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ON THE REPRESENTATION OF A NUMBER AS THE SUM OF ANY NUMBER OF SQUARES, AND IN PARTICULAR OF FIVE OR SEVEN

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1. The formulae concerning the representation of a number as the sum of 5 or 7 squares belong to one of the most unfamiliar and difficult chapters in the Theory of Numbers, and only one proof of them has been given. The proof depends on the general arithmetic theory of quadratic forms, initiated by Eisenstein and perfected by Smith and Minkowski. This theory, of which a systematic account will be found in the fourth volume of Bachmann's *Zahlentheorie* gives a complete solution of the problem of any number s of squares not exceeding 8. Beyond $s = 8$ it fails.

When s is *even* there is an alternative method. This method, which depends on the theory of the elliptic modular functions, is much simpler in idea than the method of Smith and Minkowski; and it has another very important merit, that it can be used—within the limits of human capacity for calculation—for *any* even value of s. Thus Jacobi solved the problem for 2, 4, 6 and 8. In these cases the number of representations can be expressed in terms of the divisors of n. Suppose, *e.g.*, that $s = 8$; and let us write, generally,

$$1 + \sum_{1}^{\infty} r_s(n) q^n = (1 + 2q + 2q^4 + \ldots)^s = \{\vartheta_3(0, \tau)\}^s = \vartheta^s,$$

where $q = e^{\pi i \tau}$. Then

$$\vartheta^8 = 1 + 16\left(\frac{1^3 q}{1+q} + \frac{2^3 q^2}{1-q^2} + \frac{3^3 q^3}{1+q^3} + \frac{4^3 q^4}{1-q^4} + \ldots\right),$$

and $r_8(n)$ is $16 \Sigma \delta^3$ if n is odd and $8 \Sigma \delta_0^3 - 8 \Sigma \delta_1^3$ if n is even, δ denoting a divisor of n, δ_0 a even, and δ_1 an odd divisor. When s exceeds 8 the formulae are less simple, and involve arithmetical functions of a more recondite nature. Liouville gave formulae concerning the cases $s = 10$ and $s = 12$, and Glaisher[1] has worked out systematically all cases up to $s = 18$. More recently important papers on the subject, to which I shall refer later, have been published by Ramanujan[2] and Mordell.[3] In the latter paper the whole subject is exhibited as a corollary of the general theory of modular invariants.

The primary object of my own researches has been *to deduce the formulae for $s = 5$ and $s = 7$ from the theory of elliptic functions*, and so to place the cases in which s is odd and even, so far as may be, on the same footing. The methods which I use have further important applications, but this is the one which I wish to emphasize at the moment. The formulae which I take as my goal are the formulae

$$r_5(n) = \frac{Bn\sqrt{n}}{\pi^2}\sum\left(\frac{n}{m}\right)\frac{1}{m^2}, \tag{1}$$

$$r_7(n) = \frac{Cn^2\sqrt{n}}{\pi^3}\sum\left(\frac{-n}{m}\right)\frac{1}{m^3}, \tag{2}$$

given by Bachmann (pp. 621, 655). Here n as an odd number not divisible by any square (so that there is no distinction between primitive and imprimitive representations); m runs through all odd numbers prime to n; B is 80, 160, 112, or 160, according as n is congruent to 1, 3, 5 or 7 (mod. 8); and C is 448, 560, 448 or 592 in similar circumstances. These formulae are the central formulae of the theory: they involve infinite series, but these series are readily summed in finite terms by the methods of Dirichlet and Cauchy. With them should be associated the formula

$$r_3(n) = \frac{A\sqrt{n}}{\pi}\sum\left(\frac{-n}{m}\right)\frac{1}{m}, \tag{3}$$

where A is 24, 16, 24, or 0: but this formula, as we shall see, stands in some ways on a different footing.

