or $mk + 1$, is an even entire function of order 1 and type $p < \pi$, which can be shown to be $O(z^{-1})$ as $z \to \infty$ through real values; it is real on the real axis, and changes sign at all integers not of the form mk or $mk + 1$. By a theorem of Paley and Wiener,[6]

$$F(z) = \int_0^p \cos zu\, h(u)\, du,$$

where $h(u)$ belongs to L^2. Write $g(u) = h(u) \cos {}^1/_2 u$; then

$$\begin{aligned} F(n + {}^1/_2) &= \int_0^p g(u) \cos (n + {}^1/_2)u \sec {}^1/_2 u\, du \\ &= (-1)^n \int_{\pi-p}^{\pi} g(\pi - u) \sin (n + {}^1/_2)u \csc {}^1/_2 u\, du. \end{aligned}$$

Thus $(-1)^n F(n + {}^1/_2)$ is the nth partial sum of the Fourier series of the even function which is $-\pi g(\pi - u)$ for $\pi - p < u < \pi$ and zero for $0 \leqq u < \pi - p$. Furthermore, $(-1)^n F(n + {}^1/_2) \geqq 0$ when n is not a multiple of k, and so for a sequence of integers of density arbitrarily close to 1, if k is large enough.

[1] "On sait fort peu de choses sur l'approximation orientée dans les espaces fonctionnels réticulés": Favard, J., "Remarques sur l'approximation des fonctions continues," *Acta Sci. Math., Szeged*, **12**, part A, 101–104 (1950).

[2] Fejér, L., "Gestaltliches über die Partialsummen und ihrer Mittelwerte bei der Fourierreihe und der Potenzreihe," *Z. angew. Math. u. Mech.*, **13**, 80–88 (1933).

[3] Levinson, N., *Gap and Density Theorems*, New York, 1940, chap. II.

[4] Levinson, N., op. cit., chap. VII.

[5] Duffin, R. J., and Schaeffer, A. C., "Power Series with Bounded Coefficients," *Am. J. Math.*, **67**, 141–154 (1945).

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*THE MATHEMATICS OF SECOND QUANTIZATION**

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In order to make a beginning on the problem of constructing a mathematically rigorous foundation for quantum field theory, we define the annihilation and creation operators and the particle- and field-observables as transformations on Hilbert space, and investigate their domains, adjoints, commutation relations, normality and other properties. The resulting formalism is given a physical interpretation which is illustrated by applications.

The state of an elementary particle is represented by a point in the Hilbert space $\mathfrak{R}$ and an observable by the operator A on $\mathfrak{R}$. Then the state of a system of n such particles is represented by a point in the tensor product $\mathfrak{R}^{(n)} = \mathfrak{R}\otimes \ldots \otimes\mathfrak{R}$ and the observable by $(\sum_{i=1}^{n} A^{\delta(i,1)}\otimes \ldots \otimes A^{\delta(i,n)})^{\sim}$, where $T^{\sim}$ is the closure[1] of T. In field theory, $\mathfrak{F} = \sum_{n=0}^{\infty} \oplus\mathfrak{R}^{(n)}$ is the state space[2] and $\Omega(A) = \sum_{n=0}^{\infty} \oplus (\sum_{i=1}^{n} A^{\delta(i,1)}\otimes \ldots \otimes A^{\delta(i,n)})^{\sim}$ the corresponding observable. $\Omega(A)$ exists when A is densely defined, closed and linear. It is defined to be O on $\mathfrak{R}^{(o)}$, the one-dimensional space of no-particle states. Ω preserves commutation, order and adjoint relations, and normality. If H is self-adjoint, then $\exp(i\Omega(H))\Omega(A)\exp(-i\Omega(H)) = \Omega(\exp(iH)A\exp(-iH))$. Under certain conditions (e.g., when A and B are bounded), $\Omega(aA + bB) = (a\Omega(A) + b\Omega(B))^{\sim}$ and $\Omega([A, B]) = [\Omega(A), \Omega(B)]^{\sim}$ (where $[A, B] = AB - BA$). If P is a projection, the eigenvalues of $\Omega(P)$ are the occupation numbers m of $P\mathfrak{R}$. In the corresponding eigenstates, exactly m of the particles have the property P. With the spectral theorem, this gives us the standard energy expressions $\hbar\Sigma_k\omega_k N_k$ except that there is no infinite null-point energy $\hbar\Sigma_k\omega_k/2$ to be subtracted.[3] A similar result holds for the null-point momentum, no artificial summation to zero being necessary.

