hole resonator is practically unresponsive to antinodes and to wave trains except within ranges from the pipe which are small as compared with the dimensions of the room as a whole.

* Advance note from a Report to the Carnegie Institution of Washington, D. C.

ON STEERING AN AUTOMOBILE AROUND A CORNER

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The question of so arranging the steering gear of an automobile that the front wheels shall skid as little as possible owing to axes not intersecting has been dealt with by A. L. Candy (*American Mathematical Monthly*, June 1920) and leads to a geometrical discussion of a high degree of complexity. The present somewhat trifling paper has no such ambitious purpose, but was suggested by a question asked on the witness-stand as to how much an automobile must turn out in order to pass another. It was necessary to make some assumption, and this led to the question of steering in general, and the principles that should govern it. We shall consider only the question of passing from a straight course to another at right angles therewith at constant velocity, and inquire how we must steer in order not to skid.

The layman ignorant of geometry and dynamics will suppose that we describe a circle, but this is impossible, not only because it is impossible to put the helm over suddenly so as to change from zero to a finite curvature, but also because the sudden change in the centrifugal force from zero to a finite quantity would produce a wholly intolerable shock. Inasmuch as the question of transition railroad curves has received so much attention, it is thought that the treatment of this question for an automobile may have some interest to engineers. We shall for the present neglect the fact that the rear wheels do not track with the forward ones, taking that up later as a correction. Thus we shall examine the curve described by a machine of zero wheel-base, like a wheelbarrow, measuring wheel or monocycle. The condition is evidently that the curvature shall begin at zero, increase to a maximum until the tangent has turned through an angle of 45 degrees, and then return by a symmetical process.

The centrifugal force is equal to mv^2/ρ, where m is the mass, ρ the radius of curvature, v the velocity, and if there is to be no skidding this must be less than $mg\mu$ where μ is the coefficient of friction, consequently we must have at all times

$$v^2/\rho < \mu g.$$

This does not, however, determine the shape of the curve, to obtain which some assumption must be made. If we assume that the mean value of the centrifugal force or, what is the same thing, of the curvature (since v is constant) is to be a minimum we shall get no result, except a straight line, which would be a discontinuous solution, and inadmissible. We shall therefore assume that the mean square of the curvature is a minimum, that is

$$\delta\int\kappa^2 ds = \delta\int(d\theta/ds)^2 ds = 0,$$

subject to the condition

$$\int dx = \int\cos\theta ds = \text{constant, the length of chord.}$$

But this gives

$$d^2\theta/ds^2 + \lambda\sin\theta = 0,$$

where λ is some constant. But this is the equation of the elastica or curve assumed by a bent flexible rod, ruler, or spline. In fact, in order to obtain the equation of the elastica, we have to make the potential energy of bending a minimum, and as this, at each point, is proportional to the square of the curvature, the problem is mathematically the same.

If we take the chord for the X-axis, we shall find the equation satisfied if we take the curvature proportional to y, and introducing the elliptic functions of Jacobi, with c a constant of homogeneity,

$$dx/ds = \cos\theta, \quad dy/ds = \sin\theta,$$

$$\kappa = d\theta/ds = -y/c^2, d\kappa/ds = d^2\theta/ds^2 = -1/c^2 dy/ds = -1/c^2\sin\theta,$$

$$y = a\mathrm{cn}s/c = a\mathrm{cn}u, \ dy/ds = -a/c\,\mathrm{sn}u\mathrm{dn}u = -2k\mathrm{sn}u\mathrm{dn}u,$$

where

$$s/c = u, \quad a/c = 2k,$$

from which

$$dx/ds = \sqrt{1 - 4k^2\mathrm{sn}^2u\mathrm{dn}^2u} = 2\mathrm{dn}^2u - 1,$$

$$x = 2c\int_0^{s/c}\mathrm{dn}^2u du - c\int_0^{s/c}du = c[2E(u)-u] = c[2E(k,\varphi)-F(k,\varphi)],$$

where F and E are Legendre's elliptic integrals of the first and second kind, respectively. The value of k is determined by the fact that at the point of inflexion, where $\theta = -45°$, we have $u = \overline{K}$, the complete elliptic integral, and since $\mathrm{sn}\overline{K} = 1$ and $\mathrm{dn}\overline{K} = \sqrt{1-k_2}$ we have $2k\sqrt{1-k_2} = \sin 45°$. Putting $k = \sin\alpha$ we see that $\alpha = 22°.5$. With this value for α taking values of φ at intervals of 10° from 0° to 90°, values of x and y are calculated by means of Legendre's table and the curve drawn, which is shown on the inside of figure 1. It fits very accurately a spline made of celluloid, held fast at the vertex, and merely by contact at the point of inflexion. We find the ratio of the maximum ordinate to the whole chord or base is .2757 and that the radius of maximum curvature is the chord divided by 2.1429.

