ERGODIC PROPERTY OF THE BROWNIAN MOTION PROCESS

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1. Introduction.—In a previous paper ("On the Equidistribution of Sums of Independent Random Variables," to appear elsewhere; an abstract will appear in *Bull. Am. Math. Soc.*, **59**, (May, 1953); we shall refer to this paper as [1]) we considered some properties of the sequence S_n of partial sums of independent and identically distributed random variables or random vectors in two dimensions. Here we show in Theorems 1–4 that the results of [1] carry over to the Brownian motion process in one and two dimensions.

2. The Two-Dimensional Case.—Let $X(t)$ denote the Brownian motion (Wiener) process on the line: $X(0) \equiv 0$, $X(t)$ is continuous for all t with probability 1, and for any $t_0 < t_1 < \ldots < t_n$ the random variables $X(t_j) - X(t_{j-1})$, $j = 1, \ldots, n$, are independent and normally distributed with zero means and variances $t_j - t_{j-1}$. Let $V(t) = (X(t), Y(t))$ denote the Brownian motion process in the plane, the two components of $V(t)$ being independent one-dimensional Brownian motion processes. Suppose that $f(x, y)$, $g(x, y)$ are real valued functions which are bounded and summable in the plane $-\infty < x < \infty$, $-\infty < y < \infty$, and set $\bar{f} = \int\int f(x, y)\,dxdy$, $\bar{g} = \int\int g(x, y)\,dxdy$, where here and in the sequel an integral sign without limits denotes integration over $(-\infty, \infty)$. We shall prove the following two theorems for plane Brownian motion. The corresponding results for the one-dimensional case involve no essentially new arguments and will be stated without proof at the end of the paper.

THEOREM 1. *If $\bar{f} \neq 0$ then for every u,*

$$\lim_{T \to \infty} \Pr\left[\frac{2\pi}{\bar{f}\log T}\int_0^T f(V(t))\,dt \leq u\right] = G(u), \tag{2.1}$$

where $G(u) = 1 - e^{-u}$ for $u \geq 0$, $\quad = 0$ for $u < 0$.

THEOREM 2. *If $\bar{g} \neq 0$ then*

$$\lim_{T \to \infty} \frac{\int_0^T f(V(t))\,dt}{\int_0^T g(V(t))\,dt} = \frac{\bar{f}}{\bar{g}} \text{ in probability.} \tag{2.2}$$

Proofs of Theorems 1 and 2: Assume $\bar{f} \neq 0$ and $\bar{g} \neq 0$ and define

$$Z_n(f) = \frac{2\pi}{\bar{f}\log n}\int_0^n f(V(t))\,dt, \qquad W_n(f) = \frac{2\pi}{\bar{f}\log n}\sum_{j=1}^n f(V(j)),$$

with corresponding definitions for $Z_n(g)$ and $W_n(g)$. For any positive integer n, each component of $V(n)$ is the sum of n independent random variables, each distributed normally with zero mean and unit variance, since

$$X(n) = \sum_{j=1}^{n} [X(j) - X(j-1)], \qquad Y(n) = \sum_{j=1}^{n} [Y(j) - Y(j-1)].$$

A theorem in [1] gives

$$\lim_{n \to \infty} \Pr[W_n(f) \leq u] = G(u). \tag{2.3}$$

We shall later prove as Lemma 1 that

$$\lim_{n \to \infty} E[Z_n(f) - W_n(f)]^2 = 0. \tag{2.4}$$

It follows from (2.4) that $Z_n(f) - W_n(f)$ tends to zero in probability, and hence from (2.3) that $Z_n(f)$ has the same limiting distribution, $G(u)$, as $W_n(f)$. This proves (2.1) as $T \to \infty$ through integer values, and the extention to arbitrary T is immediate, proving Theorem 1.

