.*[3]

Proof: Let $\varphi_i (1 \leq i \leq n)$ be an orthonormal set of eigenvectors of H: $H\varphi_i = \lambda_i\varphi_i$. For each j, we write

$$(Hx_j, x_j) = \lambda_q \sum_{i=1}^{n} |(x_j, \varphi_i)|^2 + \sum_{i=1}^{q} (\lambda_i - \lambda_q) |(x_j, \varphi_i)|^2 + \sum_{i=q+1}^{n} (\lambda_i - \lambda_q) |(x_j, \varphi_i)|^2. \quad (3)$$

If $\|x_j\| = 1$, then

$$(Hx_j, x_j) \leq \lambda_q + \sum_{i=1}^{q} (\lambda_i - \lambda_q) |(x_j, \varphi_i)|^2$$

and therefore

$$\sum_{i=1}^{q} \lambda_i - \sum_{j=1}^{q} (Hx_j, x_j) \geq \sum_{i=1}^{q} (\lambda_i - \lambda_q) \{1 - \sum_{j=1}^{q} |(x_j, \varphi_i)|^2\}. \quad (4)$$

If x_j $(1 \leq j \leq q)$ are orthonormal, then $\sum_{j=1}^{q} |(x_j, \varphi_i)|^2 \leq \|\varphi_i\|^2 = 1$, so that the right-hand side of (4) is ≥ 0. But the left-hand side vanishes for $x_j = \varphi_j$ $(1 \leq j \leq q)$. This proves the maximum property.

THEOREM 2. *Let λ_i be the eigenvalues of a normal transformation N so arranged that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$. Let s, q be two positive integers $(q \leq n)$. Then $\sum_{i=1}^{q} |\lambda_i|^{2s}$ is the maximum of $\sum_{j=1}^{q} \|(UN)^s x_j\|^2$, when U runs over all unitary transformations and x_j $(1 \leq j \leq q)$ runs over all sets of q orthonormal vectors in C_n.*

Proof: We need only prove the inequality

$$\sum_{j=1}^{q} \|(UN)^s x_j\|^2 \leq \sum_{i=1}^{q} |\lambda_i|^{2s}. \tag{5}$$

As $\|UNx_j\|^2 = (N^*Nx_j, x_j)$, the case $s = 1$ of (5) follows from Theorem 1. We proceed by induction, assuming that (5) is true for s. Let $\varphi_i (1 \leq i \leq n)$ be an orthonormal set of eigenvectors of N: $N\varphi_i = \lambda_i\varphi_i$. Consider a unitary transformation U and q orthonormal vectors x_j $(1 \leq j \leq q)$. We have $\|(UN)^{s+1}x_j\|^2 = \sum_{i=1}^{n} |\lambda_i|^2 \cdot |((UN)^s x_j, \varphi_i)|^2$. If we split this sum into three parts in a way similar to (3), we see that for each j:

$$\|(UN)^{s+1}x_j\|^2 \leq |\lambda_q|^2 \cdot \|(UN)^s x_j\|^2 + \sum_{i=1}^{q} (|\lambda_i|^2 - |\lambda_q|^2) \cdot |((UN)^s x_j, \varphi_i)|^2. \tag{6}$$

As $x_j (1 \leq j \leq q)$ are orthonormal, we have for each i:

$$\sum_{j=1}^{q} |((UN)^s x_j, \varphi_i)|^2 \leq \|(N^*U^*)^s \varphi_i\|^2 = \|(U^*N^*)^s U^* \varphi_i\|^2. \tag{7}$$

Using first (6), then (7) and our assumption of induction (i.e. (5) is true for s), we get

$$\sum_{i=1}^{q} |\lambda_i|^{2s+2} - \sum_{j=1}^{q} \|(UN)^{s+1}x_j\|^2 \geq \sum_{i=1}^{q} (|\lambda_i|^2 - |\lambda_q|^2)[|\lambda_i|^{2s} - \|(U^*N^*)^s U^* \varphi_i\|^2]. \tag{8}$$

