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Ag is used as the symbol of a gene for agglutinogene A of Burhoe (published herewith) to avoid confusion with *A*, agouti, dominant allele of *a*.

AN EXPRESSION OF HOPF'S INVARIANT AS AN INTEGRAL*

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1. Let $S^2 \subset R^2$ and $S^3 \subset R^4$ be spheres in the sense of Euclidean geometry, S^2 having unit radius, and let $f: S^3 \rightarrow S^2$ be a twice differentiable map. Let x^1, x^2, x^3 be local coördinates for S^3, let λ, μ be local coördinates for S^2 and let $\sigma(\lambda, \mu)$ be the area density on S^2. Let

$$u_{ij} = \frac{1}{4\pi}\,\sigma(\lambda, \mu)\,\frac{\partial(\lambda, \mu)}{\partial(x^i, x^j)} \quad (i, j = 1, 2\ 3), \tag{1.1}$$

where λ, μ, in (1.1) stand for the functions $\lambda(x^1, x^2, x^3)$, $\mu(x^1, x^2, x^3)$, by means of which f is expressed locally. Then u_{ij} are the components of an alternating tensor in S^3. It may be verified that the divergence of this tensor vanishes. That is to say

$$\partial u_{23}/\partial x^1 + \partial u_{31}/\partial x^2 + \partial u_{12}/\partial x^3 = 0.$$

Hence, by de Rham's theorem[1] there is a covariant vector-field (v_1, v_2, v_3), defined over the whole of S^3, such that

$$\partial v_i/\partial x^j - \partial v_j/\partial x^i = u_{ij}. \tag{1.2}$$

The main object of this note is to prove that

$$\frac{1}{4\pi}\int\!\!\int\!\!\int_{S^3} \sigma \begin{vmatrix} v_1 & v_2 & v_3 \\ \dfrac{\partial\lambda}{\partial x^1} & \dfrac{\partial\lambda}{\partial x^2} & \dfrac{\partial\lambda}{\partial x^3} \\ \dfrac{\partial\mu}{\partial x^1} & \dfrac{\partial\mu}{\partial x^2} & \dfrac{\partial\mu}{\partial x^3} \end{vmatrix} dx^1dx^2dx^3 = \gamma, \tag{1.3}$$

where γ is the Hopf invariant of the map f. Notice that this integral can also be expressed in the form

$$\int_{S^3} \omega_1 \square \omega_2 = \gamma, \tag{1.4}$$

where ω_1 and ω_2 are the forms whose coefficients are v_i and u_{ij}, and $\square$ denotes Grassmann multiplication.

Before starting the proof we re-state the theorem in topological terms. Let w^2 be the basic co-cycle on S^2, defined as a function of singular 2-simplexes, and let $u^2 = f_* w^2$, where f_* stands for the map of co-chains, which is dual to f. Since $\delta u^2 = 0$, because $\delta w^2 = 0$, and since the second co-homology group of S^3 is trivial, there is a co-chain, v^1, such that $\delta v^1 = u^2$. Our theorem is equivalent to the statement

$$v^1 \smile u^2 = \gamma c^3,$$

where c^3 is the basic co-cycle on S^3. This form of the result has been obtained independently by N. Steenrod, as an application of a process, which he has developed but has not yet published.

2. We shall prove (1.3) with the help of a theorem concerning 3-dimensional manifolds, which are fibred in the original sense of H. Seifert.[2] Let M^3 be such a manifold, which we assume to be orientable and closed. Let $f: M^3 \to M^2$ be a fibre mapping of M^3 on an orientable, 2-dimensional manifold M^2. Let M^3, M^2, the map f and the fibres be twice differentiable. Let each fibre be oriented so as to have a positive intersection with a transverse 2-cell, which takes its orientation from a given orientation of M^2. We recall Seifert's definition of a *fibre neighborhood*, T, of a given fibre H (p. 150), and the number n, which is the degree of the map $f|E^2$ in the point $f(H)$, where E^2 is a cross section of T. We shall call n the *order* of the fibre H, and H will be described as a *simple* or a *multiple* fibre, according as $n = 1$ or $n > 1$. In either case we may take $T = f^{-1}(E_1^2)$, where $E_1^2 \subset M^2$ is any sufficiently small 2-element, of which $f(H)$ is an inner point.