To prove Theorem 2 we shall later prove as Lemma 2 that

$$\lim_{n \to \infty} E[Z_n(f)Z_n(g) - W_n(f)W_n(g)] = 0, \tag{2.5}$$

and we make use of the fact, proved in [1], that

$$\lim_{n \to \infty} EW_n^2(f) = \lim_{n \to \infty} EW_n^2(g) = \lim_{n \to \infty} EW_n(f)W_n(g) = \int_0^\infty u^2 dG(u) = m_2, \text{ say.} \tag{2.6}$$

From (2.4)–(2.6) it follows that

$$\lim_{n \to \infty} EZ_n^2(f) = \lim_{n \to \infty} EZ_n^2(g) = \lim_{n \to \infty} EZ_n(f)Z_n(g) = m_2,$$

$$\text{and hence } \lim_{n \to \infty} E[Z_n(f) - Z_n(g)]^2 = 0. \tag{2.7}$$

As in [1], given $\epsilon > 0$ we choose $\delta = \delta(\epsilon) > 0$ such that $\delta \leq \epsilon$, $G(\delta) \leq {}^1/_2\epsilon$, and $N = N(\epsilon)$ (by (2.1) with f replaced by g) such that

$$n \geq N \text{ implies } \Pr[Z_n(g) > \delta] \geq 1 - G(\delta) - {}^1/_2\epsilon \geq 1 - \epsilon.$$

Since $$\Pr[\{Z_n(f) - Z_n(g)\}^2 < \delta^3] \geq 1 - \frac{E[Z_n(f) - Z_n(g)]^2}{\delta^3},$$

if we choose (by (2.7)) $K = K(\epsilon)$ such that

$$n \geq K \text{ implies } E[Z_n(f) - Z_n(g)]^2 \leq \delta^4,$$

then if $n \geq \max.(N, K)$,

$$\Pr[Z_n(g) > \delta] \geq 1 - \epsilon, \qquad \Pr[\{Z_n(f) - Z_n(g)\}^2 < \delta^3] \geq 1 - \epsilon,$$

which in turn imply that

$$\Pr\left[\left\{\frac{Z_n(f)}{Z_n(g)} - 1\right\}^2 < \epsilon\right] \geq \Pr[Z_n(g) > \delta, \{Z_n(f) - Z_n(g)\}^2 < \delta^3] \geq 1 - 2\epsilon.$$

Since ϵ was arbitrary this proves (2.2) as $T \to \infty$ through integer values. Again, the extension to arbitrary T is immediate. Finally, the restriction that $\bar{f} \neq 0$ can be dropped by a simple argument and the proof of Theorem 2 is complete.

3. *Proof of the Lemmas.*—We shall prove Lemma 2 first. We have

$$Z_n(f) = \frac{2\pi}{\bar{f}\log n}\sum_{j=1}^{n}\int_{j-1}^{j} f(V(t))\,dt,\quad W_n(f) = \frac{2\pi}{\bar{f}\log n}\sum_{j=1}^{n}\int_{j-1}^{j} f(V(j))\,dt.$$

Set $D_n = E[Z_n(f)Z_n(g) - W_n(f)W_n(g)]$

$$= \frac{4\pi^2}{\bar{f}\bar{g}\,(\log n)^2}\sum_{j,k=1}^{n}\int_{j-1}^{j}\int_{k-1}^{k} E[f(V(t))g(V(u)) - f(V(j))g(V(k))]\,dudt$$

$$= \frac{4\pi^2}{\bar{f}\bar{g}\,(\log n)^2}\sum_{j,k=1}^{n} a_{jk},$$

where

$$a_{jk} = \iint_{R_{jk}} [\theta(t, u) - \theta(j, k)]\,dudt,\qquad R_{jk} = \{j - 1 < t < j;\ k - 1 < u < k\},\qquad \theta(t, u) = E[f(V(t))g(V(u))].$$

We want to show that $D_n \to 0$ as $n \to \infty$, i.e., that

$$\lim_{n\to\infty} (\log n)^{-2}\sum_{j,k=1}^{n} a_{jk} = 0. \qquad (3.1)$$

To evaluate $\theta(t, u)$ we observe that for $t_1 < t_2$ the random vector $V(t_2) - V(t_1) = (X(t_2) - X(t_1), Y(t_2) - Y(t_1))$ has the joint probability density

$$\frac{1}{2\pi(t_2 - t_1)}e^{-\frac{x^2 + y^2}{2(t_2 - t_1)}}.$$

Hence for $0 < t < u$,

$$\theta(t, u) = \frac{1}{4\pi^2 t(u - t)}\iiiint f(x, y)g(x + x', y + y')\, e^{-\frac{x^2 + y^2}{2t} - \frac{x'^2 + y'^2}{2(u - t)}}\,dxdydx'dy'$$

$$= \frac{1}{4\pi^2 t(u - t)}\iiiint f(x, y)g(\zeta, \eta)e^{-Q(t, u)}\,dxdyd\zeta d\eta,$$

where we have set

$$Q(t, u) = \frac{1}{2}\left\{\frac{x^2 + y^2}{t} + \frac{(x - \zeta)^2 + (y - \eta)^2}{u - t}\right\}.$$

