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A new modified estimator of population variance in calibrated survey sampling

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In survey statistics, estimating and reducing population variation is crucial. These variations can occur in any sampling design, including stratified random sampling, where stratum weights may increase the variance of estimators. Calibration techniques, which use additional auxiliary information, can help mitigate this issue. This paper examines three calibration-based estimators—calibration variance, calibration ratio, and calibration exponential ratio estimators—within the framework of stratified random sampling. The study generates data from normal, gamma, and exponential distributions to test these estimators. Results demonstrate that the proposed calibration estimators offer more accurate estimates of population variance and outperform existing methods in estimating population variance under stratified random sampling, providing more accurate and reliable estimates.

Keywords Auxiliary information, Calibration, Stratified random sampling technique

In survey sampling sometimes the sample is divided into subgroups of interest, which are homogenous in nature. These homogenous subgroups are called strata and this technique to formulate the homogenous samples is called stratified sampling technique. The variability in the subgroup is lower than the whole individuals, so that using this technique a statistician can get greater precision than simple random sampling. The main interest of the researchers is to get the estimators of population parameters, such as mean and variance in such a way that they are less costly, more efficient and flexible to apply in real life situations. Many authors have provided estimates of the population mean and variance using what is called auxiliary information (AI). Graunt[1] was the first one who obtained the estimates for the total population of France using the birth rate as AI. The AI is useful at both the design and estimation stages, respectively. Most statisticians have used the AI at the estimation stage to improve the efficiency of estimators. Calibration technique is also used to improve the precision of estimators of the population parameters using some AI. Calibration is a method which is used to adjust the sampled unit's weights with respect to known standards (totals) or conditions. The process of calibration method includes the evaluation of the sample by assigning the values to the response tool or to selected measures[2]. The calibration estimation has been adopted by several researchers. For example, Deville and Särndal[3] provided new estimators using calibrated weights, which have the smallest distance from sampling design weights. Berge[4] extended the use of calibration estimators in survey sampling. The calibration method has many advantages in survey sampling for instance calibration estimators are consistent and provide more efficient estimators for population total as compared to other estimators. The calibration method minimizes the distance between original and calibrated weights. Estevao and Sarndal[5] developed the functional form of calibration estimation. The calibration estimators in survey sampling are proposed be Arnab and Singh[6] and Kott[7]. Kim et al.[8] stated that calibration is commonly used by including auxiliary data so that the estimation of the population parameters can be more precise. Kim and Park[9] used different calibration constraints and distance measures to produce calibration estimators. Koyuncu and Kadilar[10] developed estimators for the population mean using calibration techniques. Bhushan et al.[11] examined variance estimation methods within an efficient class of estimators for simple random sampling. Lone et al.[12] introduced variance estimators that incorporate artificial intelligence techniques in simple random sampling. Jabeen et al.[13] proposed several calibration-based estimators, including calibration mean, calibration ratio, and calibration exponential estimators, employing diverse calibration constraints and distance measures. The objective of the study is to propose calibration variance estimators following the works of Koyuncu and Kadilar[10] and Jabeen et al.[13]. All previous works in the literature focused on the mean estimation in calibration methodology. To the

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best of our knowledge, this is the first article that focuses on the variance estimation in calibration technique under stratified sampling method. The study proposes three calibration variance estimators namely, the simple calibration variance estimator, calibration variance ratio estimator, and calibration variance exponential ratio using chi-square distance measure and different calibration constraints. The R language is used to compare the proposed estimators with some existing estimators in literature.

The rest of the paper is organized into five sections. In "Some existing estimators" section provides an overview of several existing estimators. In "Some proposed calibration variance estimators" section presents the mathematical derivation of three calibration variance estimators. In "Simulation study" section details a simulation study designed to assess the efficiency of the proposed estimators across various distributions. In "Real life application" section applies these estimators to real-life data to verify their practical effectiveness. Finally, "Conclusion" section concludes the paper, summarizing the key findings and implications of the study.