Let x^1, x^2, x^3 be local coördinates for M^3, let λ, μ be local coördinates for M^2 and let[3] $\sigma(\lambda, \mu)$ be an area density for M^2, which may be given abstractly or defined in terms of a Riemannian metric. Let α be the reciprocal of the total area of M^2, which is compact since M^3 is compact, and let u_{ij} be defined by (1.1), with $1/4\pi$ replaced by α. We assume the existence of a covariant vector-field v_i, defined over the whole of M^3 and satisfying (1.2). As in Section 1 we write, using the summation convention,

$$(a) \quad \omega_2 = \alpha\sigma(\lambda, \mu)(d\lambda\delta\mu - \delta\lambda d\mu) = u_{ij}dx^i\delta x^j, \qquad (2.1)$$
$$(b)\ \omega_1 = v_i dx^i.$$

I say that

$$\int_H \omega_1 = \int_{H'} \omega_1 = \gamma, \text{ say}, \qquad (2.2)$$

where H and H' are any two simple fibres and the integrals are both taken in the positive sense and γ is thus defined. For $d\lambda = d\mu = 0$ along a fibre whence $\int \omega_2 = 0$ over any surface which is generated by fibres. Let $\Delta \subset M^2$ be a smooth, non-singular arc joining $f(H)$ to $f(H')$ and not containing the image of any multiple fibre. Then $\Sigma = f^{-1}(s)$ is generated by fibres and is obviously bounded by $\pm(H - H')$. Therefore

$$\int_H \omega_1 - \int_{H'} \omega_1 = \pm \int_\Sigma \omega_2 = 0,$$

which establishes (2.2).

3. Let D denote the determinant in (1.3). The theorem referred to in Section 2 is that

$$\alpha \iiint_{M^3} \sigma D dx^1 dx^2 dx^3 = \gamma, \qquad (3.1)$$

where γ is defined by (2.2). Let H be a simple fibre and T a fibre neighborhood of H. We may represent T as the topological product $E^2 \times H$ where E^2 is a cross-section of T, which is mapped topologically on $f(E^2)$, with non-degenerate Jacobian. We assume that $f(E^2)$ is contained in the domain of a local coördinate system, (λ, μ), for M^2, and use the same coördinates for $p \in E^2$ as for $f(p)$. Thus T may be referred to coördinates (λ, μ, θ), such that $f(\lambda, \mu, \theta) = (\lambda, \mu)$ and $\theta \in < 0, 2\pi >$ is a periodic coordinate for H. Let $\omega_1 = w_\lambda d\lambda + w_\mu d\mu + w_\theta d\theta$ and let $H(\lambda, \mu) = f^{-1}(\lambda,\mu)$. Then $D = w_\theta(\lambda, \mu, \theta)$ and

$$\gamma = \int_{H(\lambda, \mu)} \omega_1 = \int_0^{2\pi} w_\theta(\lambda, \mu, \theta)d\theta.$$

Hence the integral in (3.1), extended over T, becomes

$$\begin{aligned}
&\alpha \iiint_T \sigma(\lambda, \mu) w_\theta(\lambda, \mu, \theta) d\lambda d\mu d\theta \\
&\quad = \alpha \iint_{E^2} \left\{ \sigma(\lambda, \mu) \int_0^{2\pi} w_\theta(\lambda, \mu, \theta) d\theta \right\} d\lambda d\mu \\
&\quad = \alpha\gamma \iint_{E^2} \sigma(\lambda, \mu) d\lambda d\mu \qquad (3.2) \\
&\quad = \alpha\gamma \times \text{area of } f(E^2).
\end{aligned}$$

There are but a finite number of singular fibres in M^3. Let $q_1, \ldots, q_m \subset M^2$ be their images under f and let K^2 be a triangulation of $M^2 - (q_1 \smile \ldots \smile q_m)$, which will be an infinite complex if there are multiple fibres. Let the simplexes in K^2 be so small that $T = f^{-1}(\tau^2)$ is a fibre neighborhood of $H = f^{-1}(q)$, for each 2-simplex τ^2 in K^2 and any inner point $q \in \tau^2$. We may also suppose that each τ^2 is in the domain of a local coördinate system for M^2. Then (3.1) is established by summing the equalities (3.2), with $T = f^{-1}(\tau^2)$, for each τ^2 in K^2.