2. My new proof of the formulae (1) and (2) was arrived at incidentally in the course of researches undertaken with a different end, that of finding *asymptotic* formulae (valid for *all* values of s) for $r_s(n)$ and other arithmetical functions which present themselves as coefficients in the expansions of elliptic modular functions. In a paper[4] shortly to appear in the *Proceedings* of the London Mathematical Society, Mr. Ramanujan and I have developed an exceedingly powerful method for the solution of problems of this character, and applied it to the study of $p(n)$, the number of (unrestricted) partitions of n. This method, when applied to our present problem, introduces the function

$$\Theta_s(q) = 1 + \frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} \sum_{h,k} \left(\frac{S_{h,k}}{k}\right)^s F(qe^{-h\pi i/k}), \tag{4}$$

where

$$S_{h,k} = \sum_{p=0}^{k=1} e^{p^2 h\pi i/k}, \quad F(q) = \sum_{1}^{\infty} n^{\frac{1}{2}s-1} q^n,$$

and the summation applies to $k = 1, 2, 3, \quad . \quad . \quad . \quad .$, and all positive values of h less than, of opposite parity to, and prime to k ($h = 0$ being associated with $k = 1$ alone). The coefficient of q^n in $\Theta_s(q)$ is

$$\chi_s(n) = \frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} n^{\frac{1}{2}s-1} \sum_{h,k} \left(\frac{S_{h,k}}{k}\right)^s e^{-nh\pi i/k}; \tag{5}$$

and our method leads to the conclusion that

$$r_s(n) = \chi_s(n) + O(n^{\frac{1}{4}s}), \tag{6}$$

at any rate for every value of s exceeding 4.

When s is *even*, $F(q)$ is an elementary function; and $(S_{h,k})^s$ is easily expressible in a form which does not involve the 'Legendre-Jacobi symbol' $\left(\frac{a}{b}\right)$. The function $X_s(n)$ is then susceptible of a variety of elementary transformations and it was shown by Ramanujan, in the second of his two papers quoted above, that $X_s(n)$ is *identical with* $r_s(n)$ when $s = 4$, 6 or 8. In what follows I confine myself to the case in which s is *odd*, merely remarking that my method (which is entirely unlike that used by Ramanujan) leads directly to an alternative proof of his results.

3. When s is odd, $F(q)$ is not an elementary function. But it is not difficult to prove that

$$\frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} F(q) = \sum_{-\infty}^{\infty} \frac{1}{[(2n-\tau)i]^{\frac{1}{2}s}}, \tag{7}$$

every term on the right hand side having an argument numerically less than $\frac{1}{4}s\pi$. Further, $S^s_{h,k} = S^{s-1}_{h,k}\, S_{h,k}$; and the first factor can always be expressed in a simple form. Suppose, to fix our ideas, that $s = 5$. Then $S^4_{h,k} = (-1)^h k^2$. Substituting from this equation and from (7) into (4), and effecting some obvious simplications, we obtain

$$\Theta_5(q) = 1 + \sum_{h,k} \frac{(-1)^h S_{h,k}}{\sqrt{k}} \frac{1}{[(h-k\tau)i]^{\frac{5}{2}}} \tag{8}$$

where now h assumes all values of opposite parity to and prime to k. This formula may be simplified further by multiplying each side by

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}.$$

We then find

$$\Theta_5(q)=\frac{8}{\pi^2}\sum_{h,k}\frac{(-1)^h S_{h,k}}{\sqrt{k}}\frac{1}{[(h-k\tau)i]^{\frac{5}{2}}},\tag{9}$$

the summation now extending to $k = 0, 1, 2, \ldots$ and *all h* of opposite parity to k. This is our fundamental formula, when $s = 5$. Two steps remain: first, to prove the identity of $\Theta_5(q)$ and ϑ^5; secondly, to deduce the formulae of Smith and Minkowski.

4. The first step presents no very serious difficulty, for it involves nothing beyond an adaptation of the ideas used by Mordell in his paper quoted in §1. We prove first that Θ_5 behaves like ϑ^5 in respect to the linear modular transformations $\tau = T + 2$, $\tau = -1/T$; so that Θ_5/ϑ^5 is an invariant of the modular sub-group called by Klein-Fricke and Mordell Γ_3. Secondly, by studying the transformation $\tau = (T - 1)/T$, we prove that Θ_5/ϑ^5 is *bounded* in the 'fundamental polygon' associated with Γ_3. It then follows that the quotient is a constant which is easily seen to be unity. In all this the only difficulty arises from the use of certain reciprocity-formulae satisfied by Gauss's sums.

