then p, q, r, s form what may be called a *pseudo-linear statistical quadruple*, i.e., a quadruple which cannot be ordered by means of the between-relation though for each three of the four points one lies between the other two.

If a statistical T-metric S metrized by means of $\mathfrak{P}$ contains more than four points, then by virtue of the properties of betweenness this relation can be used to order S. Moreover, the other ideas of metric geometry (convexity, geodesics, etc.) can be applied.

The three principal applications of statistical metrics are to macroscopic, microscopic and physiological spatial measurements. Statistical metrics are designed to provide us (1) with a method removing conceptual difficulties from microscopic physics and transferring them into the underlying geometry, (2) with a treatment of thresholds of spatial sensation eliminating the intrinsic paradoxes of the classical theory. For a given point p_0 the number $\Pi(0;\ p_0, q)$ considered as a function of the point q indicates the probability that q cannot be distinguished from p_0. The study of this function should replace the attempt to determine a definite set of points q which cannot be distinguished from p_0. This function could also be used advantageously instead of a relation of physical identity for which, as Poincaré emphasized on several occasions, we always have triples p, q, r for which

$$p = q,\ q = r, \text{ and } p \neq r.$$

Experiments indicate that q sometimes can and sometimes cannot be distinguished from p_0. Hence, the adequate description of the situation seems to arise from counting the relative frequency of these occurrences.

NATURAL ISOMORPHISMS IN GROUP THEORY

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1. Introduction.—Frequently in modern mathematics there occur phenomena of "naturality": a "natural" isomorphism between two groups or between two complexes, a "natural" homeomorphism of two spaces and the like. We here propose a precise definition of the "naturality" of such correspondences, as a basis for an appropriate general theory. In this preliminary report we restrict ourselves to the natural isomorphisms of group theory; with this limitation we can present the basic concepts of our theory without developing the axiomatic approach necessary for a general treatment applicable to various branches of mathematics.

Properties of character groups (see the definitions in § 5 below) may serve to illustrate the ideas involved. Thus, it is often asserted that the character group of a finite group G is isomorphic to the group itself, but not in a "natural" way. Specifically, if G is cyclic of prime order p, there is for each generator of G an isomorphism of G to its character group, so that the proof furnishes $p - 1$ such isomorphisms, no one of which is in any way distinguished from its fellows. However, the proof that the character group of the character group of G is isomorphic to G itself is considered "natural," because it furnishes for each G a unique isomorphism, not dependent on any choice of generators.

To give these statements a clear mathematical meaning, we shall regard the character group $Ch(G)$ of G as a function of a variable group G, *together* with a prescription which assigns to any homomorphism γ of G into a second group G',

$$\gamma : G \rightarrow G',$$

the induced homomorphism (see (5) below)

$$Ch(\gamma) : Ch(G') \rightarrow Ch(G).$$

The functions $Ch(G)$ and $Ch(\gamma)$ jointly form what we shall call a "functor"; in this case, a "contravariant" one, because the mapping $Ch(\gamma)$ works in a direction opposite to that of γ. A natural isomorphism between two functions of groups will be an isomorphism which commutes properly with the induced mappings of the functors.

With our description of a natural isomorphism, practically all the general isomorphisms obtained in group theory and its applications (homology theory, Galois theory, etc.) can be shown to be "natural." This results in added clarity in such situations. Furthermore, there are definite proofs where the naturality of an isomorphism is needed, especially when a passage to the limit is involved. In fact, our condition ($E2$) below appears in the definition of the isomorphism of two direct or two inverse systems of groups.[1]

2. *Functors.*—The definition of a functor will be given for the typical case of a functor T which depends on two groups as arguments, and is *covariant* in the first argument and *contravariant* in the second. Such a functor is determined by two functions. The *group* function determines for each pair of topological groups G and H (contained in a given legitimate set of groups) another group $T(G, H)$. The *mapping function* determines for each pair of homomorphisms[2] $\gamma : G_1 \rightarrow G_2$ and $\eta : H_1 \rightarrow H_2$ a homomorphism $T(\gamma, \eta)$, such that

$$T(\gamma, \eta) : T(G_1, H_2) \rightarrow T(G_2, H_1). \tag{1}$$

We require that $T(\gamma, \eta)$ be the identity isomorphism whenever γ and η are identities, and that, whenever the products $\gamma_2\gamma_1$ and $\eta_2\eta_1$ are defined,

$$T(\gamma_2\gamma_1, \eta_2\eta_1) = T(\gamma_2, \eta_1)T(\gamma_1, \eta_2). \qquad (2)$$

Some functors will be defined only for special types of groups (e.g., for abelian groups) or for special types of homomorphisms (e.g., for homomorphisms "onto").