Some existing estimators

This section reviews key estimators developed by survey sampling statisticians, detailing their functionality and formulations. We also introduce the notation that will be used in the following sections.

W_h^* is the calibrated weight for each stratum (where $h = 1, 2, \ldots, l$).
s_y^2 is the sample variance of the sample study variable.
S_y^2 denotes the population variance of the study variable.
S_x^2 is the population variance of the auxiliary variable.
s_x^2 is the sample variance of the auxiliary variable.
Ω_h refers to the weights, which minimize the distance measure.
Q_h measure of size for sampled units.
y_{hy} represents the observed values for the study variable of the *h*th stratum.
x_{hx} denotes the observed values for auxiliary variable of *h*th stratum.
$W_h = \frac{N_h}{N}$ is the stratum weight.
$R = \frac{S_y^2}{S_x^2}$ is the ratio estimator.
ρ_{xy} is the correlation coefficient between *x* and *y*.
S_{xy} presents the covariance among *x* and *y*.

Variance ratio estimator

To overcome the problem of the estimation of variance of any population, Isaki[14] first proposed usual variance ratio estimator, say, $\boldsymbol{S_R^2}$, which is given as

$$\boldsymbol{S_R^2} = \frac{\boldsymbol{s_y^2}}{\boldsymbol{s_x^2}}\boldsymbol{S_x^2}.$$

The mean square error (MSE) of $\boldsymbol{S_R^2}$ reduces to

$$MSE(S_R^2) = \left[S_y^2 + S_x^2 - 2RS_{xy}\right].$$

Exponential variance estimator

Building on Isaki's[14] work, Bahl and Tuteja[15] developed the variance exponential estimator for situations where the study variable is not linearly related to the auxiliary information. The variance exponential estimator, say, $\hat{S}_{BT}$, and its MSE are defined by

$$\hat{S}_{BT} = s_y^2 \exp\left[\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right]$$

and

$$MSE\left(S_{BT}^2\right) = \theta S_y^4\left[\lambda_{40}^* + \frac{\lambda_{04}^*}{4} - g_o\lambda_{22}^*\right].$$

where

$$\lambda_{40}^* = C_y^4 = \frac{S_y^4}{\overline{Y}^4}, \quad \lambda_{04}^* = \frac{C_x^4}{4} = \frac{S_x^4}{4\overline{X}^4}, \quad \lambda_{22}^* = C_yC_x, \quad g_o = \rho_{xy} = \frac{S_{xy}}{S_xS_y}$$

and

$$\theta = \frac{1}{n} - \frac{1}{N}.$$

Variance ratio estimator

Upadhyaya and singh[16] proposed the variance exponential estimator, which is defined by

$$S_{us}^2 = s_y^2 \exp\left[\frac{S_x - \lambda_{04}}{S_x + \lambda_{04}}\right].$$

Its MSE takes the form

$$MSE\left(S_{us}^2\right) = \theta S_y^4\left[\lambda_{40}^* + g_o^2\lambda_{04}^* - 2g_o\lambda_{22}^*\right],$$

Calibration estimators
In stratified random sampling, new calibration estimators for estimating the population mean using AI are proposed by Koyuncu and Kadilar[17]. The classic unbiased estimator of the population mean is given, under this stratified random sampling scheme, by

$$\overline{y}_1 = \sum_{h=1}^{l} W_h^* \overline{y}_h,$$

where W_h^* are the calibration weights that reduce the chi-square distance measure to the smallest possible value. The chi-square distance measure is given by

$$L_1 = \frac{\sum_h^l (\Omega_h - W_h)^2}{Q_h - W_h}.$$

Calibration ratio and exponential estimators
Jabeen et al.[13] proposed the calibration estimators by taking motivation from Kim et al.[8]. The calibration estimator is given by

$$y_2 = \sum_h^l W_h \overline{y}_{hy}^*,$$

where $\overline{y}_{hy}^*$ is assumed to be a variable with a usual variance estimator exponential and ratio estimators, i.e.

