C_n without cutting either C_m or C_n or passing through the position of the line infinity. This is a region which can be easily marked out on the complex plane and will have inside of it neither of the contours C_m and C_n.

Thus our theorem asserts that if the zeros of P_n are inside of one branch of an hyperbola and the zeros of P_m are inside the other branch, all the zeros of Φ' are *inside* of the hyperbola, or again, if all the zeros of P_n are real and lie in the interval (1) $x > a$, while all the zeros of P_m are real and lie in the interval (2) $x < b \leq a$, then $\Phi'(z)$ has no complex zeros and all of its zeros lie in the intervals (1) and (2).

[1] *J. Ec. Polytech., Paris,* 28.

[2] All these proofs save one by Hayashi (*Annals of Mathematics,* March, 1914) are based on dynamical considerations. Fejér, Ueber die Wurzel vom kleinstein absoluten Betrage, etc., *Leipzig, Math. Ann.,* 65, 417, attributes the theorem to Gauss and gives a bibliography for it.

[3] If this is not at once intuitionally evident it can be shown by resolving the vectors in question into components parallel to the arms of the angle above mentioned.

INTERPRETATION OF THE SIMPLEST INTEGRAL INVARIANT OF PROJECTIVE GEOMETRY

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If $y = f(x)$ is the cartesian equation of a plane curve, the integral

$$s = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx, \tag{1}$$

which represents the length of the arc of this curve between the points $P_0(x_0, y_0)$ and $P_1(x_1, y_1)$, obviously remains unchanged when the curve is subjected to a plane motion. Therefore we may speak of s as an integral invariant of the group of motions, or as a *metric integral invariant.*

In the present paper we shall show how to find integrals connected with a given plane curve, whose values are not changed when the points of the plane are subjected to an arbitrary projective transformation. We shall speak of these integrals as *projective integral invariants.*

Let y_1, y_2, y_3, be the homogeneous coördinates of a point P_y, and let y_1, y_2, y_3 be given as linearly independent analytic functions of a parameter x. As x changes P_y will describe a non-rectilinear analytic curve C_y. There exists a uniquely determined linear homogeneous differential equation of the third order

$$y''' + 3p_1y'' + 3p_2y' + p_3y = 0 \tag{2}$$

of which y_1, y_2, y_3 form a fundamental system of solutions, and of which C_y shall be said to be an integral curve. All other integral curves of (2), associated with different fundamental systems of solutions, are projective transforms of C_y.

Since the coordinates y_1, y_2, y_3 are homogeneous, only those combinations of the coefficients p_1, p_2, p_3 can be of interest for the geometry of the curve C_y which depend only upon the ratios $y_1 : y_2 : y_3$. These combinations, the so-called *seminvariants* of (2), are all expressible as functions of

$$P_2 = p_2 - p_1^2 - p_1', \quad P_3 = p_3 - 3p_1p_2 + 2p_1^3 - p_1'' \qquad (3)$$

and of their derivatives. The seminvariants of (2) are not altered if (2) is transformed by putting $\bar{y} = \lambda(x)y$ where $\lambda(x)$ is an arbitrary function of x.[1]

Although the seminvariants depend only upon the ratios $y_1 : y_2 : y_3$, they are still not adequate to represent the purely geometric properties of the curve C_y. The values of P_2 and P_3 depend also upon the special parametric representation which has been chosen for C_y. We may change this parametric representation in the most general way by putting $\bar{x} = \xi(x)$, where $\xi(x)$ is an arbitrary function of x. Those combinations of the seminvariants, called *absolute projective differential invariants*, which are left unaltered by all possible transformations of this sort, express intrinsic properties of the curve C_y. Moreover these properties are projective properties, since any projective transform of C_y may be regarded as an integral curve of (2).

