In general if $u_1, \ldots, u_{Rm}$ is any system of integrals of the second kind on A, and $v_1, \ldots, v_{Rm}$ is the corresponding system on B

$$U = \omega^{-1}u, \quad V = \omega^{-1}v$$

where ω is the period matrix of the integrals and we have the result that if α is a matrix

$$\alpha = \bar{\omega}^{-1}f^m a^m \bar{\omega}^{-1}$$

$$\sum_{ij} \alpha_{ij}u_i \times v_j$$

is improper.

9. It is possible to deduce a similar relation between the p- and q-fold integrals of total differentials of the second kind by considering the cycle Γ_r^h of §1. Until, however, some application of such a result arises there is no point in carrying through the analysis, which does not introduce any new idea.

[1] Hodge, *J. Lon. Math. Soc.*, **5**, p. 283 (1930).
[2] Lefschetz, *Colloquium Lectures on Topology*, p. 266 (1930).
[3] Loc. cit.
[4] Hurwitz, *Mat. Ann.*, **28**, 561–585 (1887).
[5] Cf. Baker, *Abel's Theorem and the Allied Theory*, p. 185.
[6] Lefschetz, *Trans. Amer. Math. Soc.*, **22**, 337 (1921).

PROOF OF A RECURRENCE THEOREM FOR STRONGLY TRANSITIVE SYSTEMS

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Let

$$\frac{dx_i}{dt} = X_i(x_1, \ldots, x_n) \qquad (i = 1, 2, \ldots, n) \tag{1}$$

be a system of n differential equations of the first order, valid in a closed analytic n-dimensional manifold without singularity, M. The points of M are taken to be represented by a finite number of such sets of variables (x) in overlapping domains. For definiteness, the right-hand members X_i as well as the transformations of connection between the sets (x) are taken to be analytic. Finally it will be assumed that there is a volume integral invariant, $\int dx_1\, dx_2 \ldots dx_n$ in suitable coördinates.

If we select any $(n - 1)$ dimensional analytic surface σ in M, which cuts the trajectories in one and the same sense throughout at an angle $\theta \geqq d > 0$, the points of σ whose trajectories cut M infinitely often as the time t increases and decreases, fill all of σ save at most a set of measure 0 in the sense of Lebesgue. This is essentially the significance of the classic work of Poincaré on the recurrence of trajectories.[1]

Now it is probably true that in general such systems (1) are *strongly transitive* in the sense that any measurable set of complete trajectories in M has either the measure 0 or V, where V is the total volume of M. The fact that such strong transitivity may be realized has been shown in a simple example by E. Hopf, who has first defined this type of transitivity.

We propose in this note to prove the following simple recurrence theorem, if t_n denotes time of the nth crossing of σ by a trajectory which issues from a point P of σ, then we have, for a certain constant τ,

$$\lim_{n=\infty} \frac{t_n(P)}{n} = \tau, \tag{1}$$

for all points P save those which belong to a set of measure 0. In other words there is a fixed "mean time" of crossing on a general trajectory.

Very recently von Neumann,[2] by an application of abstract integral equation theory in a direction suggested by Koopman,[3] has obtained results which would show that $t_n(P)/n$ converges *in the mean* toward τ; but this does not show convergence nor a mean time in the usual sense. E. Hopf[2] has established his results directly.

I propose to establish (1) here, and in the following note to establish a general recurrence theorem and thence the "ergodic theorem."

The method of proof is one which I tried to use nearly ten years ago in order to show that there was some uniformity of recurrence when there was merely regional transitivity. That attempt would seem to have failed because the hypothesis was not exacting enough. It is to be remarked that the demonstration of the strong transitivity condition in any except very simple cases appears to be extraordinarily difficult.

Consider an "infinitesimal" cylinder made up of arcs of trajectories with a base $d\sigma$ at P in σ, and of height dn normal to $d\sigma$. Its volume is then $d\sigma dn$ or $v\cos\theta d\sigma dt$, where v denotes the velocity, and dt denotes the corresponding time.

Suppose now that the tube of trajectories with this base $d\sigma$ (t increasing) cuts σ again for a first time in the base $d\bar{\sigma}$. A doubly closed tube is thus formed having a total volume $t(P)v\cos\theta d\sigma$, where $t(P)$ stands for the time interval between the crossing at P and at $\bar{P}$. Let t increase further by dt; the tube then advances to a new position, differing from the former in that the cylinder of volume $v\cos\theta d\sigma dt$ has been subtracted, and the cylinder of volume $\bar{v}\cos\bar{\theta} d\bar{\sigma} dt$ has been added. But these volumes are equal, since volumes are conserved. In consequence if we designate the analytic point function $v \cos \theta > 0$ by $\omega(P)$ it is clear that $\omega(P)d\sigma = \omega(\bar{P})d\bar{\sigma}$; in other words, $\int \omega(P)d\sigma$ is conserved by the $(n-1)$-dimensional transformation $T(P)$ which takes P in σ to $\bar{P}$ in σ.