To every permutation π in the symmetric group Π_n corresponds the unitary operator U_π on $\mathfrak{R}^{(n)}$ defined by $U_\pi\psi_1\otimes \ldots \otimes\psi_n = \psi_{\pi(1)}\otimes \ldots \otimes\psi_{\pi(n)}$. These operators U_π generate a ring $\mathfrak{G}_n$ isomorphic to the group algebra of Π_n. For $G_n \in \mathfrak{G}_n$ and $\phi \in \mathfrak{R}$, we construct the densely defined, closed, linear transformations $\omega(\phi) = (\sum_{n=0}^{\infty} \oplus G_n)(\phi\otimes)$, $\omega^*(\phi) = ((\phi\otimes)^*(\sum_{n=0}^{\infty} \oplus G_n^*))^{\sim}$ on $\mathfrak{F}$, where $(\phi\otimes)$ maps $\mathfrak{R}^{(n)}$ into $\mathfrak{R}^{(n+1)}$ by $(\phi\otimes)\phi_1\otimes \ldots \otimes\phi_n = \phi\otimes\phi_1\otimes \ldots \otimes\phi_n$. The mapping ω obeys the rules $\omega(\phi) = \omega^*(\phi)^*$, $\omega(\phi)^* = \omega^*(\phi)$, $\omega(a\phi + b\psi) = (a\omega(\phi) + b\omega(\psi))^{\sim}$, $\omega^*(a\phi + b\psi) = (a^*\omega^*(\phi) + b^*\omega^*(\psi))^{\sim}$, $\exp(-i\Omega(H))\omega(\phi)\exp(i\Omega(H)) = \omega(\exp(-iH)\phi)$, $\exp(-i\Omega(H))\omega^*(\phi)\exp(i\Omega(H)) = \omega^*(\exp(-iH)\phi)$, $[\Omega[A], \omega(\phi)]^{\sim} = \omega(A\phi)$, and $[\Omega(A), \omega^*(\phi)]^{\sim} = -\omega^*(A^*\phi)$. The center of $\mathfrak{G}_n$ is spanned by a set of orthogonal projections $P_\tau = \sum_{\pi \in \Pi_n}\tau(\pi)U_\pi/n!$ indexed by the characters τ of Π_n. The alternating and symmetric characters a_n and s_n give us the subspaces $\mathfrak{A} = (\sum_{n=0}^{\infty} \oplus P_{a_n})\mathfrak{F}$ and $\mathfrak{S} = (\sum_{n=0}^{\infty} \oplus P_{s_n})\mathfrak{F}$ of antisymmetric and symmetric wave-functions. Setting G_n equal to $\sqrt{n}P_{a_n}$ or $\sqrt{n}P_{s_n}$, we get the creation operators $\omega_a(\phi)$ or $\omega_s(\phi)$ and annihilation operators $\omega_a^*(\phi)$ or $\omega_s^*(\phi)$, on $\mathfrak{A}$ or $\mathfrak{S}$ (to which they must be restricted), for the Fermi-Dirac or Bose-Einstein cases, respectively. Both $\omega_a(\phi)$ and $\omega_a^*(\phi)$ are bounded: $\|\omega_a(\phi)\| = \|\omega_a^*(\phi)\| = \|\phi\|$. The domains of $\omega_s(\phi)$ and $\omega_s^*(\phi)$ are the same as that of $\sqrt{\Omega(P_{[\phi]})}$ (on $\mathfrak{S}$), where $P_{[\phi]}$ is the projection of $\mathfrak{R}$ on the subspace $[\phi]$ spanned by ϕ.

