gives an easy method of characterizing the unit point of the coördinate system. If we impose the conditions, for example, that the coefficients of the first, second and third degree terms in (4) shall all be unity, then the unit point lies on Q and its projection from (0, 0, 0, 1) upon $x_4 = 0$ lies on a tangent of Segre. This point, the two points of intersection of the tangent with the above cubic surface, and the point of intersection of the tangent and the line $x_1 = x_4 = 0$ have a cross ratio equal to -2. The three choices of the tangent of Segre correspond to the cube roots of unity which appear when the above conditions on the coefficients are imposed.

[1] Wilczynski, *Trans. Amer. Math. Soc.*, **9**, 79–120 (1908).

[2] Green, *Ibid.*, **20**, 79–153 (1919).

[3] Stouffer and Lane, *Bull. Amer. Math. Soc.*, **34**, 460 (1928).

[4] Bompiani, *Rend. Acc. Lincei*, [6] **6**, 187–190 (1927), and *Math. Zeitschrift*, **29**, 678–683 (1929).

[5] Fubini e Cech, *Geometria Proiettiva Differenziale*, **1**, 148 (1926).

[6] Stouffer, *Bull. Amer. Math. Soc.*, **34**, 301 (1928).

DYNAMICAL SYSTEMS OF CONTINUOUS SPECTRA

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1. In a recent paper by B. O. Koopman,[1] classical Hamiltonian mechanics is considered in connection with certain self-adjoint and unitary operators in Hilbert space $\mathfrak{H}$ (= $\mathfrak{L}_2$). The corresponding canonical resolution of the identity $E(\lambda)$, or "spectrum of the dynamical system," is introduced, together with the conception of the spectrum revealing in its structure the mechanical properties of the system.[2] In general, $E(\lambda)$ will consist of a discontinuous part (the "point spectrum") and of a continuous part. The case of a pure point spectrum, and the other extreme, that in which the inner product $(E(\lambda)f, g)$ is, for every f and g in $\mathfrak{H}$, the Lebesgue integral of one of its derivatives, may readily be treated by known analytical tools.[3] The present paper is devoted to the case where $E(\lambda)$ is continuous ($\lambda \neq 0$), but without $(E(\lambda)f, g)$ being necessarily equal to the integral of its derivative. It will further be assumed that the system is non-integrable in the sense that any f in $\mathfrak{H}$ such that $U_t f = f$ almost everywhere on Ω must be almost everywhere constant. In other words, we are assuming the following hypothesis:

$$C.\quad E(\lambda + 0) - E(\lambda - 0) = 0, \text{ for } \lambda \neq 0:$$

If $E_0 = E(+0) - E(-0)$, then $E_0 f =$ almost everywhere constant.

Suppose that ϕ is a characteristic function of a non-integrable system: $U_t\phi = e^{i\lambda t}\phi$. Then $U_t|\phi| = |\phi|$, so that $|\phi|$ must be almost everywhere constant, and hence we may take $\phi = e^{i\theta}$, θ being defined almost everywhere (mod 2π).[4] From $U_t\phi = e^{i\lambda t}\phi$ follows that $U_t\theta = \theta + \lambda t$ (mod 2π); thus θ is an "angle variable." Since ϕ is in $\mathfrak{H}$, $||\phi|| = \int_\Omega e^{i\theta}e^{-i\theta}dv = \mu\Omega$ must be finite, so that a non-integrable system of infinite $\mu\Omega$ can have no characteristic function. These considerations permit the following restatement of our hypothesis:

C'. The system possesses no invariant subset of Ω of positive finite measure, and in the case where $\mu\Omega$ is finite, no angle variables.

We will show that under this hypothesis all the initially observed properties of the system are obliterated by the lapse of time: the method of elementary mechanics of computing the final from the initial state must be replaced by the methods of the theory of probability. Contrary to the case of classical statistical mechanics, this situation is not dependent upon the system's having an enormous number of degrees of freedom: this number may perfectly well reduce to two.