4. Now let a simple fibre, H, bound[4] a surface, C, which we assume to be piecewise differentiable. Let C be given locally by $x^i = x^i(\xi, \eta)$, where $x^i(\xi, \eta)$ are differentiable functions of the parameters ξ, η. Then, by (1.2), (1.1) and Stokes' Theorem we have

$$\gamma = \int_H v_i dx^i = \alpha \int\int_C \sigma \left\{ \frac{\partial(\lambda, \mu)}{\partial(x^2, x^3)} \frac{\partial(x^2, x^3)}{\partial(\xi, \eta)} + \frac{\partial(\lambda, \mu)}{\partial(x^3, x^1)} \times \frac{\partial(x^3, x^1)}{\partial(\xi, \eta)} + \frac{\partial(\lambda, \mu)}{\partial(x^1, x^2)} \frac{\partial(x^1, x^2)}{\partial(\xi, \eta)} \right\} d\xi d\eta$$

$$= \alpha \int\int_C \sigma \frac{\partial(\lambda, \mu)}{\partial(\xi, \eta)} d\xi d\eta,$$

which is the degree of the map $f|C$. Hence γ is the Hopf invariant of the map f, and (1.3) is established in case $f: S^3 \to S^2$ is a fibre mapping.

5. Let S^2 be given by $x^2 + y^2 + z^2 = 1$ and S^3 by $|\xi|^2 + |\eta|^2 = 1$, where x, y, z are Cartesian coördinates for R^3 and ξ, η are complex coördinates for R^4. Let $f: S^3 \to S^2$ be given by

$$x + iy = 2\rho\xi^m\bar{\eta}^n, \quad z = \rho(|\xi|^{2m} - |\eta|^{2n}),$$

where $\rho = 1/(|\xi|^{2m} + |\eta|^{2n})$ and m, n are any (positive, zero or negative) integers, which are prime to each other (unless one or both is zero). Then, f is a fibre map, with fibres given by[5] $\xi = \xi_0 e^{in\theta}$, $\eta = \eta_0 e^{im\theta}$, and its Hopf invariant[6] is $\pm mn$, the sign depending on the orientation of S^3. Thus m and n may be chosen, say with $m = \pm\gamma$, $n = 1$, so that f has a given invariant, and is therefore homotopic[7] to a given map $S^3 \to S^2$.

6. We complete the proof of (1.3) by showing that the integral is an invariant of the homotopy class of the map f. This will be included in a more general result, which has no reference to fibre maps. Let M^3 and M^2 be twice differentiable manifolds and $f: M^3 \to M^2$ any twice differentiable map. Let ω_2 be the form in M^3 which is given by (2.1), and assume that ω_2 is an exterior derivative. Let $\omega_1' = \omega_2$, where ω' denotes the exterior derivative of ω. Then we prove that

(*A*) *The integral*

$$I(f) = \int_{M^3} \omega_1 \square \omega_2$$

is independent of the choice of the form ω_1, *satisfying* $\omega_1' = \omega_2$.

(*B*) *Let* $F:M^3 \to M^2$ *be any twice differentiable map, which is homotopic to* f. *Then the form* Ω_2, *given by* (2.1) *in terms of* F, *is also an exterior derivative, and*

(*C*) $I(F) = I(f)$.

The statement (*A*) follows at once from the fact that, if $\omega_1' = \bar{\omega}_1' = \omega_2$, then $\bar{\omega}_1 - \omega_1$ is a closed form. Since ω_2 is an exterior derivative, so is $(\bar{\omega}_1 - \omega_1)\Box\omega_2$, whence

$$\int_{M^3} \bar{\omega}_1\Box\omega_2 - \int_{M^3} \omega_1\Box\omega_2 = \int_{M^3} (\bar{\omega}_1 - \omega_1)\Box\omega_2 = 0.$$

In order to prove (*B*) and (*C*) we imbed M^2 as an analytic surface in R^3. Let N be the open set consisting of points whose distance from M^2 is less than a positive ρ_1, which is so small that no two normals to M^2 intersect in N. Let $\rho > 0$ be so small that, if $\delta(q, q') \leq \rho$, where $q, q' \subset M^2$ and $\delta(q, q')$ is the Euclidean distance from q to q', then the linear segment, which is given in vector notation by $(1 - t)q + tq'$, for $-1 \leq t \leq 2$, lies in N. This being so, let $\phi(q, q', t)$ be the normal projection of $(1 - t)q + tq'$ on M^2.