If $\psi_1, \psi_2 \in \mathfrak{R}$, we define $\psi_1\psi_2^*$ to be the transformation on $\mathfrak{R}$ such that $(\psi_1\psi_2^*)\phi = (\phi, \psi_2)\psi_1$. Then $\omega_a(\phi)\omega_a^*(\psi) = \Omega(\phi\psi^*)$, $\omega_a^*(\psi)\omega_a(\phi) = (\phi, \psi)I - \Omega(\phi\psi^*)$, and $(\omega_s(\phi)\omega_s^*(\psi))^{\sim} = \Omega(\phi\psi^*)$, $(\omega_s^*(\psi)\omega_s(\phi))^{\sim} = (\phi, \psi)I + \Omega(\phi\psi^*)$; from which it follows, if $[A, B]_+ = AB + BA$, that $[\omega_a(\phi), \omega_a^*(\psi)]_+ = [\omega_s(\phi), \omega_s^*(\psi)]^{\sim} = (\phi, \psi)I$, and $[\omega_a(\phi), \omega_a(\psi)]_+ = [\omega_s(\phi), \omega_s(\psi)]^{\sim} = [\omega_a^*(\phi), \omega_a^*(\psi)]_+ = [\omega_s^*(\phi), \omega_s^*(\psi)]^{\sim} = 0$. If $\{\phi_i\}$ is an orthonormal basis of $\mathfrak{R}$, then $\{\omega_a(\phi_i)\}$ is irreducible on $\mathfrak{A}$ and $\{\omega_s(\phi_i)\}$ is irreducible on $\mathfrak{S}$. By purely formal manipulations it can be seen that $\Omega(A)$ corresponds to the expression[2] "$\Sigma_{m,n}\omega(\phi_m)(A\phi_n, \phi_m)\omega^*(\phi_n)$" if ω is ω_a or ω_s.