2. By a P-set we shall mean a set of points of Ω, by a t-set, a set of points on the time axis. The P-set for which $f(P) > a$, etc., shall be denoted as usual by $[f(P) > a]$, etc., and similarly for t-sets and functions of t. Along with the μ-measure μM of a P-set M defined with respect to the volume element $dv = \rho d\omega$, we shall define the τ-measure τI of a t-set I as follows:

$$\tau I = \lim_{T=+\infty} \frac{m(I\cdot[\,|t| \leqq T])}{2T}.$$

Here m denotes the ordinary Lebesgue measure, and the customary set-multiplication is referred to.[5] By a zero P-set or a zero t-set we shall mean one for which the μ-measure or the τ-measure, respectively, is zero. The "characteristic function" of a P-set of a t-set Θ shall be denoted, as usual, by $\chi_\Theta : \chi_\Theta(P) = 1$ or 0 according as P is or is not on Θ, etc. In these terms we are able to state the fundamental theorem of this paper:

THEOREM I. Under the hypothesis C, there exists a zero t-set I such that, for any two P-sets M and N of finite μ-measure, as $t \longrightarrow +\infty$ (or $t \longrightarrow -\infty$) through values not on I,

$$\mu(M_t\cdot N) \longrightarrow \frac{\mu M\cdot\mu N}{\mu\Omega}, \tag{1}$$

M_t being the image of M after the lapse of time t. When $\mu\Omega = \infty$, the right-hand member is to be replaced by zero.

Proof.—It has been shown[6] that in non-integrable systems,

$$(E_0\chi_M, \chi_N) = \frac{\mu M \cdot \mu N}{\mu\Omega};$$

hence (1) is equivalent to

$$\lim_{\substack{t \to \pm\infty \\ t \text{ not on } I}} (U_t\chi_M, \chi_N) = (E_0\chi_M, \chi_N),$$

which, in turn, may be replaced by

$$\lim_{\substack{t \to \pm\infty \\ t \text{ not on } I}} (U_t f, g) = (E_0 f, g), \qquad (1')$$

since the aggregate of χ-functions and their linear combinations is everywhere dense in $\mathfrak{H}$. That is, we must prove that as $t \to \pm\infty$ through values not on I, $U_t f$ converges "weakly" to $E_0 f$.[7] Now $(U_t f, g) \to (E_0 f, g)$ for all f, g if and only if $(U_t f, f) \to (E_0 f, f)$ for all f.[8] Using, now, the known properties:[9] $U_t E_0 = E_0 U_t = U_t$, $E_0^2 = E_0$, etc., we have $((U_t - E_0)f, f) = ((U_t - U_t E_0)f, f) = (U_t(I - E_0)f, f) = (U_t(I - E_0)^2 f, f) = (U_t(I - E_0)f, (I - E_0)f)$, so that our problem is to show that $(U_t(I - E_0)f, (I - E_0)f) \to 0$. That is, we are to show that a zero t-set I exists such that, as $t \to \pm\infty$, t not on I, $(U_t g, g) \to 0$ whenever $E_0 g = 0$. Since U_t is a bounded operator, it is sufficient to show this only for an everywhere dense sequence $g_1, g_2, g_3, \ldots$ of functions g.

We now introduce the expression

$$\frac{1}{2T}\int_{-T}^{T} |(U_t g, g)^2|\, dt.$$

It has the value

$$\frac{1}{2T}\int_{-T}^{T}\left|\int_{-\infty}^{+\infty} e^{i\lambda t}\, d_\lambda \,\|E(\lambda)g\|^2\right|^2 dt$$

$$= \frac{1}{2T}\int_{-T}^{T} dt \int_{-\infty}^{+\infty} e^{i\lambda t}\, d_\lambda\,\|E(\lambda)g\|^2 \cdot \int_{-\infty}^{+\infty} e^{-i\mu t}\, d_\mu\,\|E(\mu)g\|^2$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\frac{1}{2T}\int_{-T}^{T} e^{i(\lambda-\mu)t}\,dt\right) d_\lambda\,\|E(\lambda)g\|^2 \cdot d_\mu\,\|E(\mu)g\|^2$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{e^{i(\lambda-\mu)T} - e^{-i(\lambda-\mu)T}}{2i(\lambda-\mu)T}\cdot d_\lambda\,\|E(\lambda)g\|^2 \cdot d_\mu\,\|E(\mu)g\|^2$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\sin(\lambda-\mu)T}{(\lambda-\mu)T}\cdot d_\lambda\,\|E(\lambda)g\|^2 \cdot d_\mu\,\|E(\mu)g\|^2.$$