If γ and η are both isomorphisms,[3] it follows from these conditions that $T(\gamma, \eta)$ is also an isomorphism. Consequently, if the groups G_1 and G_2 and the groups H_1 and H_2 are isomorphic, the functor T gives rise to isomorphic groups $T(G_1, H_1)$ and $T(G_2, H_2)$.

3. Examples.—The direct product $G \times H$ of two groups may be regarded as the group function of a functor. The corresponding mapping function specifies, for each pair of homomorphisms $\gamma: G_1 \to G_2$ and $\eta: H_1 \to H_2$, an induced homomorphism $\gamma \times \eta$, defined for every element (g_1, h_1) in $G_1 \times H_1$ as

$$[\gamma \times \eta](g_1, h_1) = (\gamma g_1, \eta h_1).$$

Then

$$\gamma \times \eta : G_1 \times H_1 \to G_2 \times H_2, \qquad (3)$$

and, whenever $\gamma_2\gamma_1$ and $\eta_2\eta_1$ are defined, one has

$$(\gamma_2\gamma_1) \times (\eta_2\eta_1) = (\gamma_2 \times \eta_2)(\gamma_1 \times \eta_1). \qquad (4)$$

Except for the absence of contravariance, these conditions are parallel to (1) and (2), hence $G \times H$, $\gamma \times \eta$ define a functor, covariant in both G and H.

Whitney's tensor product[4] $G \circ H$ of two discrete groups[5] G and H is the group function of a functor. The elements of this group are all finite sums $\Sigma g_i \circ h_i$ of formal products $g_i \circ h_i$; the group operation is the obvious addition, and the relations are $g \circ (h + h') = g \circ h + g \circ h'$ $(g + g') \circ h = g \circ h + g' \circ h$. Given two homomorphisms $\gamma: G_1 \to G_2$ and $\eta: H_1 \to H_2$, there is an induced homomorphism $\gamma \circ \eta$ of $G_1 \circ H_1$ into $G_2 \circ H_2$, defined for any generator $g_1 \circ h_1$ of $G_1 \circ H_1$ as

$$[\gamma \circ \eta](g_1 \circ h_1) = (\gamma g_1) \circ (\eta h_1) \in G_2 \circ H_2.$$

Formulae (3) and (4), with the cross replaced by the circle, again hold, so that $G \circ H$, $\gamma \circ \eta$ determine a functor of discrete groups, covariant in both arguments.

In a similar fashion, the free product of two groups leads to a functor.

An important functor is given by the group of all homomorphisms ϕ of a fixed locally compact topological abelian group G into another topological abelian group H. The sum of two such homomorphisms ϕ_1 and ϕ_2 is defined for each $g \epsilon G$ by setting $(\phi_1 + \phi_2)(g) = \phi_1(g) + \phi_2(g)$. Under this operation, all $\phi: G \rightarrow H$ constitute a group $Hom(G, H)$; it carries an appropriate topology, the description of which we omit. For given $\gamma: G_1 \rightarrow G_2$ and $\eta: H_1 — H_2$ and for each $\phi \epsilon Hom(G_2, H_1)$ we have

$$G_1 \xrightarrow{\gamma} G_2 \xrightarrow{\phi} H_1 \xrightarrow{\eta} H_2.$$

Consequently we define $Hom(\gamma, \eta)(\phi) = \eta\phi\gamma$, and verify that

$$Hom(\gamma, \eta): Hom(G_2, H_1) \rightarrow Hom(G_1, H_2),$$

$$Hom(\gamma_2\gamma_1, \eta_2\eta_1) = Hom(\gamma_1, \eta_2)\ Hom(\gamma_2, \eta_1).$$

Clearly when γ and η are identity mappings of G and H the induced mapping $Hom(\gamma, \eta)$ is the identity mapping of $Hom(G, H)$ on itself. Hence the functions $Hom(G, H)$ and $Hom(\gamma, \eta)$ determine for abelian groups a functor Hom, covariant in H and contravariant in G.