The question now arises, how nearly does the curve actually described correspond with the above description? We may take as the simplest assumption that when the driver gets ready to turn he puts the steering wheel over with a constant velocity until he is half-way around, and then

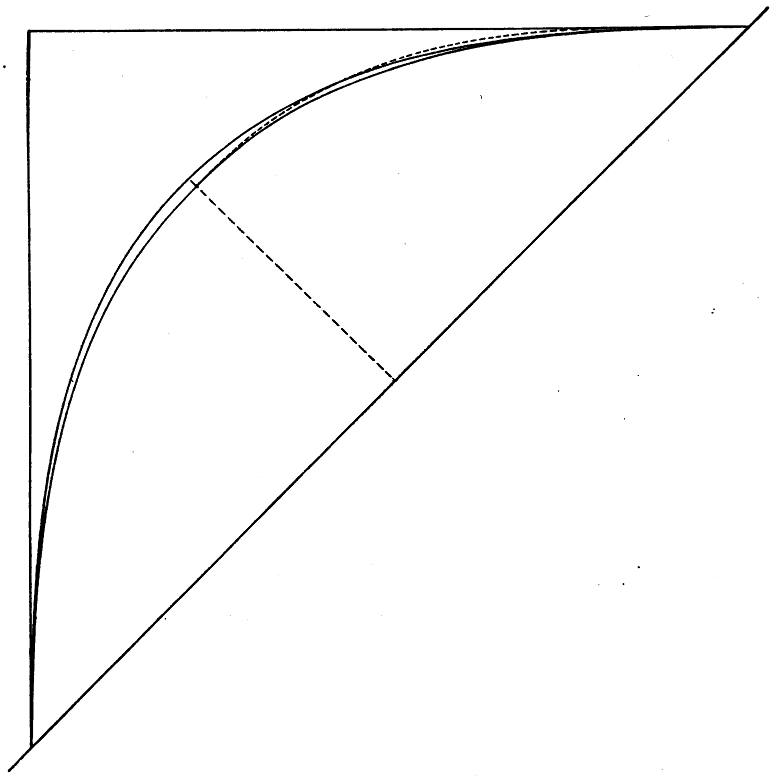

FIG. 1

turns it back with the same velocity. This is to be sure not exactly true, but as the wheel is dominated rather by friction than by inertia it will do for an approximation, and is besides about what observation shows. If the effect of this is to make the curvature of the path proportional to the time or to the distance travelled since turning (which will be shown below to be nearly the case), we have

$$\kappa = d\theta/ds = as$$

as the differential equation of the curve, where a is a constant. But a curve whose curvature is proportional to the length of the arc is the well-known spiral of Cornu, used in the theory of diffraction in optics. The curve may be constructed graphically from the differential equation, but more accurately by the use of tables of Fresnel's integrals. If we put

$$x = \int_0^v \cos\pi/_2 v^2 dv, \quad y = \int_0^v \sin\pi/_2 v^2 dv,$$

we have

$$dx = \cos\pi/_2 v^2 dv, \quad dy = \sin\pi/_2 v^2 dv, \quad ds^2 = dx^2 + dy^2 = dv^2,$$
$$dy/dx = \tan\theta = \tan\pi/_2 v^2, \quad \theta = \pi/_2 v^2 = \pi/_2 s^2 = as^2/2.$$

The integrals above have been tabulated by Gilbert, and are reproduced in Verdet's *Traité d'Optique Physique.* From them the spiral has been plotted, and is shown in the outer curve in figure 1. It will be seen that it differs very little from the elastica. As a matter of fact the ratio of the maximum ordinate at right angles to the chord is .2861 compared with .2757 for the elastica. For the circle this ratio would be only .2071. The reason for the near agreement of the two curves, in one of which the curvature is proportional to the arc, and in the other to the ordinate, is of course the fact that for a considerable distance the curve is nearly straight.

