$$p_1^h + \ldots + p_s^h = N_h, \qquad 1 \leq h \leq k.$$

He has now succeeded in extending its range of validity from

$$s \geq 4.14k(k+1)(k+2) \log k \qquad (s \geq 11)$$

to

$$s \geq k^2(4 \log k + 2 \sqrt{\log k^2} + \log \log k + 9) \qquad (s \geq 11). \quad (4)$$

Concerning Waring's problem Vinogradow[2] shows that the Hardy-Littlewood's asymptotic formula for the number of solutions of

$$x_1^k + \ldots + x_s^k = N, \qquad x_v \geq 0$$

holds for $s \geq 20\, k^2 \log k$. His method is hardly able to go beyond the order of magnitude $k^2 \log k$, but the numerical factor may still be improved. Indeed the author has replaced Vinogradow's inequality by a sharper one

$$s \geq 4k^2(\log k + {}^1/_2 \sqrt{\log k^2} + {}^1/_4 \log \log k + 1). \qquad (5)$$

Vinogradow's asymptotic formula for Waring-Goldbach's problem also holds within the range (5).

[1] Hua, *Additive Prime Number Theory.* This booklet was accepted for publication by the Acad. of USSR in 1940. The appearance was delayed by World War II.

[2] *Comptes Rendus of USSR,* 1942, No. 7.

ON THE MULTIPLICITY OF STEADY GAS FLOWS HAVING THE SAME STREAMLINE PATTERN

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The streamline pattern for any one steady flow of an ideal gas is the streamline pattern for a great many different modes of flow of such gas. Such trivial variants of the given flow as those obtainable by changing all pressures and densities in the same proportion are but special cases of wide classes of substitution flows having a common streamline pattern with the original flow. One such class of substitution flows of particular interest will be discussed here.

We shall consider steady flows of an ideal gas in which changes in entropy occur only in infinitely thin shock-front regions. This means that in regions between shocks the flow is *isentropic,* but not necessarily *homentropic.* That is, the entropy in a shock-free region is *constant along any given streamline* but is not necessarily *constant throughout the flow.* Only if the flow is homentropic and not merely isentropic will the density be a unique function of the pressure throughout the shock-free region. The existence of such a function is the necessary and sufficient condition that the acceleration field, $-\frac{1}{\rho}$ grad p, possess a potential (be irrotational). Since irrotationality of the acceleration field is necessary (but not sufficient) for irrotationality of the flow field, only such isentropic flows as are also homentropic may possibly be irrotational. Hence in considering flows which are isentropic but not necessarily homentropic we include rotational as well as irrotational flows.

We shall first limit our attention to regions between shock fronts, and consider that class of substitution flows for a given steady streamline pattern for which the pressures remain unchanged. That is for which

$$p' = p \tag{1}$$

Now the dynamic equilibrium of the force components normal to the streamline requires that the normal component of the pressure gradient balance the centrifugal reaction of the flow, or

$$\frac{\partial p'}{\partial n} = \frac{\rho' v'^2}{R} \tag{2a}$$

$$\frac{\partial p}{\partial n} = \frac{\rho v^2}{R} \tag{2b}$$

where R represents the local radius of curvature of the streamline, v the flow velocity and ρ the mass density. Since $p' = p$, the condition

$$\rho' v'^2 = \rho v^2 \tag{3}$$

follows.

Now, for gases of constant specific heats, C_p and C_v,

$$p \propto \rho^{\gamma} \exp.\ (s/C_v) \quad \begin{bmatrix} \gamma \equiv C_p/C_v \\ s = \text{entropy (specific)} \end{bmatrix} \tag{4}$$

For such gases the "velocity of sound" is given by

$$c \equiv \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} = \sqrt{\frac{\gamma p}{\rho}} \tag{5}$$

and the Mach number, defined by

$$M \equiv \frac{v}{c} \tag{6}$$

becomes

$$M = \sqrt{\frac{\rho v^2}{\gamma p}} \tag{7}$$

Since $p' = p$ and $\rho' v'^2 = \rho v^2$, the additional condition

$$M' = M \tag{8}$$

is obtained.

A further condition restricting the possible substitution flows is the necessity of dynamic equilibrium of the force components *along* each streamline. That is to say, Bernoulli's equation must hold along each individual streamline:

$$\frac{\gamma}{\gamma - 1} p' + \frac{1}{2} \rho' v'^2 = H' \rho' \tag{9}$$

where H' is a quantity (the "total enthalpy") which is constant along any particular streamline. The original flow satisfies a similar equation

$$\frac{\gamma}{\gamma - 1} p + \frac{1}{2} \rho v^2 = H \rho \tag{10}$$

Hence, from (1) and (3) it follows that

$$H' \rho' = H \rho, \text{ or } \rho' = \frac{H}{H'} \rho = m \rho \tag{11}$$

where m is a parameter constant along any given streamline but variable from streamline to streamline. Since there are no further conditions to be imposed on the substitution flows, we still have at our disposal the value of the arbitrary parameter, m, for each streamline.