Since $E(\lambda)$ has no point-spectrum except possibly for $\lambda = 0$, for which value precisely we have $(E(+0) - E(-0))g = E_0 g = 0$, $\|E(\lambda)\|^2$ is a continuous function of λ, which, as λ goes from $-\infty$ to $+\infty$, goes in a non-decreasing manner from 0 to $\| g \|^2$. If $\lambda = \phi(x)$ is the inverse of $x = \| E(\lambda) \|^2$, $\phi(x)$ increases monotonically from $-\infty$ to $+\infty$ as x goes from 0 to $\| g \|^2$, it is constant on no interval, although it may have finite jumps. Hence the above expression may be written as

$$\int_0^{\| g \|^2} \int_0^{\| g \|^2} \frac{\sin\{\phi(x) - \phi(y)\}T}{\{\phi(x) - \phi(y)\}T} dx\, dy.$$

Since the region of integration is finite and the integrand in absolute value less than 1, and since (except for the zero set $x = y$) it approaches zero, the expression approaches zero as $T \longrightarrow \pm \infty$. (This would not be true if $\phi(x)$ were constant on certain intervals, i.e., if $\| E(\lambda)g \|^2$ had discontinuities, and thus, $E(\lambda)$ had a point spectrum.)

Thus,

$$\lim_{T \longrightarrow +\infty} \frac{1}{2T} \int_{-T}^{+T} |(U_t g, g)|^2\, dt = 0. \qquad (2)$$

Consider the series

$$\pi\,(t) = \sum_{n=1}^{\infty} \frac{1}{2^n \, \|g_n\|^2} |(U_t g_n, g_n)|^2.$$

It is uniformly convergent for $|t| < +\infty$, since it is dominated by the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$, on account of the inequality $\|(U_t g_n, g_n)\|^2 \leqq \| U_t g \| \cdot \| g \| = \| g \|^2$. Hence, in virtue of (2),

$$\lim_{t \longrightarrow +\infty} \frac{1}{2T} \int_{-T}^{T} \pi(t)\, dt = 0. \qquad (3)$$

Now if we can show that a zero t-set I can be found such that

$$\lim_{\substack{t \longrightarrow \pm\infty \\ t \text{ not on } I}} \pi(t) = 0, \qquad (4)$$

then, on account of the inequality

$$0 \leqq |(U_t g_n, g_n)|^2 \leqq 2^n \, \| g \|^2 \pi(t),$$

we shall have the consequence that

$$\lim_{\substack{t \longrightarrow \pm\infty \\ t \text{ not on } I}} (U_t g_n, g_n) = 0,$$

that is, the theorem will be proved.

Let $I_m = \left[\pi(t) \geqq \frac{1}{m}\right]$. Evidently $I_1 \subset I_2 \subset I_3 \subset \ldots$, and, in virtue of (3),

$$\lim_{T \longrightarrow +\infty} \frac{m(I_m \cdot [|t| \leqq T])}{2T} = 0.$$

We now choose $T_1 < T_2 < \ldots$ such that the left-hand member of the above is, for all $m = 2, 3, \ldots$, and $T \geqq T_{m-1}$, less than $\frac{1}{m}$. Then we set

$$I = I_1 \cdot [|t| \leqq T_1] + I_2 \cdot [T_1 < |t| \leqq T_2] + \ldots$$
$$\ldots + I_m \cdot [T_{m-1} < |t| \leqq T_m] + \ldots.$$

For $T_{m-1} < T \leqq T_m$ we have

$$I \cdot [|t| \leqq T] \subset I_{m-1} \cdot [|t| \leqq T_{m-1}] + I_m \cdot [|t| \leqq T],$$

hence,

$$\frac{m(I \cdot [|t| \leqq T])}{2T} \leqq \frac{m(I_{m-1} \cdot [|t| \leqq T_{m-1}]) + m(I_m \cdot [|t| \leqq T])}{2T}$$
$$\leqq \frac{m(I_{m-1} \cdot [|t| \leqq T_{m-1}])}{2T_{m-1}} + \frac{m(I_m \cdot [|t| \leqq T])}{2T}$$
$$\leqq \frac{1}{m-1} + \frac{1}{m}.$$

As $m \longrightarrow +\infty$, that is, as $T \longrightarrow +\infty$, this approaches zero; hence I is a zero t-set. Since, further, when $|t| > T_{m-1}$, a t on I_m is on I, it follows that outside I, $\pi(t) < \frac{1}{m}$. Hence as $m \longrightarrow +\infty$, or $t \longrightarrow \pm\infty$, $\pi(t) \longrightarrow 0$, so that (4) is established.