It is clearly sufficient to prove (*B*) and (*C*) in case $\delta\{f(p), F(p)\} \leq \rho$ for each $p \in M^3$. Let this be so and let M^4 be the open manifold $M^3 \times (-1, 2)$. Then a twice-differentiable map, $g:M^4 \to M^2$, is defined by

$$g(p \times t) = \phi\{f(p), F(p), t\},$$

and $g(p \times 0) = f(p)$, $g(p \times 1) = F(p)$. Let x^1, x^2, x^3 be local coördinates for M^3 and let the map g be given locally by $\lambda = \lambda(x^1, x^2, x^3, t)$, $\mu = \mu(x^1, x^2, x^3, t)$. Let $\bar{\omega}_2$ be the form in M^4, which is given by (2.1). Any cycle in M^4 is homologous to a cycle in $M^3 \times 0$, on which $\bar{\omega}_2$ reduces to the form, $\omega_2(0)$, associated with the map f. Since the latter is an exterior derivative its integral over any cycle in M^3 is zero. Therefore the integral of $\bar{\omega}_2$ over any cycle in M^4 is zero.

Let $M_0^4 = M^3 \times \langle 0, 1 \rangle$, so that M_0^4 is a bounded manifold in M^4. On p. 66, *et seq.*, of de Rham's paper let A be a complex covering M_0^4. Then, with other minor adjustments in his arguments, it follows that there is a form $\bar{\omega}_1$, defined throughout an open set, $U \subset M^4$, which contains M_0^4, such that $\bar{\omega}_1' = \bar{\omega}_2$ in U. Since M_0^4 is compact we may take $U = M^3 \times (-\epsilon, 1 + \epsilon)$ for some $\epsilon > 0$.

Since $\bar{\omega}_1' = \bar{\omega}_2$ in U we have $\omega_1'(t) = \omega_2(t)$ $(-\epsilon < t < 1 + \epsilon)$, where $\omega_1(t)$ and $\omega_2(t)$ are the forms which are obtained from $\bar{\omega}_1$ and $\bar{\omega}_2$ by writing $dt = 0$. This establishes (*B*), since $\omega_2(0) = \omega_2$, $\omega_2(1) = \Omega_2$, regarding $\omega_1(t)$ and $\omega_2(t)$ as forms in M^3, which depend on the parameter t.

In order to prove (*C*) it is sufficient to prove that $\partial\omega_3(t)/\partial t$ is an exterior derivative, where $\omega_3(t) = \omega_1(t)\Box\omega_2(t)$ $(-\epsilon < t < 1 + \epsilon)$. For, treating $\omega_1(t), \omega_2(t), \omega_3(t)$ as forms in M^3, which depend on the parameter t, we have

$$\frac{d}{dt}\int_{M^3} \omega_1(t)\Box\omega_2(t) = \int_{M^3} \partial\omega_3(t)/\partial t = 0$$

if $\partial\omega_3(t)/\partial t$ is an exterior derivative. To prove that it is we write $\bar{\omega}_3 = \bar{\omega}_1\Box\bar{\omega}_2$. Then $\bar{\omega}_3' = \bar{\omega}_1'\Box\bar{\omega}_2 = \bar{\omega}_2\Box\bar{\omega}_2$, and it is clear from (2.1) that $\bar{\omega}_2\Box\bar{\omega}_2 = 0$. Let u_{abc} $(a, b, c = 1, \ldots, 4;\ x^4 = t)$ be the coefficients of $\bar{\omega}_3$. Since $\bar{\omega}_3' = 0$ we have

$$\frac{\partial u_{jk4}}{\partial x^i} - \frac{\partial u_{ik4}}{\partial x^j} + \frac{\partial u_{ij4}}{\partial x^k} - \frac{\partial u_{ijk}}{\partial t} = 0 \ (i, j, k = 1, 2, 3).$$

But u_{ijk} $(i, j, k = 1, 2, 3)$ are the coefficients of $\omega(t)$. Therefore $\partial\omega_3(t)/\partial t = \theta_2'$, where $\theta_2 = u_{ij4}dx^i\delta x^j$, and the proof is complete.