The simplest assumption that could have been made for a rough calculation would have been that of a parabola, $y = ax^2$, but the ordinary parabola of order two would not do since its curvature does not vanish at the origin, and diminishes as we leave the origin. A cubical parabola would be the most obvious, but we shall do better by putting $y = ax^n$ and determining the exponent n so that the curve shall pass through the vertex of the elastica and be tangent to it. We find $n = 3.458$. The curve is drawn dotted on figure 1, and lies between the other two curves. The ratio of the extreme distance from the chord to the length of the chord in the case $n = 3$ would have been .25. We consequently see that any of the four curves, the elastica, spiral of Cornu, parabola of order 3.458, or cubical parabola, will give a very good approximation to the actual path.

We have now to consider the effect of the finite length of the wheel-base, and this leads us to consider the question of steering in general. We shall first suppose for generality that both the front and rear wheels can be steered, as in a hook and ladder truck, but we shall still neglect the effect of the finite width of the track. The case contemplated is exactly realized in the bicycle. In figure 2, let l be the length of the wheel-base, let s, θ, α, ρ be the distance run, the angle made by the tangent with a fixed direction, the angle through which the wheel has been turned (that is, the angle made by the tangent to the path with the chassis, and the radius of curvature of the path, respectively, for the front wheel, and let the same letters with accents denote the corresponding quantities for the rear wheel. (When we come to consider the dynamics of the matter we shall need corresponding quantities for the center of mass, and shall denote them by two accents.) Then since the whole machine turns about the instantaneous center which is the intersection of the axes of the two wheels, we see on reference to figure 2, ψ being the angle made by the chassis with the fixed direction,

$$\rho d\theta = ds, \ \rho' d\theta' = ds', \ \theta + \alpha = \theta' + \alpha' = \psi.$$

We also have

$$\cos\alpha/\rho = \cos\alpha'/\rho' = \sin(\alpha - \alpha')/l,$$

which equations give ρ and ρ' as functions of α, α'. If then α and α' are given as functions, say of s and s', this makes seven equations between

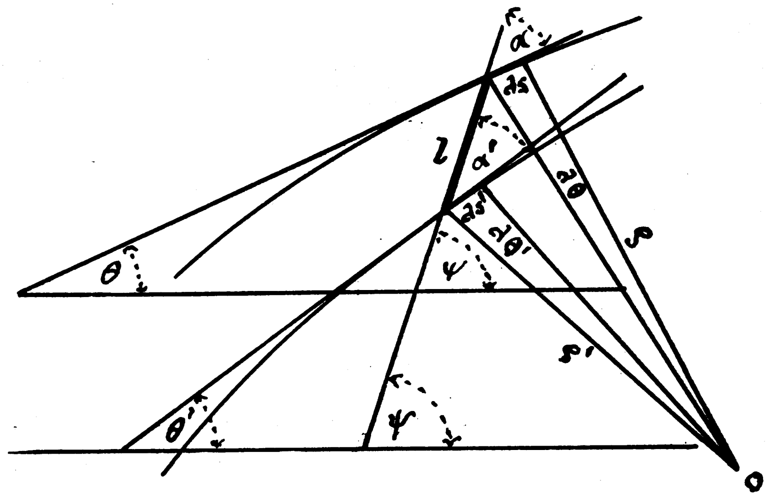

FIG. 2

the variables α, α', θ, θ', s, s', ρ, ρ', so that we may find the differential equations of the path of either the front or rear wheel. In particular, if α and α' are constant and equal, we shall have merely a movement of translation, while if $\alpha' = 0$, α = constant we shall describe two concentric circles. If $\alpha' = 0$, $\alpha = 90°$, the machine will describe a circle about the rear wheel. If, with these values of α, α' the length l be made variable, and a given function of s, the front wheel will describe a curve and the rear wheel its evolute. Thus a curve may be mechanically constructed from its intrinsic equation. This principle is used in certain integrating machines, and the author has invented an integrator for drawing trajectories involving this principle, which will be described later if he succeeds in having it built.

In the case of the automobile we have

$$\alpha' = 0, \qquad \rho = l\mathrm{ctn}\alpha, \qquad ds = l\mathrm{ctn}\alpha d\theta,$$

and if as before we assume the mode of steering to be $\alpha = as$ we may integrate and obtain the intrinsic equation of the curve,

$$ld\theta = \tan(as)ds, \qquad la\theta = \log \sec as.$$

Developing the tangent in series we may see how much this differs from the Cornu spiral. The curve has been drawn graphically from the differential equation, but since on a sixty-foot chord it differs but slightly from the spiral (which may be judged from the fact that for $\theta = 45°$, $\alpha = 23°.2'$; $\rho = 22.35$ compared with $\rho = 22.28$ for the spiral, and that the rear wheel's track is less than two feet distant at the maximum) it is not here reproduced.