$$\begin{aligned}
\overline{y}_{hy}^* &= \overline{y}_{hy},\\
\overline{y}_{hy}^* &= \overline{y}_{hy}\exp\left[\frac{\overline{x}_{hx} - \overline{X}_{hx}}{\overline{x}_{hx} + \overline{X}_{hx}}\right],\\
\overline{y}_{hy}^* &= \overline{y}_{hy}\frac{\overline{X}_{hx}}{\overline{x}_{hx}}.
\end{aligned}$$

The calibration constraints, which define the relationship between the study variables and auxiliary variables, are defined by

$$\begin{aligned}
\sum_h^l \Omega_h s_{hx}^2 &= \sum_h^l W_h s_{hx}^{2*},\\
\sum_h^L \Omega_h \overline{x}_h &= \sum_h^L W_h \overline{x}_h^*
\end{aligned}$$

and

$$\sum_h^L \Omega_h = \sum_h^L W_h.$$

Some proposed calibration variance estimators
In this section, we propose some new variance estimators to estimate the variation in the population using calibration technique.

Proposed calibration variance estimator
The following calibrated estimator is proposed to estimate the population variance, and it is defined by

$$t_1 = \sum_h^l \Omega_h s_{hy}^{2*},$$

where

$$s_{hy}^* = s_{hy}^2,$$

with following constraints where $h = 1, 2 \ldots l$ stratum.

$$\sum_{h}^{l} \Omega_h \overline{x}_h = \sum_{h}^{l} W_h \overline{X}_h, \tag{1}$$

$$\sum_{h}^{l} \Omega_h s_{hx}^2 = \sum_{h}^{l} W_h S_{hx}^2 \tag{2}$$

and

$$\sum_{h}^{l} \Omega_h = \sum_{h}^{l} W_h, \tag{3}$$

More details about the proof of the estimator t_1 are given in "Appendix 1".

Proposed calibration ratio estimator

We use the concept of Jabeen et al.[13] and propose the initiated following calibration ratio estimator. The calibration ratio estimator is defined by

$$t_2 = \sum_{h}^{l} \Omega_h S_{hy}^{2*},$$

where

$$S_{hy}^{*2} = S_{yh}^2 \cdot \frac{S_{xh}^2}{S_{xh}^2},$$

with the constraints mentioned in Eqs. (1) to (3), $h = 1, 2, \ldots k$ stratum. The proof of proposed estimator t_2 is given in "Appendix 2".

Proposed calibration exponential estimator

Following Jabeen et al.[13], we propose the following calibration exponential estimator, which is defined as

$$t_3 = \sum_{h}^{l} \Omega_h S_{hy}^{2*},$$

where

$$S_{hy}^{*2} = s_{hy}^2 \exp\left[\frac{S_{hx}^2 - s_{hx}^2}{S_{hx}^2 + s_{hx}^2}\right],$$

with constraints given above in Eqs. (1), (2) and (3), where $h = 1, 2, \ldots k$ stratum. Proof of the proposed estimator t_3 can be found in "Appendix 3".

Simulation study

We produce distinctive simulated population where y_{ji}^* and x_{ji}^* values for various distributions, as given in Table 1. To obtain different levels of relationship among investigation and helping variable, we apply few transformations, which are given in Table 2.

Each population comprises of three strata and each stratum contains 5 units. We choose $\boldsymbol{n_i} = 2, 3, 4$ units from every stratum separately respectively, we get $\binom{5}{2}\binom{5}{3}\binom{5}{4} = 2500$ samples. The correlation coefficients between study variable and auxiliary variable for each stratum are taken as 0.5, 0.7 and 0.9 as per Tracy et al.[17]. The MSE is computed using following formula.

$$MSE\left(\hat{y}_i\right) = \frac{\sum_{k=1}^{\binom{N}{n}} \left(\hat{y}_i - \bar{y}\right)^2}{\binom{N}{n}}, \quad i = 1, 2, 3 \ldots$$

The results of the MSE are presented in Tables 3, 4, 5 and 6. The results of simulated data presented in Table 3, 4, 5 and 6 are obtained for different distributions. In our analysis, we calculated the MSE for three proposed estimators across four different distributions with varying parameters, as detailed in Tables 1, 2, 3, 4, 5 and 6. The results, presented in Table 7, demonstrate that our proposed estimators consistently outperform the existing estimators by Isaki[14] and Bahl and Tuteja[15]. Specifically, the MSE values for our estimators are consistently lower compared to those of the conventional estimators.