According to the result of Poincaré, the transformation from P to $\bar{P}$ is one-to-one in σ except at a set of points of measure 0. Of course, $t(P)$ is defined except at such points. More precisely, σ may be broken up into a numerable set of open continua, in which the transformation $T(P)$ and the function $t(P)$ are analytic, together with a further set of measure 0.

If the time to the nth crossing of σ be denoted by $t_n(P)$ (defined except for a set of measure 0), we have the fundamental functional identity

$$t_n(P) = t(T^{n-1}(P)) + t_{n-1}(P), \tag{2}$$

which states that the time to the nth crossing is the time beyond the $(n-1)$th crossing together with the time to the $(n-1)$th crossing. Here $T^k(P)$ denotes the kth transformed point of P. By successive use of the above identity we derive further

$$t_n(P) = t(T^{n-1}(P)) + t(T^{n-2}(P)) + \ldots + t(P). \tag{3}$$

From this equation we obtain

$$\int_\sigma t_n(P)dP = \int_\sigma t(T^{n-1}(P)dP + \ldots + \int_\sigma t(P)dP, \tag{4}$$

where dP stands for the $(n-1)$-dimensional volume element $\omega(P)d\sigma$.

But $\int t(P)dP$ extended over σ is the total volume V of M, according to the hypothesis of strong transitivity. For, this integral represents the measure of all the trajectories which issue from σ, and the remaining measurable set of trajectories is therefore of measure 0 by this hypothesis. Moreover, since $\int dP$ is conserved by T, and T transforms σ into itself except over a set of measure 0, we have

$$\int_\sigma t(T^k(P))dP = \int_\sigma t/T^{k-1}(P)dP = \ldots = V.$$

Thus (4) gives us

$$\frac{\int_\sigma t_n(P)dP}{n\int_\sigma dP} = \frac{V}{\int_\sigma v\cos\theta d\sigma} = \alpha. \tag{5}$$

In other words the *mean time* of the nth crossing of σ is precisely the ratio α of the total volume to the rate of flux across the surface σ.

Now consider the set S_δ of points P such that for a definite $\delta > 0$ and for infinitely many values of n, we have

$$t_n(P) > n(\alpha + \delta) \quad (n = 1, 2, \ldots, n) \tag{6}$$

The set $S_{k,\delta}$ of points P for which this inequality holds for *some* $n \geqq k$ is a measurable set, which diminishes (or at least does not increase) with increase of k, toward the limiting measurable set S_δ. Moreover, the set S_δ has the property of invariance under $S{:}T(S) = S$. Hence S_δ has either the measure 0 or that of σ, for, according to the hypothesis of strong transitivity, the measure of the trajectories through S_δ is $\int_{S\delta} t(P)dP = 0$ or V.

Similarly the set S'_δ of points P such that for a definite $\delta > 0$ and for infinitely many values of n, we have

$$t_n(P) < n(\alpha - \delta) \quad (n = 1, 2, \ldots), \tag{7}$$

is an invariant measurable set of measure 0 or σ.

If S_δ and S'_δ are both of measure 0 for δ arbitrarily small, we would conclude at once that, for almost all points P of σ, and for n sufficiently large,

$$n(\alpha - \delta) < t_n(P) < n(\alpha + \delta),$$

no matter how small δ be taken. In this case, of course, the stated theorem is true.

If this is not the case, suppose for example that S_δ has the measure of σ for some $\delta > 0$. Certainly in that event, the set $S_{k,\delta}$ for $k = 1$ will also have the measure of σ, since $S_{1,\delta}$ is the set for which (6) holds for some n.

Now $S_{1,\delta}$ can be broken up into the sequence of distinct classes U_1, U_2, ... of points P defined as follows:

$U_1 : t(P) > \alpha + \delta$;
$U_2 : t_2(P) > 2(\alpha + \delta)$, P not in U_1;
$U_3 : t_3(P) > 3(\alpha + \delta)$, P not in U_1 or U_2;
. . . . .

In consequence, if P is a point of U_k we have

$$t_k(P) > k(\alpha + \delta) \quad t_l(P) \leqq l(\alpha + \delta) \quad (1 \leqq l < k)$$

whence, by subtraction and use of (3),

$$t_{k-l}(T^l(P)) > (k - l)(\alpha + \delta).$$

We infer that, if P is a point of U_k, then $T^l(P)$ for $l < k$ is a point of one of the sets U_{k-l}, U_{k-l-1}, ..., U_1. It follows that, as l increases, $T^l(P)$ falls successively in sets U_{k-1}, ..., U_1, with lower subscripts, not more than $k - 1$ points being required before a point of this set falls in U_1.

Thus it is seen that one may separate U_k into $k - l$ distinct measurable sets U_{kj} $(j = 1, 2, \ldots, k - 1)$, such that, if P lies in U_{kj}, the points $T(P)$, ..., $T^j(P)$ fall in U_i's with decreasing subscripts $i < k$, the last point $T^j(P)$ only being in U_1.