Every absolute projective differential invariant can be expressed as a quotient of two relative invariants. The simplest of these relative invariants is[2]

$$\theta_3 = P_3 - \frac{3}{2}P_2'. \qquad (3)$$

The property of θ_3 which justifies us in speaking of it as a relative invariant, is the following. Let us transform (2) by putting

$$\bar{x} = \xi(x), \ \bar{y} = \lambda(x)y, \qquad (4)$$

where $\xi(x)$ and $\lambda(x)$ are arbitrary functions of x. From the coefficients of the resulting differential equation between $\bar{x}$ and $\bar{y}$ let us form the quantity $\theta_3(x)$ according to the same rule which was used in forming θ_3 from the coefficients of (2). We shall find[3]

$$\bar{\theta}_3(\bar{x}) = \frac{\theta_3(x)}{(\xi')^3}. \qquad (4)$$

This equation may be written

$$\bar{\theta}_3(\bar{x})\ (d\bar{x})^3 = \theta_3(x)dx^3.$$

Consequently the integral

$$p = \int_a^b \sqrt[3]{\theta_3\ (x)}\, dx \tag{5}$$

will not change its value under the transformations (4). Thus we see that the integral p is intrinsically connected with some geometric property of that arc of the curve C_y which corresponds to the interval $a \leqq x \leqq b$. It is also clear that this integral and its geometric significance will remain unaltered by any projective transformation of the plane, since it is expressed entirely in terms of the coefficients p_1, p_2, p_3 of (2) which are invariants of the projective group.

Therefore *the integral p, defined by (5), is a projective integral invariant.* If I is any absolute differential invariant of the curve C_y, the integral $\int I dp$ is again an integral invariant, and all integral invariants are expressible in terms of those obtained in this way.

We wish to explain the geometrical significance of the invariant integral p. For this purpose we need one further preliminary notion, namely that of the eight-pointic nodal cubic of a given point of a given curve.

A cubic curve is in general determined by nine of its points. If eight points only are given, there exist infinitely many cubics, forming a pencil, which pass through these points. In particular there exists a pencil of cubics, such that each cubic of the pencil has eight-pointic or seventh-order contact with the given curve C_y at a specified non-singular point P_y. One and only one of the cubics of this pencil has P_y, the point of contact, as double point. We call this cubic the *eight-pointic nodal cubic* of the point P_y, or the *penosculating nodal cubic* of P_y.[4]

The significance of the integral p is contained in the following theorem which we shall state without proof, but all of the terms of which have now been explained.

Consider an arc of an analytic curve corresponding to the interval $a \leqq x \leqq b$ of the independent variable. Divide this interval into n parts by means of the values $x_0 = a, x_1, x_2, \ldots x_{n-1}, x_n = b$, such that $\lim \delta x_k = \lim (x_{k+1} - x_k) = 0$ as n grows beyond bound. Let $A, P_1, P_2, \ldots P_{n-1}, B$ be the points on the curve which correspond to these $n + 1$ values of x. Let t_k be the tangent and C_k the eight-pointic nodal cubic of P_k. The three points of inflection of the cubic C_k are on a line i_k which interesects t_k in a point I_k. Denote by t_k one of the three inflectional tangents of C_k and let T_k be its intersection with

t_k. The line P_kP_{k+1} will intersect i_k and t_k in two points, I_k' and T_k', and the cross-ratio $(I'_k, T_k', P_k, P_{k+1})$ turns out to be equal to

$$1 - \frac{3}{\sqrt[3]{10}} \sqrt[3]{\theta_3 (x_k)}\, \delta x_k \tag{6}$$

except for terms of higher than the first order in δx_k.