Corollary.—The set I of Theorem I can be taken as a set of intervals such that there are only a finite number of intervals of the set in any arbitrarily chosen finite interval.

Proof.—Since the expressions $(U_t g, g)$ are continuous function of t, $\pi(t)$ is likewise, so that every I_n is a closed set, and, likewise, I. Hence the Lebesgue and that Jordan measure of I are equal. In each of the regions $|t| \leqq 1$, $|t| \leqq 2$, ..., $m - 1 < |t| \leqq m$, ... we can replace I by a finite set of overlapping intervals without increasing its measure by more than, for example, $\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^m}, \ldots$,—thus, in all, by a quantity < 1, so that it remains a zero t-set.

THEOREM II. When the hypothesis C is not realized, the conclusion of Theorem I is invalid.

Proof.—To begin with, suppose the system to be integrable. Then there will exist an invariant $\Lambda \subset \Omega : \Lambda_t = \Lambda$, with $\mu\Lambda > 0$ and finite, and

$\mu(\Omega - \Lambda) > 0$. If $\mu\Omega$ is finite, take $M = \Lambda$, $N = \Omega - \Lambda$ in (1), when $\mu(M_t \cdot N) = \mu(\Lambda_t \cdot (\Omega - \Lambda)) = \mu(\Lambda \cdot (\Omega - \Lambda)) = 0$, whereas $\frac{\mu\Lambda \cdot \mu(\Omega - \Lambda)}{\mu\Omega} \neq 0$. If $\mu\Omega = \infty$, take $M = N = \Lambda$, in which case $\mu(M_t \cdot N) = \mu(\Lambda_t \cdot \Lambda) = \mu\Lambda > 0$, whereas $\frac{\mu\Lambda \cdot \mu\Lambda}{\mu\Omega} = 0$. Thus, in either case, (1) is violated.

Suppose, on the other hand, that the system is non-integrable, but that it has a characteristic function φ of characteristic number $\lambda > 0$: $U_t\varphi = e^{i\lambda t}\varphi$. In this case, as we have seen, we may take $\varphi = e^{i\theta}$, and $\mu\Omega$ will be finite. In (1′) take $f = g = \varphi$; then $(U_tf, g) = (e^{i\lambda t}\varphi, \varphi) = e^{i\lambda t}(\varphi, \varphi) = e^{i\lambda t}\mu\Omega$, so that (1′), which in the non-integrable case is a consequence of (1), is impossible.

Theorem II is thus proved.

Now let us consider to what extent the presence of the exceptional t-set I appears necessary in Theorem I. We have:

THEOREM III. There exist spectra $E(\lambda)$ (not necessarily belonging to a dynamical system[10]) such that, under the hypothesis C, the conclusion (1′) of Theorem I fails to hold when I is empty.

Proof.—Take f such that $E_0f = 0$; then it is sufficient to show that $\lim_{\to \pm\infty} (U_tf, f)$ does not exist, or else that it is not zero. We have[11]

$$(U_tf, f) = \int_{-\infty}^{+\infty} e^{i\lambda t} d\, ||E(\lambda)f||^2,$$

and since $x = ||E(\lambda)f||^2$ can evidently be taken to be an arbitrarily given continuous non-decreasing function, increasing from zero to a finite a $(= ||f||^2)$ as t goes from $-\infty$ to $+\infty$, its inverse $\lambda = \varphi(x)$ can obviously be taken to be equal to an arbitrary monotonically increasing function, possibly with finite jumps, in the interval of definition $0 \leqq x \leqq a$, and going from $-\infty$ to $+\infty$. Hence

$$(U_tf, f) = \int_0^a e^{i\varphi(x)t}\, dx.$$

Now let $a = 1$, and $\varphi(x)$ have always such a value that its hexadic development contains only the digits 0 and 5. For instance, let $\varphi(0) = -\infty$, $\varphi(1) = +\infty$, and, for $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$ ($a_n = 0$ or 1, the latter infinitely often), let $\varphi(x) = \sum_{n=1}^{\infty} \frac{5a_n}{6_n}$. For $t = 2\pi \cdot 6^n$, the fractional part of $\frac{1}{2\pi}\varphi(x) \cdot t$ will thus start with the hexadic digit 0 or 5, i.e., it is $\geqq 0$, $\leqq \frac{1}{6}$, or $\geqq \frac{5}{6}$,