No.	Parameters and distribution of SV	Parameters and distribution of AV
1	$f\left(y_{ji}^{*}\right)=\frac{1}{\beta^{\alpha}\Gamma(\alpha)}y_{ji}^{*\alpha-1}\exp\left(-\frac{y_{ji}^{*}}{\beta}\right)$	$f\left(x_{ji}^{*}\right)=\frac{1}{\beta^{\alpha}\Gamma(\alpha)}x_{ji}^{*\alpha-1}\exp\left(\frac{-x_{ji}^{*}}{\beta}\right)$
2	$f\left(y_{ji}^{*}\right)=\frac{1}{\lambda}\exp\left(-\frac{y_{ji}^{*}}{\lambda}\right)$	$f\left(x_{ji}^{*}\right)=\frac{1}{\lambda}\exp\left(\frac{-x_{ji}^{*}}{\lambda}\right)$
3	$f\left(y_{ji}^{*}\right)=\frac{1}{y_{ji}^{*}\sqrt{2\Pi}}\exp\left(\frac{-(\ln y_{ji}^{*})^{2}}{2}\right)$	$f\left(x_{ji}^{*}\right)=\frac{1}{x_{ji}^{*}\sqrt{2\Pi}}\exp\left(\frac{-(\ln x_{ji}^{*})^{2}}{2}\right)$
4	$f\left(y_{ji}^{*}\right)=\frac{1}{\sqrt{2\Pi}}\exp\left(\frac{-y_{ji}^{*2}}{2}\right)$	$f\left(x_{ji}^{*}\right)=\frac{1}{\sqrt{2\Pi}}\exp\left(\frac{-x_{ji}^{*2}}{2}\right)$

Table 1. Parameters and distribution of study (SV) and auxiliary variable (AV).

Strata	Study variable	Auxiliary variable
1 Stratum	$y_{1i}=50+y_{1i}^{*}$	$x_{1i}=15+\sqrt{(1-\rho_{xy1}^{2})x_{1i}^{*}}+\rho_{xy1}\frac{s_{1x}}{s_{1y}}y_{1i}^{*}$
2 Stratum	$y_{2i}=150+y_{2i}^{*}$	$x_{2i}=100+\sqrt{(1-\rho_{xy2}^{2})x_{2i}^{*}}+\rho_{xy2}\frac{s_{2x}}{s_{2y}}y_{2i}^{*}$
3 Stratum	$y_{3i}=100+y_{3i}^{*}$	$x_{3i}=300+\sqrt{(1-\rho_{xy3}^{2})x_{3i}^{*}}+\rho_{xy3}\frac{s_{3x}}{s_{3y}}y_{3i}^{*}$

Table 2. Properties of each stratum.

α,β	Estimator	MSE
$\alpha=2,\beta=3$	t_1	95,965.23
	t_2	177,237.1
	t_3	79,867.68
$\alpha=2.5,\beta=3.75$	t_1	90,936.87
	t_2	177,145
	t_3	74,138.4
$\alpha=3,\beta=4.1$	t_1	78,387.8
	t_2	141,730.5
	t_3	44,512.85

Table 3. The MSE of the proposed estimators from gamma distribution.

ρ	Estimator	MSE
0.5	t_1	15,717.72
	t_2	4021.978
	t_3	9922.135
0.7	t_1	17,091.06
	t_2	2969.764
	t_3	9631.747
0.9	t_1	13,366.41
	t_2	2703.995
	t_3	8005.757

Table 4. The MSE of the proposed estimators from normal distribution.