Denote by d_q the expression on the right-hand side of (8), we have

$$d_{q+1} - d_q = (|\lambda_q|^2 - |\lambda_{q+1}|^2)[\sum_{i=1}^{q} |\lambda_i|^{2s} - \sum_{i=1}^{q} \|(U^*N^*)^s U^* \varphi_i\|^2]. \tag{9}$$

As $U^*\varphi_i (1 \leq i \leq q)$ are orthonormal, our assumption of induction shows that the right-hand side of (9) is ≥ 0, and $d_{q+1} \geq d_q$. But $d_1 = 0$, hence $d_q \geq 0$. This proves that (5) is also true for $s + 1$.

THEOREM 3. *Let A be an arbitrary linear transformation and s be any positive integer. Let the eigenvalues of $(A^s)^*A^s$ be denoted by $\kappa_i^{(s)}$ $(\kappa_i^{(1)} = \kappa_i)$ and so arranged that $\kappa_1^{(s)} \geq \kappa_2^{(s)} \geq \ldots \geq \kappa_n^{(s)}$. Then for any positive integer $q \leq n$, we have*

$$\sum_{i=1}^{q} \kappa_i^{(s)} \leq \sum_{i=1}^{q} \kappa_i^s. \tag{10}$$

Proof: Let $A = UH$ be the polar decomposition of A, where U is unitary and H is the non-negative square root of A^*A. The eigenvalues of

H are $\kappa_i^{1/s}$. By Theorem 2, any q orthonormal vectors $x_j (1 \leq j \leq q)$ satisfy

$$\sum_{j=1}^{q} ((A^s)^* A^s x_j, x_j) = \sum_{j=1}^{q} \|(UH)^s x_j\|^2 \leq \sum_{i=1}^{q} \kappa_i^s. \qquad (11)$$

But by Theorem 1, $\sum_{i=1}^{q} \kappa_i^{(s)}$ is the maximum of the first $\sum$ sum in (11), when the q orthonormal vectors x_j vary. Thus (10) is proved.

We now prove the case $\omega(t) = t^s (s = 1, 2, 3, \ldots)$ of Weyl's theorem. Here we use the same notation as in the theorem stated at the beginning of this note. Using Schur's superdiagonal form of matrices, it is clear that there exist n orthonormal vectors $y_i (1 \leq i \leq n)$ such that $|\lambda_i|^2 \leq \|Ay_i\|^2 (1 \leq i \leq n)$ and therefore $\sum_{i=1}^{q} |\lambda_i|^2 \leq \sum_{i=1}^{q} \|Ay_i\|^2$. But applying Theorem 1 to A^*A, we find $\sum_{i=1}^{q} \|Ay_i\|^2 \leq \sum_{i=1}^{q} \kappa_i$. Hence

$$\sum_{i=1}^{q} |\lambda_i|^2 \leq \sum_{i=1}^{q} \kappa_i. \qquad (12)$$

As in Theorem 3, we denote by $\kappa_i^{(s)}$ the eigenvalues of $(A^s)^* A^s$ arranged in descending order (in particular, $\kappa_i^{(1)} = \kappa_i$). Applying (12) to the transformation A^s, we get

$$\sum_{i=1}^{q} |\lambda_i|^{2s} \leq \sum_{i=1}^{q} \kappa_i^{(s)},$$

which together with (10) gives the case $\omega(t) = t^s$ of (2).

* This work was supported in part by the Office of Naval Research.

[1] Weyl, H., "Inequalities between the Two Kinds of Eigenvalues of a Linear Transformation," these PROCEEDINGS, **35**, 408–411 (1949).

[2] Fan, K., "On a Theorem of Weyl concerning Eigenvalues of Linear Transformations. II," to be published in these PROCEEDINGS.

[3] An alternative form of Theorem 1: $\sum_{i=1}^{q} \lambda_i$ and $\sum_{i=1}^{q} \lambda_{n+1-i}$ are, respectively, the maximum and minimum of the trace of HP, when P runs over all projections on q-dimensional linear subspaces. There is also a similar alternative form of Theorem 2.