In what follows C will denote any constant whose numerical value is immaterial, and $F = \sup|f(x, y)|$, $G = \sup|g(x, y)|$. Then if $(t, u)\epsilon R_{jj}$ or $\epsilon R_{j,\, j+1} (j \geq 2)$,

$$|\theta(t, u)| \leq \frac{G}{4\pi^2 t(u - t)} \int\int |f(x, y)| e^{-\frac{x^2 + y^2}{2t}} dxdy \cdot$$
$$\int\int e^{-\frac{x'^2 + y'^2}{2(u - t)}} dx'dy' \leq \frac{G}{2\pi t}\int\int |f(x\ y)| dxdy \leq \frac{C}{j - 1}. \quad (3.2)$$

Also, for $0 < t < u$,

$$|\theta(t, u)| \leq \frac{1}{4\pi^2 t(u - t)} \int\int\int\int |f(x, y)\cdot g(\zeta, \eta)|$$
$$e^{-\frac{x^2 + y^2}{2t}} dxdyd\zeta d\eta \leq \frac{F}{2\pi(u - t)} \int\int |g(\zeta, \eta)| d\zeta d\eta \leq \frac{C}{u - t}. \quad (3.3)$$

We write $\sum_{j,\, k\, =\, 1}^{n} a_{jk}$ as the sum of the following terms:

(a) $\sum_{j\, =\, 2}^{n} a_{jj}$, (b) $\sum_{k\, =\, 3}^{n} a_{1k}$, (b′) $\sum_{=\, 3}^{n} a_{j1}$, (c) $\sum_{j\, =\, 2}^{n\, -\, 1} a_{j,\, j+1}$,

(c′) $\sum_{k\, =\, 2}^{n\, -\, 1} a_{k+1,k}$, (d) $\sum_{2\leq j\leq k-2\leq n-2} a_{jk}$, (d′) $\sum_{2\leq k\leq j-2\leq n-2} a_{jk}$, and

(e) $a_{11} + a_{12} + a_{21}$.

To prove (3.1) it will suffice to show that each of (a), (b), (c), (d) is $o\{(\log n)^2\}$.

(a) From (3.2), if $(t, u)\epsilon R_{jj}, j \geq 2$, $|\theta(t, u)| \leq \frac{C}{j - 1}$.

Also, $|\theta(j, j)| = \frac{1}{2\pi j}\left|\int\int f(x, y)g(x, y)e^{-\frac{x^2 +}{2j}} dxdy\right| \leq \frac{C}{j - 1}$.

$$\text{Hence } \left|\sum_{j=2}^{n} a_{jj}\right| \leq C\cdot\sum_{j=2}^{n} \frac{1}{j - 1} = 0(\log n). \quad (3.4)$$

(b) From (3.3), if $(t, u)\epsilon R_{1k}$, $k \geq 3$,

$$|\theta(t, u)| \leq \frac{C}{u - t} \leq \frac{C}{k - 2}.$$

$$\text{Also, } |\theta(1, k)| \leq \frac{C}{k - 2}.$$

$$\text{Hence } \left| \sum_{k=3}^{n} a_{1k} \right| \leq C \cdot \sum_{k=3}^{n} \frac{1}{k-2} = 0(\log n). \tag{3.5}$$

(c) Again from (3.2),

$$\sum_{j=2}^{n-1} a_{j,\, j+1} = 0(\log n). \tag{3.6}$$

(d) If $(t, u) \in R_{jk}$, $2 \leq j \leq k - 2 \leq n - 2$, then $Q(t, u) \geq 0$ and

$$\theta(t, u) - \theta(j, k) = \frac{1}{4\pi^2 t(u-t)} \int \ldots \int f(x, y) g(\zeta, \eta) \left[e^{-Q(t,u)} - e^{-Q(j,k)}\right] dx \ldots d\eta + \frac{1}{4\pi^2}\left[\frac{1}{t(u-t)} - \frac{1}{j(k-j)}\right] \int \ldots \int \times$$
$$f(x,y) g(\zeta, \eta) e^{-Q(j,k)}\, dx \ldots d\eta = J_1 + J_2.$$