The measurable sets in U_k, ..., U_1,

$$U_{kj}, T(U_{kj}), \ldots, T^{k-1}(U_{kj}) \quad (j = 1, 2, \ldots. k - 1), \tag{8}$$

are all distinct from one another. In fact, if there were a point P in common to

$$T^{j_1'}(U_{kj_1}) \text{ and } T^{j_2'}(U_{kj_2}) \qquad (j_1' \leqq j_2'),$$

the transformation $T^{-j_1'}$ would give a corresponding point in common to

$$U_{kj_1} \text{ and } T^{j_2'-j_1'}(U_{kj_2}).$$

But this is obviously not possible for $j_2' - j_1' > 0$; and is only possible for $j_2' - j_1' = 0$ if $j_1 = j_2$. Hence there are no such common points.

Next let us consider the measurable part of U_{k-1} made up of points P *not* in any such set U_{kj}. This part may be likewise separated in measurable parts $U_{k-1,j}$ $(j = 1, \ldots, k - 2)$ such that if P lies in $U_{k-1,j}$, then $T(P), \ldots, T^j(P)$ fall in sets $U_i (i < k - 1)$ with decreasing subscripts, the last point $T^j(P)$ only being in U_1.

The sets

$$U_{k-1,j},\ T(U_{k-1,j}),\ \ldots,\ T^{k-2}(U_{k-1,j}) \qquad (j = 1, \ldots, k - 2) \qquad (9)$$

so obtained are again distinct from one another. Furthermore they are entirely distinct from the previous sets. For if there were a point P in common to

$$T^{j_1'}(U_{k,j_1}) \text{ and } T^{j_2'}(U_{k-1,j_2}),$$

a transformation by T^{-j}, where j is the lesser of the integers j_1', j_2' or else is their common value, would give a corresponding point common to

$$U_{k,j_1} \text{ and } U_{k-1,j_2} \text{ if } j = j_1' = j_2',$$

or common to

$$U_{kj_1} \text{ and } T^{j_2'-j_1'}(U_{k-1,j_2}) \text{ if } j = j_1' < j_2',$$

or common to

$$T^{j_1'-j_2'}(U_{kj_1}) \text{ and } U_{k-1,j_2} \text{ if } j = j_2' < j_1'.$$

The first and second cases are obviously impossible. The last case could only arise for $j_1' - j_2' = 1$; but this is also impossible since U_{k-1,j_2} $(j_2 = 1, \ldots, k - 1)$ falls in the part of U_{k-1} not in $T(U_k)$ and so not in any $T(U_{kj_1})$.

Proceeding in the same manner, we may define a set of entirely distinct measurable sets

$$\begin{array}{ll} U_{k,k-1},\ U_{k,k-2} \cdots & U_{k,1}, \\ \cdots\cdots U_{k-1,k-2}, \ \cdots & U_{k-1,1}, \\ & \cdots \\ & U_{21}, \\ & U_{10}, \end{array}$$

of which the last U_{10} consists of the points of U_1 not in any preceding set. These will have the property that the finite set of measurable sets

$$T^m(U_{l,j}) \quad (l = 2, \ldots k, j = 1, 2, \ldots l - 1, m = 0, 1, \ldots l - 1)$$

are entirely distinct from one another and, together with U_{10}, exhaust $S_{1,\delta,k} = U_1 + U_2 + \ldots + U_k$.

Consider now the integral

$$\int_{S_{1,\delta,k}} t(P)dP = \sum_{j,l,n} \int_{T^m(U_{l,j})} t(P)dP + \int_{U_{10}} t(P)dP.$$

This integral may be written

$$\sum_{j,l} \int_{U_{lj}+\ldots+T^{l-1}(U_{lj})} t(P)dP$$

which is the same as

$$\sum_{j,l} \int_{U_{l,j}} t_l(P)\, dP.$$

Since $U_{l,j}$ is a part of U_l, each partial integral in $\sum$ exceeds $l(\alpha + \delta)$ $\int_{U_{lj}} dP$, by definition of U_l. This quantity is the same as

$$(\alpha + \delta) \int_{U_{l,j}+\ldots+T^{l-1}(U_{l,j})} dP$$

since $\int dP$ is conserved by T. Likewise, $\int_{U_{10}} t(P)dP$ exceeds $(\alpha + \delta)$ $\int_{U_{10}} dP$. Hence we deduce the inequality

$$\int_{S_{1,\delta,k}} t(P)dP > (\alpha + \delta) \int_{S_{1,\delta,k}} dP$$

for $k = 1, 2, \ldots$. But, inasmuch as $S_{1,\delta}$ has the measure of σ and is the limit of $S_{1,\delta,k}$ for $k = 1, 2, \ldots$, we would then conclude

$$\int_\sigma t(P)dP \geqq (\alpha + \delta) \int_\sigma dP,$$

which is manifestly impossible.

Evidently the argumentation just given applies equally well for any numerable set of distinct measurable elements of surface, σ, which make an angle $\theta > d > 0$ with the trajectories and have a finite $\int dP$.

Thus the theorem is proved not only for any single surface σ but for any such measurable aggregate.

[1] *Méthodes nouvelles de la Mécanique Celeste*, t. 3.

[2] Not yet published.

[3] "Hamiltonian Systems and Transformations in Hilbert Space," these PROCEEDINGS, 315–318 (May, 1931).