By a perspective correspondence the three points $I'_{n-1}, T'_{n-1}, P_{n-1}$ of $P_{n-1} B$ may be projected into the points $I'_{n-2}\, T'_{n-2}, P_{n-1}$ of $P_{n-2} P_{n-1}$. Let B_{n-1} be the point of $P_{n-2} P_{n-1}$ which, in this perspective, corresponds to B. Then project similarly $I'_{n-2}, T'_{n-2}, P_{n-2}, B_{n-1}$ into the four points $I'_{n-3}, T'_{n-3}, P_{n-2}, B_{n-2}$ of $P_{n-3} P_{n-2}$, and continue in this way. We shall finally obtain upon the line AP_1 a point B_1 determined from B by this sequence of perspectives. As n grows beyond bound, B_1 will approach a limiting position Q on the initial tangent t_0 of the arc AB. The cross-ratio

$$k = (I_0, T_0, A, Q) \tag{7}$$

will be the limit which the product

$$\prod_{k=1}^{n} \left(1 - \frac{3}{\sqrt[3]{10}} \sqrt[3]{\theta_3 (x_k)}\, \delta x_k\right) \tag{8}$$

approaches when n grows beyond bound. Consequently we find

$$\log k = \frac{3}{\sqrt[3]{10}} \int_a^b \sqrt[3]{\theta_3 (x)}\, dx. \tag{9}$$

This equation contains the desired interpretation of the integral p.

From a theoretical point of view the expression (5) for the integral p is the simplest and most general. We shall however give, in conclusion, three other expressions for p in terms of more familiar variables.

If the curve is given by means of its cartesian equation in the form $y = f(x)$, we may write

$$p = -\frac{1}{3\sqrt[3]{2}} \int \frac{\sqrt[3]{9\,(y'')^2 y^{(5)} - 45\, y'' y''' y^{(4)} + 40\, y^{(3)3}}}{y'' \sqrt{1 + y'^2}}\, dx, \tag{10}$$

where $y' = dy/dx$, $y'' = d^2y/dx^2$, and so on.

If the curve is given by means of parametric equations of the form $x = \varphi(s)$, $y = \psi(s)$, where s denotes the length of arc, and if r is the radius of curvature at the point which corresponds to the value s of the parameter, we find

$$p = \frac{1}{\sqrt[3]{6}} \int \sqrt[3]{\frac{r'''}{r} - \frac{r'r''}{r^2} + \frac{4}{9}\frac{r'^3}{r^3} + 4\frac{r'}{r^3}}\, ds, \tag{11}$$

where $r' = dr/ds$, $r'' = d^2r/ds^2$, etc..

Finally we may write

$$p = \int \frac{1}{\rho_0} \sqrt[8]{\frac{r_0}{2r}}\, ds, \tag{12}$$

where r has the same meaning as in (11), where ρ_0 is the distance from the point P of the curve to the center M of the corresponding osculating conic, and where r_0 is the radius of curvature at M of the locus which M describes when P moves along the given curve.

[1] E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, p. 58.
[2] Loc. cit., p. 59. [3] Loc. cit., p. 60. [4] Loc. cit., pp. 67-68.

SIZE INHERITANCE IN GUINEA-PIG CROSSES

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For several years my pupils and I have been engaged in studying the inheritance of size and weight differences among animals, these being characteristics of much economic importance and of peculiar theoretical interest. Preliminary studies published in 1909 showed that size and weight in rabbits do not follow the Mendelian rules of dominance and segregation as unit-characters. But Lang subsequently suggested that multiple Mendelian factors may be concerned in such cases, extending to animals a principle already recognized by Nilsson-Ehle in dealing with certain categories of characters in plants. Punnett and Bailey (1914) accept this principle in explaining weight inheritance in crosses of bantam fowls with those of ordinary size. They believe that four differential factors are concerned in a particular cross studied, three dominant factors which tend to increase size being found in the larger race, one such factor being found in the bantam race. By recombination in F_2 some individuals are obtained smaller than the bantam race, and others in F_3 larger than the larger race. But there are some reasons for questioning the validity of this analysis which assigns very definite quantitative values to the several hypothetical factors, without however making any allowance for physiological changes of size due to non-genetic causes, or for possible quantitative variation in the factors themselves. Moreover, let it be granted for the sake of argument that these four Mendelizing factors exist and that each is an independent agency for increasing size. On the Mendelian hypothesis there should be obtained from the cross in question individuals which lack *all four* of these factors. What, it may be asked, will their size be? Will they be with-