7. We conclude with some additional observations. First notice that (3.2) is valid even if H is a fibre of order $n > 1$. For, provided the total, area of M^2 is finite, (3.1) is true for open as well as for closed manifolds, the proof being the same in each case. Therefore (3.2), with $n > 1$, follows from (3.1), applied to the interior of T.

Also

$$\gamma = n\int_H \omega_1 \tag{7.1}$$

if H is a fibre of order n. For if H' is a simple fibre in a fibre neighborhood of H it is obvious from the definition of a fibre neighborhood that there is a surface which is generated by fibres and bounded by $\pm(H' - nH)$. Therefore (7.1) is proved in the same way as (2.2).

Also if H is a fibre of order n and E^2 is a cross-section of a fibre neighborhood, T, of H, then f maps E^2 on $f(E^2)$ with degree n. Combining these three results, we have

$$\begin{aligned}\int_T \omega_1\Box\omega_2 &= \alpha\gamma \times \text{area of } f(E^2)\\ &= \alpha n\int_H \omega_1 \times \frac{1}{n}\int_{E^2}\sigma(\lambda, \mu)d\lambda d\mu\\ &= \int_H \omega_1 \times \int_{E^2}\omega_2\\ &= \int_H \omega_1 \times \int_m \omega_1,\end{aligned} \tag{7.2}$$

where m is the boundary of E^2.

8. Let us now start with ω_1, that is to say with a covariant vector-field V, which is defined all over M^3. Then curl V is an alternating covariant tensor, which is the same as a contravariant vector density. Therefore "lines of flow" or "lines of force" are defined by curl V and we assume them to be the fibres discussed in Section 2. Let H be a given

fibre and T a fibre neighborhood of H. Let λ, μ, θ be the coördinates for T, which were defined in Section 3, except that we restrict θ to the interval $-1/n < \theta < 1/n$. Then these coördinates are valid even if H is a multiple fibre, of order n. In these coördinates curl V has components of the form 0, 0, $u(\lambda, \mu, \theta)$, and

$$\frac{\partial u}{\partial \theta} = div(\text{curl } V) = 0.$$

From the transformation law of the components, u_{ij}, of curl V, namely

$$V_{ab} = u_{ij}\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b},$$

it follows that $\sigma(\lambda, \mu) = \mu(\lambda, \mu, \theta)$ is unaltered by coördinate transformations of the form $\theta' = \theta'(\lambda, \mu, \theta)$, $\lambda' = \lambda$, $\mu' = \mu$ and transforms in the same way as a scalar density in M^2 under transformations of the form $\lambda' = \lambda'(\lambda, \mu)$, $\mu' = \mu'(\lambda, \mu)$, $\theta' = \theta$. Hence curl V, whose components in the coördinates (λ, μ, θ) are 0, 0, $\sigma(\lambda, \mu)$, is related to the fibre mapping $(\lambda, \mu, \theta) \rightarrow (\lambda, \mu)$ by the equations (1.1), with the factor $1/4\pi$ discarded. In the notation of vector calculus (7.2) becomes

$$\int\int_T\int V\cdot\text{curl } V\, d\tau = \int_H V\cdot ds \times \int_M V\cdot ds.$$

* Hopf, H., *Math. Annalen*, **104**, 637–665 (1931).

[1] de Rham, G., *Journal de Math p. et a.*, **96**, 185 (1931).

[2] *Acta Mathematica*, **60**, 147–238 (1933).

[3] $\sigma > 0$ in positively oriented coördinate systems.

[4] In any case the fact that ω_2 is an exterior derivative implies that H bounds with division.

[5] Cf. Seifert, *loc. cit.*, p. 159.

[6] Let H_1, H_2 be fibres of orders m, n. Then $H \sim mH_1$ in $S^3 - H_2$ and $H' \sim nH_2$ in $S^3 - H$, where H, H' are simple fibres near H_1, H_2, respectively, and it follows that $\gamma = \pm mn\, L(H_1, H_2)$, where L denotes the linking coefficient. In this case H_1, given by $\eta = 0$, is of order n and H_2, given by $\xi = 0$, is of order m, and $L(H_1, H_2) = \pm 1$ (cf. Hopf, *loc. cit.*, p. 655).

[7] Hurewicz, W., *Proc. Akad. Amsterdam*, **38**, 119 (1935). See also H. Freudenthal, *Compositio Math.*, **5**, 299–314 (1937).