$\leqq 1$—i.e., $\frac{1}{2\pi}\varphi(x)t$ has a value (mod 1) $\geqq -\frac{1}{6}, \leqq \frac{1}{6}$. Hence

$$\mathfrak{R}\int_0^a e^{i\varphi(x)t}\,dx = \int_0^1 \cos\left(2\pi\cdot\frac{1}{2\pi}\varphi(x)t\right)dx$$
$$\geqq \int_0^1 \cos\left(2\pi\cdot\frac{1}{6}\right)dx$$
$$= \cos\frac{\pi}{3} = \frac{1}{2} > 0.$$

Thus $\lim_{t\to\pm\infty}(U_t f, f) = 0$ is excluded, and our theorem is proved.

III. Theorems I, II, III, delimit a situation which could scarcely be rendered more precise: the hypotheses and diverse restrictions seem all inherent in the nature of the case. It would, of course, be extremely interesting if a proof of Theorem I could be found without the use of the spectrum $E(\lambda)$, etc., for example, along the lines of the recent paper of Birkhoff's on the ergodic hypothesis.[12]

Theorem I expresses the fact that in a system for which C holds, consecutive states at two sufficiently different epochs are almost certainly statistically independent. For $\frac{\mu(M_t\cdot N)}{\mu\Omega}$ is the probability that, in the time t, the system go from M to N—and this is $=\frac{\mu M\cdot\mu N}{(\mu\Omega)^2}$, i.e., the product of the probabilities of the system's being in M and of its being in N. (We say "almost certainly," to correspond with the fact that the exceptional zero t-set I is excluded.) Such a system can have, *a la longue*, no physical properties. Indeed, the only properties which any system can have *a la longue* are those which violate C, namely:

1. Invariant sub-sets: barriers that are never passed.
2. Angle variables: clocks that never change.[13]

Theorem I may also be expressed by saying that the states of motion corresponding to any set M of Ω become more and more spread out into an amorphous everywhere dense chaos. Periodic orbits, and such like, appear only as very special possibilities of negligible probability.

We have already called attention to the essential difference between this situation, which may exist in very simple systems, and that envisaged in the kinetic theory of gases, in which the confused character of the motion is an intuitively evident consequence of the large number of degrees of freedom of the system.

[1] "Hamiltonian Systems and Transformations in Hilbert Space," these PROCEEDINGS, 315–318 (May, 1931); this reference will here be abbreviated to (H). Another reference of importance for us is the paper by v. Neumann on the proof of the quasi-ergodic hy-

pothesis (these PROCEEDINGS, pp. 70–82 (Jan., 1932)), which will be referred to under the abbreviation (Q).

We shall, in the present paper, assume the notation, etc., of (H) and (Q) to be known.

[2] Properties, of course, which are true "almost everywhere" on the manifold of states of motion Ω, corresponding with the nature of the functional tools. The allied conception of properties true "almost everywhere" on the time axis with respect to the time-average τ-measure here defined is an essential notion in the present paper; it is contained implicitly in J. v. Neumann's recent proof of the quasi-ergodic hypothesis (these PROCEEDINGS, pp. 70–82 (Jan., 1932), and 263–266 (March, 1932)).

[3] These cases are made the subject of as yet unpublished investigations by B. O. Koopman. In the case of a pure point spectrum, the theory of almost periodic functions is available: $(U_t f, g)$ is an almost periodic function of t, a fact having an obvious physical interpretation when $f = \chi_M(P)$ and $g = \chi_N(P)$, characteristic functions of the sets M and N ($\subset \Omega$; μM, μN finite). In the second case here noted, in virtue of the well-known Fourier integral theorem, $(U_t f, g) \longrightarrow 0$ as $t \longrightarrow \infty$, showing (on making the above choice of f, g) that any such region M will flow, in course of time, "almost entirely" out of a fixed region N of finite measure. This possibility obviously implies that $\mu\Omega$ is infinite. There is a similarly obvious interpretation when $\lambda = 0$ is a simple unique characteristic number.

[4] Cf. (H), p. 318.

[5] It is unnecessary for us to enter upon the discussion of τ-measurability or the properties of τ-measure.

[6] Cf. (Q), p. 78.

[7] The ergodic theorem (Q) states that $\frac{1}{T}\int_0^T U_t f\, dt$ converges "strongly" to $E_0 f$, thus this theorem tells us *more* than the present one from the point of view of convergence, but *less*, from the point of view of the function, since f is replaced by its time average.