It may be said that the above is not applicable to the way in which an automobile actually skids, when it is well known that the two wheels do not break the frictional constraint together, but the rear wheel generally skids first. This may be true, but I think this will not invalidate anything that I have said. One would suppose that this question would be treated in any elementary book on statics, but as I have failed to find it in any treatise, I insert a treatment here. It will now be necessary to consider a third path, namely, that of the center of gravity of the machine, which we shall still assume to be on the line joining the wheels, at distances d and d' therefrom, figure 3. Then we have $d'\text{ctn}\alpha'' = (d + d')\text{ctn}\alpha$. The three

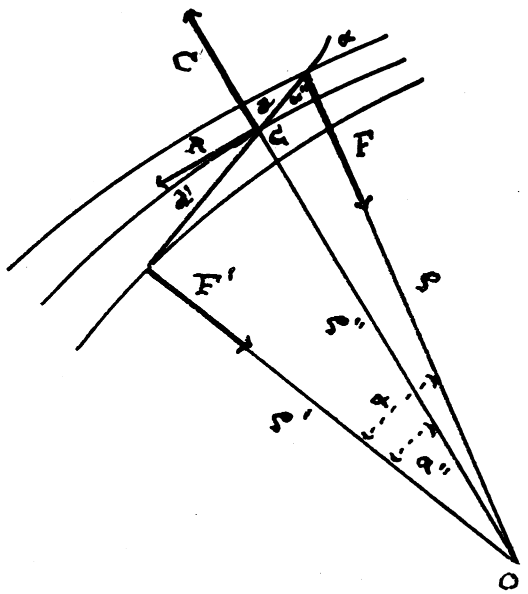

FIG. 3

paths have the same center of curvature. The constraints due to the friction can be only at right angles to the paths of the wheels, and we will call them F, F'. The kinetic reaction of the machine is described by the centrifugal force C, the tangential reaction R, and the moment S. It may be asked how the wheels can cause any reaction in the path, but it will be seen that there must be such when we consider that energy of translation being converted into energy of rotation and conversely will give rise to acceleration in the path, negative or positive, respectively. In fact we see at once that S must vanish with R, for three concurrent forces do not give rise to any moment. Resolving parallel and perpendicular to the chassis,

$$F' + F\cos\alpha = C\cos\alpha'' + R\sin\alpha'',$$
$$F\sin\alpha = C\sin\alpha'' - R\cos\alpha'',$$

and taking moments about the center of mass,

$$F'd' - Fd\cos\alpha = S,$$

or making use of the relation between α and α'',

$$F' - F\sin(\alpha-\alpha'')/\sin\alpha'' = S/d'.$$

Solving these equations we obtain

$$R = -S/d'\sin\alpha'',$$
$$F = \sin\alpha''/\sin\alpha\ (C + S\cos\alpha''/d'),$$
$$F' = \sin(\alpha - \alpha'')/\sin\alpha\ C + S/d'\ (\sin(\alpha-\alpha'')\cos\alpha''/\sin\alpha - 1).$$

Now since $C = mv^2/\rho''$ and $S = mk^2d^2\psi/dt^2 = mk^2(d^2\theta/dt^2 + d^2\alpha''/dt^2)$ and in the method of steering assumed this is $mk^2d^2\theta''/dt^2 = mk^2v^2d^2\theta/ds^2$, if v is constant, both terms are proportional to v^2. Consequently the conclusions above stated are justified. (It is to be remarked that if v is constant, the kinetic reaction above is to be replaced by the force of traction necessary to maintain the velocity constant.) The term in C is proportional to the curvature, and the term in S to the rate of increase of curvature, the latter is greatest at the beginning and end of the curve, the former at the middle.

Finally I have made some experiments to determine the coefficient of friction μ of rubber on slimy stone, with the conclusion that it is about one-fourth. Using the criterion $v^2 < \mu g\rho$ we arrive at the conclusion that the safe speed for curves as above is somewhat less than ten miles per hour, which may account for the common traffic rule of "eight miles around the corner."

THE PROBABLE ERROR OF THE VITAL INDEX OF A POPULATION[1]

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In comparing the birth-death ratios of different communities we need their probable errors to determine the significance of observed differences.

Let N = the population of a community,
B = number of births in a year,
D = number of deaths in a year.

Then the birth-death ratio or vital index is defined as $\dfrac{100\,B}{D}$.