The operators $i(\omega_s(\phi) - \omega_s^*(\phi))/\sqrt{2}$ and $(\omega_s(\phi) + \omega_s^*(\phi))/\sqrt{2}$ are essentially self-adjoint,[1] so their closures $p(\phi)$ and $q(\phi)$ exist and are self-adjoint. The commutation relations $[q(\phi), q(\psi)]^{\sim} = [p(\phi), p(\psi)]^{\sim} = ((\psi, \phi) - (\phi, \psi))I/2$ and $[q(\phi), p(\psi)]^{\sim} = i((\psi, \phi) + (\phi, \psi))I/2$ reduce to the standard ones when ϕ and ψ are elements of an orthonormal basis. Time-dependent commutation relations enable us to avoid the singular Dirac δ-function and the Jordan-Pauli invariant D-function. Field quantities have physical meaning only as averages over a region. Point-dependence introduces divergencies into the mathematics, so p and q here depend on elements of Hilbert space (as distributions with respect to which the averages are taken) rather than points of Euclidean 3-space E_3. The formalism is illustrated by a derivation of the Yukawa-potential, and by the following completely rigorizable, relativistically invariant, divergence-free (as far as it goes) derivation of Maxwell's equations: A photon is represented by ψ in $\mathfrak{R} = \mathfrak{L}_2(E_3)\otimes\mathfrak{K}_4$, where $\mathfrak{K}_4$ is a 4-dimensional Hilbert space. If $k_x = -i\hbar(\partial/\partial x)$, etc., and $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$ then the Hamiltonian is $H = ck\otimes I$. The Lorentz group acts on time-dependent elements of $\mathfrak{R}$ by having $\exp(-(itc/\hbar)k)\phi$, for $\phi \in \mathfrak{L}_2(E_3)$, transform like a scalar, and the orthonormal basis $\rho_1, \rho_2, \rho_3, \rho_4$ of $\mathfrak{K}_4$ transform contragrediently to x, y, z, ct. The four-vector wave-functions come in pairs, $\hat{\psi}$ and ψ, as co- and contravariant components for the same particle. Expectation values are written $(A\psi, \hat{\psi})$. We restrict photons to be such that their covariant wave-functions must be in the subspace $\hat{P}$ of all $\hat{\psi} = \phi_1\otimes\rho_1 + \phi_2\otimes\rho_2 + \phi_3\otimes\rho_3 + \phi_4\otimes\rho_4 (\phi_i \in \mathfrak{L}_2(E_3))$ for which $(\partial/\partial x)\phi_1 + (\partial/\partial y)\phi_2 + (\partial/\partial z)\phi_3 - c^{-1}(\partial/\partial t)\phi_4 = 0$. This eliminates the physical influence of longitudinal and scalar components from expectation values, and leaves only two effective polarization states. They are perpendicular to the direction of motion of the photon and have the desired spin properties. Now let $P(\psi) = i(\omega_s(\psi) - \omega_s^*(\hat{\psi}))^{\sim}/\sqrt{2}$ and $Q(\psi) = (\omega_s(\psi) + \omega^*(\hat{\psi}))^{\sim}/\sqrt{2}$. Then if $\phi \in \mathfrak{L}_2(E_3)$ is a real-valued function on E_3 for taking field-value averages, the covariant four-potential operators are $A_i(\phi) = \hbar Q(\sqrt{H^{-1}}\phi\otimes\rho_i), i = 1, 2, 3$, and $\Phi(\phi) = \hbar Q(\sqrt{H^{-1}}\phi\otimes\rho_4)$. Averages over mutually space-like regions commute. The total energy of the field is $\Omega(H) - c(A_1(s_1) + A_2(s_2) + A_3(s_3) + \Phi(s_4))$, where s_1, s_2, s_3, s_4 is any real, square-integrable, contravariant four-current density. Expectation values of $(\vec{\nabla}\cdot\vec{A} -. c^{-1}(\partial/\partial t)\Phi)(\phi)$ are always zero (on photons), and $(\Box^2 A_i)(\phi) = -c^{-1}(s_i, \phi)I, i = 1, 2, 3, (\Box^2\Phi)(\phi) = c^{-1}(s_4, \phi)I$, so Maxwell's equations are satisfied. A photon is polarized parallel to its electric vector and perpendicular to its magnetic vector—both perpendicular to its momentum. Its energy satisfies Planck's relation $E = h\nu$, where ν is the frequency of the induced field.

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ON A CONJECTURE OF MURRAY AND VON NEUMANN

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1. *Introduction.*—In this note the authors present a proof of a conjecture of F. J. Murray and J. v. Neumann[1] concerning normalcy of factors.

A ring of operators[2] $\mathcal{R}$ is said to be *normal* if each subring $\mathcal{S}$ of $\mathcal{R}$ coincides with the set of operators in $\mathcal{R}$ each of which commutes with every operator in $\mathcal{S}'_{\mathcal{R}}$, where $\mathcal{S}'_{\mathcal{R}}$ is the ring of operators in $\mathcal{R}$ each of which commutes with every operator in $\mathcal{S}$. In symbols, normalcy requires that $(\mathcal{S}'_{\mathcal{R}})'_{\mathcal{R}} = \mathcal{S}$ for each subring $\mathcal{S}$ of $\mathcal{R}$. The center of a normal ring $\mathcal{R}$ consists of the operators α I, α complex (put $\mathcal{S} = \{\alpha \text{ I}\}$); i.e., $\mathcal{R}$ is a *factor*. J. v. Neumann proved[3] that the factor $\mathcal{B}$ of all bounded operators is normal. The question of which factors are normal was raised by F. J. Murray and J. v. Neumann (R.O. I, p. 185). They showed that all factors in case I (the discrete case) are normal and exhibited examples of non-normal factors in case II (the continuous case). Their later results establish the non-normalcy of each member of a restricted class of factors in case II, viz.,