The streamline pattern and all pressures and Mach numbers are left unchanged if along each streamline the values of density and of velocity are multiplied respectively by m and $1/\sqrt{m}$, where m may change from streamline to streamline.

This class of substitution flows has thus far in our discussion been limited to shock-free regions, but this restriction can now be removed. Consider the flow quantities immediately in front of and immediately behind a shock front to be designated by the subscripts 1 and 2. Then the necessary conditions of conservation of mass, momentum and energy for a streamline passing through a steady shock front can be expressed in the form of equations (12–14):

$$\rho_1 v_1 = \rho_2 v_2 \tag{12}$$

$$\rho_1 v_1^2 \sin^2 \alpha_1 + p_1 = \rho_2 v_2^2 \sin^2 \alpha_2 + p_2 \tag{13}$$

$$\rho_1 v_1^2 + K p_1 = \rho_2 v_2^2 + K p_2 \tag{14}$$

where α represents the angle between the streamline and the shock front, and K is a constant depending on the kind of gas involved. It is clear that any substitution flow consistent with the requirements of equations (1), (3) and (11) will satisfy these shock conditions as well as does the original flow. Therefore this group of flow substitutions produces no change in streamline pattern even if shock fronts are present.

In the most general flows of an ideal gas the total enthalpy, H, $(\equiv i + \frac{v^2}{2}$, where i is the specific enthalpy) is constant along each particular streamline, whether or not it intersects shock fronts, but it varies from streamline to streamline. The entropy is constant along each particular streamline in regions between shock fronts, but varies from streamline to streamline and increases discontinuously whenever the streamline crosses a shock front.

In supersonic regimes ($v > c$) flows of this most general character can be computed by the use of the general method of characteristics combined with the shock-front integral relations (equations (12–14)) when suitable boundary conditions are imposed. However, the computation is considerably simpler if either the total enthalpy, H, or the entropy, s, is universally constant for a region under consideration. For if H is constant throughout the flow, a universal relation exists between v and c, as from (5) and (10):

$$\frac{c^2}{r - 1} + \frac{v^2}{2} = H \tag{15}$$

On the other hand, if s is constant throughout a region of the flow, there exists a universal relation between p and ρ. (See equation (4).)

By the use of the class of substitution flows discussed above, a flow of non-constant H and s can be readily replaced by a substitution flow having either constant H or (in a shock-free region) constant s if the values of m are suitably chosen on any surface intersecting each streamline a single time. The simpler substitute flow would then yield directly the pressures, Mach number and flow pattern of the original flow and, after a simple inversion of the conversion from original to substitute flow, the densities and velocities as well.

Similarly, if it is desirable to do so, an original flow problem involving non-constant H and constant s could be replaced by one having constant H and non-constant s, and vice versa. It is not possible, however, to substitute a flow having *both* H and s constant for one in which either is non-constant. In other words, this class of substitution flows does not in general allow the replacing of a rotational flow by an irrotational one.

The relations discussed may be arrived at more briefly, but less directly, from dimensional considerations. Along each stream tube, there exists for a perfect gas only one independent dimensional reference quantity, as for instance the "reservoir" pressure. All variables can be expressed in terms of that reference pressure in conjunction with one non-dimensional quantity as for instance the local Mach number. Thus the local pressure is equal to the reference pressure multiplied by a function of the Mach number, and of non-dimensional quantities representing the geometry involved. Hence the lateral pressure gradient is determined and invariant with respect to transformations not involving changes of the geometry or changes of the reference pressure and thus changes of any pressure. It follows that it must be possible to write all pertinent equations in terms of the local pressure, the local Mach number and the space coördinates, thus eliminating one dependent variable.

Among possible flows calling for application of these relations, there are jets from different "reservoirs" flowing together. Such is the flow when a propulsion jet issues into the rapidly moving air (relative to a missile or airplane) of the atmosphere. It may also be instructive to idealize boundary layer wakes by considering them as jets of a perfect gas with a lowered total enthalpy.

TURBULENCE AS AN ENVIRONMENTAL DETERMINANT OF RELATIVE GROWTH IN DAPHNIA

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Temporal variation or cyclomorphosis is pronounced in many limnetic races of *Daphnia*. The relative length of the head is the most variable aspect of such races of the north temperate zone. The winter and early spring generations bear short round heads resembling those characteristic of pond *Daphnia*. Individuals of midsummer generations have elongate heads, called helmets, which are often nearly as long as the rest of the body. During the period of these striking phenotypic changes reproduction is entirely a sexual, which means that the genetic constitutions of all generations are almost identical. These phenotypic differences must therefore be determined by seasonally variable cytoplasmic or environmental factors, in all probability the latter.

The aim of the present investigation is precise determination of all of the environmental factors controlling cyclomorphosis. The efforts of