The special case when H is the group P of reals modulo 1 furnishes the character group,

$$Ch(G) = Hom(G, P), \quad Ch(\gamma) = Hom(\gamma, e)$$

where e is the identity mapping of P on itself. Therefore the character group is a contravariant functor, defined for abelian groups. Explicitly, if we express the result $\chi(g)$ of applying the character χ to the element $g \epsilon G$ as the value (a real number modulo 1) of the bilinear form (g, χ), the definition of $Ch(\gamma)$ can be written as

$$(g, Ch(\gamma)\chi') = (\gamma g, \chi'), \quad g \epsilon G, \quad \chi' \epsilon Ch(G'). \qquad (5)$$

4. *Equivalence of Functors.*—Let T and S be two functors which are, say, both covariant in the variable G and contravariant in H. Suppose that for each pair of groups G and H we are given a homophorism

$$\tau(G, H): T(G, H) \rightarrow S(G, H).$$

We say that τ establishes a natural *equivalence* of the functor T to the functor S and that T is naturally *equivalent* to S (in symbols, τ: $T \longleftrightarrow S$) whenever

($E1$) Each $\tau(G, H)$ is a bicontinuous isomorphism of $T(G, H)$ onto $S(G, H)$;

($E2$) For each $\gamma: G_1 \rightarrow G_2$ and $\eta: H_1 \rightarrow H_2$,
$\tau(G_2, H_1)T(\gamma, \eta) = S(\gamma, \eta)\tau\ (G_1, H_2)$.

The first requirement insures the term-by-term isomorphism of the two group functions $T(G, H)$ and $S(G, H)$, while the second requirement is precisely the "naturality" condition. It can be shown that the condition ($E2$) is implied by two special cases; the case when η is an identity, and the case when γ is an identity.

This relation of natural equivalence between functors is reflexive, symmetric and transitive. In many cases we dispense with condition ($E1$), and obtain a more general concept of a "transformation" of a functor T into a functor S.

5. *Examples of Natural Equivalence.*—The well-known isomorphism

$$G \cong Ch(Ch(G)) \tag{6}$$

for locally compact abelian groups, can be regarded as an equivalence of functors, and is in this sense *natural*. The right-hand side of (6) suggests the covariant functor, Ch^2, defined by iteration of the functor Ch, as

$$Ch^2(G) = Ch(Ch(G)), \quad Ch^2(\gamma) = Ch(Ch(\gamma)).$$

The left-hand side of (6) suggests the identity functor, I,

$$I(G) = G, \quad I(\gamma) = \gamma.$$

The bilinear form $(g, \chi) = \chi(g)$ determines to each character $\chi \in Ch(G)$ and each $g \in G$ a real number modulo 1; similarly the form $(\chi, h) = h(\chi)$ is defined for each $h \in Ch^2(G)$. The form (g, χ), regarded as a function of χ for fixed g, is a character h in $Ch^2(G)$ which we call $[\tau(G)]g$. Explicitly, this definition of τ reads

$$(\chi, \tau(G)g) = (g, \chi), \quad g \in G, \quad \chi \in Ch(G).$$

The validity of condition ($E1$) for $\tau(G)$ is the basic theorem of character theory. The condition ($E2$) asserts that in the diagram

$$\begin{array}{ccc} G & \xrightarrow{\tau(G)} & Ch^2(G) \\ \downarrow \gamma & & \downarrow Ch^2(\gamma) \\ G' & \xrightarrow{\tau(G')} & Ch^2(G') \end{array}$$

the two paths leading from G to $Ch^2(G')$ have the same effect, or that, for each $g \in G$, both elements $\tau(G')\gamma g$ and $Ch^2(\gamma)\tau(G)g$ are identical as elements of $Ch^2(G')$. This means that, for each $\chi \in Ch(G')$, one should have

$$(\chi', \tau(G')\gamma g) = (\chi', Ch^2(\gamma)\tau(G)g).$$

By the definition of τ, the expression on the left is simply $(\gamma g, \chi')$. By successive application to the expression on the right of the definitions of Ch, τ and Ch, we obtain

$$(\chi', Ch^2(\gamma)\tau(G)g) = (Ch(\gamma)\chi', \tau(G)g) = (g, Ch(\gamma)\chi') = (\gamma g, \chi').$$

The identity of these results shows that we do have a natural equivalence $\tau(G):G \longleftrightarrow Ch^2(G)$.

When G is finite, the isomorphism $G \rightarrow Ch\, G$ cannot be "natural" according to our definitions, for the simple reason that the functor I on the left is covariant, while the functor Ch on the right is contravariant.

As other examples of equivalences between functors, we may cite the usual isomorphisms which give the associative and commutative laws for the direct product, the tensor product and the free product. Various distributive laws, such as

$$(G_1 \times G_2) \circ H \cong (G_1 \circ H) \times (G_2 \circ H),$$

$$Hom(G_1 \times G_2, H) \cong Hom(G_1, H) \times Hom(G_2, H),$$

when established with the obvious isomorphisms, are in fact equivalences between functors.