[8] If we replace in $(U_t f, f) \longrightarrow (E_0 f, f)$ f by $\frac{f+g}{2}$ and by $\frac{f-g}{2}$, and subtract, the real part of $(U_t f, g) \longrightarrow (E_0 f, g)$ results. If we replace g by ig, the imaginary part results. Thus the general formula holds.

[9] Cf. (Q), p. 73.

[10] The following question is of great interest: When will a canonical resolution of the identity $E(\lambda)$ be the spectrum of a dynamical system? An (unpublished) formula obtained by Koopman is that, when f, g, and the ordinary product fg are in $\mathfrak{H}$, then, for a dynamical system,

$$E(x)fg = \int_{-\infty}^{+\infty} E(x-s)f \cdot d_s E(s) g,$$

which is a consequence of the fundamental equation obtained in (H):

$$U_t F(f, g, \ldots) = F(U_t f, U_t g, \ldots),$$

where F is any single-valued function. Recently J. v. Neumann has shown that the first-mentioned formula is sufficient and necessary for the U_t being generated by a suitably chosen group of point-transformations $P \longrightarrow P_t$ (unpublished).

[11] If $||E(\lambda)f||^2$ is differentiable with respect to λ (the case for a type of continuous spectrum defined by Hellinger and Hahn) we have $\int_{-\infty}^{+\infty} \frac{d}{d\lambda} ||E(\lambda) f||^2 \cdot e^{it\lambda} d\lambda$. By a well-known theorem on Fourier integrals this has $\lim_{t \to \pm\infty}$ equal to zero. Cf. reference 3 above.

[12] These PROCEEDINGS, pp. 650–660 (Dec., 1931).

[13] Though it is very probable that the case C or C', in which our theorems hold is the general one for dynamical systems, it is not easy to construct effective examples. (See footnote 10.) An example, recently constructed by v. Neumann, will be published soon; it refers to a two-dimensional flow of the following type: the flow takes place in a rectangle, oriented parallel to the X and Y axes, the upper side of which has been replaced by suitable chosen curve $Y = F(X)$. The flow itself is parallel to the positive Y-axis, and each point X, $F(X)$ has to be identified with the corresponding point $X + \alpha$, 0. (The number $X + \alpha$ is to be taken mod. a, where a is the breadth of the parallelogram in the direction of the X-axis; α is a number incommensurable with a.) If $F(X)$ and α are suitably chosen this flow can be shown to fulfill C (and C').

PHYSICAL APPLICATIONS OF THE ERGODIC HYPOTHESIS

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I. In a recent issue of these PROCEEDINGS[1] the author has obtained a proof of the so-called quasi-ergodic hypothesis. The reader is referred to that paper for the precise formulation of this hypothesis, which plays so important a rôle in the foundations of classical statistical mechanics, and thus in the kinetic theory of gasses; the terminology of that paper will be used throughout this note. The exact statement of the mathematical result obtained in the previous paper by the author is as follows:

Let Ω be either the phase-space Φ of the mechanical system considered, or a sub space of Φ invariant under the transformation ($P \longrightarrow P_t$, P a point of Φ, t the time) induced by the equations of motion.[2] Let dv be the volume element defined in Ω invariant[3] under the transformation $P \longrightarrow P_t$, μN the Lebesgue measure (or weight) of $N(\subset \Omega)$ defined by means of dv : $\mu N = \int_N dv$. Let the time of sojourn of P_τ in N during the time $s < \tau < t$, divided by $t - s$, be denoted by $Z_{s,t}(N; P)$.

Then there exists a function $Z(N; P)$ such that, as $t - s \longrightarrow +\infty$, the function of P $Z_{s,t}(N; P)$ converges, in the sense of "strong convergence" in the space of functions of P, to the limit $Z(N; P)$; that is

$$\lim_{t-s \to +\infty} \int_\Omega |Z_{s,t}(N; P) - Z(N; P)|^2 dv = 0. \tag{1}$$

This property determines $Z(N; P)$, which function, in our previous paper, is studied in more detail and calculated explicitly. The condition for the validity of the so-called quasi-ergotic hypothesis is that $Z(N; P)$ be independent of P; we have shown in our earlier paper that this will be true if and only if there exists in Ω no integral of the equations of motion