Setting
$R_a = \{|x| \leq a, |y| \leq a, |\zeta| \leq a, |\eta| \leq a\}$, $R_a' =$ complement of R_a, let $\epsilon > 0$ be arbitrary and choose $a > 0$ such that

$$\underset{R_a'}{\int \ldots \int} |f(x, y) g(\zeta, \eta)|\, dx \ldots d\eta < \epsilon.$$

We can write

$$J_1 = \frac{1}{4\pi^2 t(u-t)} \underset{R_a'}{\int \ldots \int} + \frac{1}{4\pi^2 t(u-t)} \underset{R_a}{\int \ldots \int} = J_1' + J_1'',$$

where

$$|J_1'| \leq \frac{\epsilon}{4\pi^2 t(u-t)} \leq \frac{C\epsilon}{(j-1)(k-j-1)}. \tag{3.7}$$

To obtain a bound for J_1'' we observe that

$$Q(t, u) - Q(j, k) = \frac{x^2 + y^2}{2}\left(\frac{1}{t} - \frac{1}{j}\right) + \frac{(x-\zeta)^2 + (y-\eta)^2}{2}\left(\frac{1}{u-t} - \frac{1}{k-j}\right) \leq \frac{x^2+y^2}{2}\left(\frac{1}{j-1} - \frac{1}{j}\right) + \frac{(x-\zeta)^2 + (y-\eta)^2}{2}\left(\frac{1}{k-j-1} - \frac{1}{k-j}\right) = \frac{x^2+y^2}{2(j-1)j} + \frac{(x-\zeta)^2 + (y-\eta)^2}{2(k-j-1)(k-j)} = A (\geq 0), \text{ say.}$$

Also,

$$Q(t, u) - Q(j, k) \geq \frac{(x-\zeta)^2 + (y-\eta)^2}{2}\left(\frac{1}{k-j+1} - \frac{1}{k-j}\right) = -\frac{(x-\zeta)^2 + (y-\eta)^2}{2(k-j)(k-j+1)} \geq -A.$$

Hence

$$e^{-A} - 1 \leq e^{-\{Q(t,u)-Q(j,k)\}} - 1 \leq e^{A} - 1, \qquad \text{and}$$

$$\left|e^{-\{Q(t,u)-Q(j,k)\}} - 1\right| \leq \max.\ [e^{A} - 1,\ 1 - e^{-A}] = e^{A} - 1.$$

Therefore

$$|J_1''| \leq \frac{1}{4\pi^2 t(u-t)} \int \underset{R_a}{\cdots} \int |f(x,y)g(\zeta,\eta)| \cdot e^{-Q(j,k)} \cdot \left|e^{-\{Q(t,u)-Q(j,k)\}} - 1\right| dx\ldots d\eta \leq \frac{FG}{4\pi^2(j-1)(k-j-1)} \int \underset{R_a}{\cdots} \int (e^{A} - 1)\, dx\ldots d\eta.$$

Since $e^x \leq 1 + xe^b$ for $0 \leq x \leq b$, we have in R_a

$$e^{A} - 1 \leq Ae^{\left[\frac{a^2}{(j-1)j} + \frac{4a^2}{(k-j-1)(k-j)}\right]} \leq Ae^{\left[\frac{a^2}{2} + 2a^2\right]},$$

and hence

$$|J_1''| \leq \frac{C(a)}{(j-1)(k-j-1)} \left[\frac{1}{(j-1)j} + \frac{1}{(k-j-1)(k-j)}\right]. \qquad (3.8)$$

Turning to J_2, we have

$$|J_2| \leq C\left|\frac{1}{t(u-t)} - \frac{1}{j(k-j)}\right| = C\left|\frac{j(k-j) - t(u-t)}{t(u-t)j(k-j)}\right| \leq C\frac{|j(k-j) - t(u-t)|}{(j-1)j(k-j-1)(k-j)}.$$