A less obvious relation between the tensor product and the functor "*Hom*" is[6]

$$Hom(G, Hom(H, K)) \cong Hom(G \circ H, K), \qquad (7)$$

where G and H are discrete abelian groups, K a topological abelian group. This isomorphism is obtained by a correspondence $\tau(G, H, K)$ which specifies for each element $\phi \in Hom(G, Hom(H, K))$ a corresponding homomorphism in $Hom(G \circ H, K)$, defined for any generator $g \circ h$ of $G \circ H$ as

$$[\tau(G, H, K)]\ (\phi)(g \circ h) = [\phi(g)](h) \text{ in } K.$$

One may show that τ does give an isomorphism, bicontinuous in the appropriate topologies. Both sides of (7) may be treated as the group functions of functors which are obtained by composition from "*Hom*" and "$\circ$." The corresponding mapping functions, for given homomorphisms

$$\gamma:G_1 \rightarrow G_2, \quad \eta:H_1 \rightarrow H_2, \quad \kappa:K_1 \rightarrow K_2,$$

are defined by a parallel composition as

$$Hom(\gamma, Hom(\eta, \kappa)), \quad Hom(\gamma \circ \eta, \kappa).$$

Both functors are contravariant in G and H, covariant in K.

The naturality condition for the isomorphism τ reads

$$\tau(G_1, H_1, K_2)\ Hom(\gamma, Hom(\eta, \kappa)) = Hom(\gamma \circ \eta, \kappa)\tau(G_2, H_2, K_1).$$

Both sides, when applied to an element $\phi \in Hom(G_2, Hom(H_2, K_1)$ yield a homomorphism in $Hom(G_1 \circ H_1, K_2)$. If each of these homomorphisms is applied to a typical generator $g_1 \circ h_1$ of the tensor product $G_1 \circ H_1$, straightforward application of the relevant definitions shows that the same element of K_2 is obtained in both cases; namely, $\kappa\{[\phi(\gamma(g_1))](\eta(h_1))\}$.

One may also see directly that this expression represents the only way of constructing an element of K_2 from the elements g_1 and h_1 and the mappings κ, ϕ, γ and η.

The natural isomorphism (7) has some interesting consequences. If K is taken to be the group P of real numbers modulo 1, $Hom(H, K)$ becomes the character group $Ch(H)$, and the formula may be written as

$$Hom(G, Ch\, H) \cong Ch(G \circ H).$$

Applying the functor Ch to both sides and using the natural equivalence of Ch^2 and I, we obtain the equivalence

$$G \circ H \cong Ch\, Hom(G, Ch\, H).$$

Since this is "natural," this could be used as a definition of the tensor product $G \circ H$.

6. *Generalizations.*—With the appropriate definition of a normal subfunctor S of a functor T one can construct a quotient functor T/S, whose group function has as its values quotient groups (i.e., factor groups). With this operation, all the standard constructions on groups may be represented as group functions of suitable functors.

An inspection of the concept of a functor and of a natural equivalence shows that they may be applied not only to groups with their homomorphisms, but also to topological spaces with their continuous mappings, to simplicial complexes with their simplicial transformations, and to Banach spaces with their linear transformations. These and similar applications can all be embodied in a suitable axiomatic theory. The resulting much wider concept of naturality, as an equivalence between functors, will be studied in a subsequent paper.

[1] Pontrjagin, L., "Ueber den algebraischen Inhalt der topologische Dualitätsätze," *Mathematische Ann.*, **105**, 165–205 (1931). Lefschetz, S., "Algebraic Topology," *Am. Math. Soc. Colloquium Pub.*, **27**, 55 (1942).

[2] By a homomorphism we mean a *definite* pair of groups G_1 and G_2 and a (continuous) homomorphic mapping γ_1 of the first onto a subgroup of the second. The product $\gamma_2\gamma_1$ is defined for those pairs $\gamma_1: G_1 \longrightarrow G_2$, $\gamma_2: G_2' \longrightarrow G_3$ with $G_2 = G_2'$.

[3] By an isomorphism we mean a homomorphism of G_1 *onto* G_2 which is one-one and bicontinuous.

[4] Whitney, H., "Tensor Products of Abelian Groups," *Duke Math. Jour.*, **4**, 495–528 (1938).

[5] Here and subsequently the group operation in G and in H is written as addition, whether or not the groups are abelian.

[6] This isomorphism was established by the authors; cf. *Ann. Math.*, **44** (1943).