We now transform (9) by effecting the summations with respect to h, using certain contour integrals of a type common in the work of Lindelöf and other writers. We thus obtain

$$\vartheta^5 = 1+\frac{32}{3}\left\{\sum_{1,3,5,\ldots}\frac{1}{k^2}\sum_j\sum_{m=0}^{\infty}(mk+j)^{\frac{3}{2}}q^{mk+j}\right.$$
$$\left.-\sum_{2,4,6,\ldots}\frac{1}{k^2}\sum_j\sum_{m=0}^{\infty}(-1)^{m+\mu}(mk+j)^{\frac{3}{2}}q^{mk+j}\right\}\tag{10}$$

a fundamental identity which contains the whole theory of the representation of numbers by sums of 5 squares. The symbols j and μ alone require explanation; j runs through the complete set of least positive residues of $0, 1^2, 2^2, \ldots, (k - 1)^2$ to modulus k, each taken as often as it occurs; and μk is the multiple of k deducted in order to arrive at such a residue. And the remainder of the work is purely arithmetical. Picking out the coefficient of q^n, we obtain a series for $r_5(n)$ which is found, after some reduction, to be equivalent to the series given by Bachmann.

4. The formulae which correspond to (10) for $s = 7$ and $s = 3$ are

$$\vartheta^7 = 1 + \frac{256}{15}\left\{\sum_{1,3,5,\ldots} \frac{(-1)^{\frac{1}{2}(k-1)}}{k^3} \sum_j \sum_{m=0}^{\infty} (mk+j)^{\frac{5}{2}} q^{mk+j} \right. \tag{11}$$

$$\left. - \sum_{2,4,6,\ldots} \frac{1}{k^2} \sum_j \sum_{m=0}^{\infty} (-1)^{m+\mu} (mk+j)^{\frac{5}{2}} q^{mk+j} \right\},$$

$$\vartheta^3 = 1 + 8\left\{\sum_{1,3,5,\ldots} \frac{(-1)^{\frac{1}{2}(k-1)}}{k} \sum_j \sum_{m=0}^{\infty} (mk+j)^{\frac{1}{2}} q^{mk+j} \right. \tag{12}$$

$$\left. + \sum_{2,4,6,\ldots} \frac{1}{k} \sum_j \sum_{m=0}^{\infty} (-1)^{m+\mu} (mk+j)^{\frac{1}{2}} q^{mk+j} \right\}.$$

The interpretation of j and μ is as before, except that, when k is even, j is a residue of one of the numbers $\frac{1}{2}k$, $\frac{1}{2}k + 1^2$, . . . , $\frac{1}{2}k + (k-1)^2$. These identities embody the theory for 7 or 3 squares. It should be noted however, that the application of my method becomes very much more difficult when $s = 3$, as the double series used are then not absolutely convergent; and in fact the only proof of (12) which I possess consists in an identification of the results which it gives with those already known.

I conclude by a word concerning the cases in which $s > 8$. Here, when s is odd, we are on untrodden ground. We have the asymptotic formula (6); and we can evaluate $X_s(n)$ as when $s = 5$ or 7, thus obtaining a series of new results. But it is no longer to be expected that our results should be *exact*, and I have verified that, when $s = 9$, they are not exact, even when $n = 1$.

[1] Glaisher, J. W. L., *Proc. London Math. Soc.*, (Ser. 2), **5**, 1907, (479–490).

[2] Ramanujan, S., *Trans. Camb. Phil. Soc.*, **22**, 1916, (159–184); *Ibid.*, (in course of publication).

[3] Mordell, L. J., *Quart. J. Math.*, **48**, 1917, (93–104).

[4] Hardy, G. H., and Ramanujan, S., *Proc. London Math. Soc.*, (Ser. 2), **17**, 1918, (in course of publication).

THE CRYSTAL STRUCTURE OF ICE

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During the winter of 1916–1917 the crystal structure of ice was investigated by means of the X-rays. The photographic method originated by deBroglie[1] was used with certain modifications suggested privately by Dr. A. W. Hull. The source of energy was a Coolidge tube with tungsten target excited by an induction coil with mercury turbine interrupter. At first the