Since $1 - 2j = j(k-j) - jk + (j-1)^2 \leq j(k-j) - t(u-t) \leq j(k-j) - (j-1)(k-1) + j^2 = k + j - 1$,

$$|j(k-j) - t(u-t)| \leq \max.\ [k + j - 1, 2j - 1] = k + j - 1 = (k-j-1) + 2(j-1) + 2.$$

Hence

$$|J_2| \leq C\left[\frac{1}{(j-1)j(k-j)} + \frac{1}{j(k-j-1)(k-j)} + \frac{1}{(j-1)j(k-j-1)(k-j)}\right] \leq C\left[\frac{1}{(j-1)^2(k-j-1)} + \frac{1}{(j-1)(k-j-1)^2} + \frac{1}{(j-1)^2(k-j-1)^2}\right]. \qquad (3.9)$$

From (3.7)–(3.9) we obtain for $2 \leq j \leq k - 2 \leq n - 2$,

$$|a_{jk}| \leq \frac{C\epsilon}{(j-1)(k-j-1)} + C(a)\left[\frac{1}{(j-1)^3(k-j-1)} + \frac{1}{(j-1)(k-j-1)^3}\right] + C\left[\frac{1}{(j-1)^2(k-j-1)} + \frac{1}{(j-1)(k-j-1)^2} + \frac{1}{(j-1)^2(k-j-1)^2}\right]. \quad (3.10)$$

Now from [1],

$$\sum_{2\leq j\leq k-2\leq n-2} \frac{1}{(j-1)(k-j-1)} \sim (\log n)^2, \quad (3.11)$$

while

$$\sum_{=2}^{n-2} \sum_{k=j+2}^{n} \frac{1}{(j-1)^2(k-j-1)} \leq \sum_{j=2}^{n-2} \frac{1}{(j-1)^2} \cdot \sum_{m=1}^{n} \frac{1}{m} = 0\ (\log n), \quad (3.12)$$

$$\sum_{j=2}^{n-2} \sum_{k=j+2}^{n} \frac{1}{(j-1)(k-j-1)^2} \leq \sum_{j=2}^{n-2} \frac{1}{j-1} \sum_{m=1}^{\infty} \frac{1}{m^2} = 0\ (\log n). \quad (3.13)$$

From (3.4)–(3.6) and (3.10)–(3.13) it follows that

$$\limsup_{n\to\infty} (\log n)^{-2} \left| \sum_{j,k=1}^{n} a_{jk} \right| \leq C\epsilon,$$

and since ϵ is arbitrary this proves (3.1) and Lemma 2.

Turning to the proof of Lemma 1, let

$$K_n = \frac{1}{\log n}\left[\int_0^n f(V(t))\,dt - \sum_{j=1}^{n} f(V(j))\right] = \frac{1}{\log n}\sum_{j=1}^{n}\int_{j-1}^{j}\left[f(V(t)) - f(V(j))\right] dt.$$

Then $EK_n^2 = (\log n)^{-2} \cdot \sum_{j,k=1}^{n} b_{jk}$,

where

$$b_{jk} = \int\int_{R_{jk}} E[f(V(t)) - f(V(j))][f(V(u)) - f(V(k))]\,dtdu$$

$$= \int\int_{R_{jk}} [\{\theta(t, u) - \theta(t, k)\} - \{\theta(j, u) - \theta(j, k)\}]\,dtdu$$

and where we now write

$$\theta(t, u) = Ef(V(t))f(V(u))$$
$$= \frac{1}{4\pi^2 t(u - t)} \int \ldots \int f(x, y) f(\zeta, \eta) e^{-Q(t, u)}\, dx \ldots d\eta$$

for $0 < t < u$. For $2 \leq j \leq k - 2 \leq n - 2$,

$$\theta(t, u) - \theta(t, k) = \frac{1}{4\pi^2 t(u - t)} \int \ldots \int f(x, y) f(\zeta, \eta)[e^{-Q(t, u)} - e^{-Q(t, k)}]\, dx \ldots d\eta + \frac{1}{4\pi^2 t}\left[\frac{1}{u - t} - \frac{1}{k - t}\right] \int \ldots \int f(x, y) f \times (\zeta, \eta) e^{-Q(t, k)}\, dx \ldots d\eta = J_1 + J_2.$$

In R_{jk},

$$Q(t, u) - Q(t, k) = \frac{(x - \zeta)^2 + (y - \eta)^2}{2}\left(\frac{1}{u - t} - \frac{1}{k - t}\right) = \frac{(x - \zeta)^2 + (y - \eta)^2}{2} \cdot \frac{k - u}{(u - t)(k - t)} \geq 0.$$

Hence $\left|e^{-Q(t, u)} - e^{-Q(t, k)}\right| = e^{-Q(t, k)}\left|e^{-\{Q(t, u) - Q(t, k)\}} - 1\right| \leq 1 - e^{-\{Q(t, u) - Q(t, k)\}} \leq Q(t, u) - Q(t, k) \leq \frac{(x - \zeta)^2 + (y - \eta)^2}{2} \cdot \frac{k - u}{(u - t)(k - t)}$.

Thus, writing

$$J_1 = \frac{1}{4\pi^2 t(u - t)} \int \underset{R_a'}{\ldots} \int + \frac{1}{4\pi^2 t(u - t)} \int \underset{R_a}{\ldots} \int = J_1' + J_1'',$$

we have $J_1' \leq \dfrac{C\epsilon}{(j - 1)(k - j - 1)}$,

$$J_1'' \leq \frac{F^2}{8\pi^2 t(u - t)} \int_{R_a} \ldots \int [(x - \zeta)^2 + (y - \eta)^2] \times dx \ldots d\eta \cdot \frac{k - u}{(u - t)(k - t)} \leq \frac{C(a)}{(j - 1)(k - j - 1)^3},$$

$$|J_2| \leq \frac{C}{t} \cdot \frac{k - u}{(u - t)(k - t)} \leq \frac{C}{(j - 1)(k - j - 1)^2}.$$

Hence we have

$$|\theta(t, u) - \theta(t, k)| \leq \frac{C\epsilon}{(j-1)(k-j-1)} + \frac{C(a)}{(j-1)(k-j-1)^3} + \frac{C}{(j-1)(k-j-1)^2}, \quad (3.14)$$

and similarly it can be shown that the right-hand side of (3.14) is an upper bound for $|\theta(j, u) - \theta(j, k)|$. Hence, as in the proof of Lemma 2,

$$\lim_{n \to \infty} (\log n)^{-2} \sum_{2 \leq j \leq k - 2 \leq n - 2} b_{jk} = 0,$$

and the other sums occurring in EK_n^2 can be proved to be $o\{(\log n)^2\}$ as before, completing the proof of Lemma 1.

4. *The One-Dimensional Case.*—Let $f(x)$ and $g(x)$ be real valued functions which are bounded and summable in the line $-\infty < x < \infty$, and set $\bar{f} = \int f(x)dx, \quad \bar{g} = \int g(x)dx.$

THEOREM 3. *If $\bar{f} \neq 0$ then for every u,*

$$\lim_{T \to \infty} \Pr\left[\frac{1}{\bar{f}\sqrt{T}} \int_0^T f(X(t))\, dt \leq u\right] = H(u),$$

where
$$H(u) = \begin{cases} \sqrt{\frac{2}{\pi}} \int_0^u e^{-y^2/2}\, dy & \text{for } u \geq 0, \\ 0 & \text{for } u < 0. \end{cases}$$

THEOREM 4. *If $\bar{g} \neq 0$ then*

$$\lim_{T \to \infty} \frac{\int_0^T f(X(t))\, dt}{\int_0^T g(X(t))\, dt} = \frac{\bar{f}}{\bar{g}} \text{ in probability.}$$

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OMNIBUS CHECKING OF THE 61-PLACE TABLE OF DENARY LOGARITHMS COMPILED BY PETERS AND STEIN, BY CALLET, AND BY PARKHURST

BY HORACE S. UHLER

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The first reference is: *Zehnstellige Logarithmentafel. Erster Band. Herausgegeben von Reichsamt für Landesaufnahme unter wissenschaftlicher Leitung von Prof. Dr. J. Peters.* Berlin 1922. Table 14b, pages 156–162 of the appendix. The original source for Table 14b is acknowledged on page xix by the statement that "···this table contains the 61-place common