These findings indicate that the proposed estimators provide more accurate and reliable results than their predecessors. Therefore, adopting our proposed estimators is likely to enhance performance and utility in practical applications.

λ	Estimator	MSE
2	t_1	118,204.7
	t_2	238,458.8
	t_3	151,247.1
2.5	t_1	85,241.85
	t_2	171,522.5
	t_3	57,668.86
3.5	t_1	41,555.86
	t_2	82,738.78
	t_3	6635.789

Table 5. The MSE of the proposed estimators from exponential distribution.

ρ	Estimator	MSE
0.5	t_1	2,155,477
	t_2	3,011,671
	t_3	2,237,658
0.7	t_1	21,955,477
	t_2	1,291,763
	t_3	2,120,258
0.9	t_1	4,166,660
	t_2	2,458,213
	t_3	1,920,386

Table 6. The MSE of the proposed estimators from log normal distribution.

MSE of t_1	MSE of t_2	MSE of t_3	MSE of Isaki[14]	MSE of Bahl and Tuteja[15]
95,965.23	177,237.1	78,967.68	150,404.1	266,448
90,936.87	177,145	74,138.4	119,218	230,510.8
78,387.8	141,730.5	44,512.85	122,183.5	212,128.3
15,717.72	4021.978	9922.135	644,222	176,251.8
17,091.06	2969.764	9631.747	851,864.6	196,821.4
13,366.41	2703.995	8005.757	699,237.2	203,189.5
118,204.7	238,458.8	151,247.1	61,846,925	314,576.8
85,241.85	171,522.5	57,668.86	42,588,038	231,087.6
41,555.86	82,738.78	6635.789	21,837,626	132,359.2
2,155,477	3,011,671	2,237,658	4,686,898	6,131,997
2,195,253	1,291,763	2,120,258	2,401,915	6,684,994
4,166,660	2,458,213	1,920,386	6,806,558	6,790,442

Table 7. Findings of the comparison of MSE across various estimators.

Real life application

To demonstrate the performance of our proposed estimators, we use a real-life dataset from Koyuncu and Kadilar[18]. The dataset includes two variables: the number of teachers and the number of classes in both primary and secondary schools across Turkey. The data were collected from six diverse regions: Marmara, Aegean, Mediterranean, Central Anatolia, Black Sea, and East and Southeast Anatolia. A total sample size of $n = 180$ was selected, with the sample sizes for each stratum, n_h, detailed in Table 8. The MSEs and their comparisons are presented in Table 9.

Table 9 displays the MSEs obtained from the proposed calibrated variance estimators—simple, ratio, and exponential—using the data provided in Table 8. The results align with those from the simulation studies. The comparison reveals that the proposed calibrated estimators are more efficient than existing estimators, as they exhibit the lowest MSEs. The application of the calibration technique enhances the overall performance of the estimators in estimating population variation.

Table 9 confirms that the results align with those obtained from the simulation studies. The comparison reveals that the proposed estimators using the calibration technique are more efficient than the existing ones, as

$N_1=127$	$N_2=117$	$N_3=103$
$N_4=170$	$N_5=205$	$N_6=201$
$n_1=31$	$n_2=21$	$n_3=29$
$n_4=38$	$n_5=22$	$n_6=39$
$S_{y1}=883.835$	$S_{y2}=644.922$	$S_{y3}=1033.467$
$S_{y4}=810.585$	$S_{y5}=403.654$	$S_{y6}=711.723$
$\overline{Y}_1=703.74$	$\overline{Y}_2=413$	$\overline{Y}_3=573.17$
$\overline{Y}_4=424.66$	$\overline{Y}_5=267.03$	$\overline{Y}_6=393.84$
$C_{y1}=1.256$	$C_{y2}=1.562$	$C_{y3}=1.803$
$C_{y4}=1.909$	$C_{y5}=1.512$	$C_{y6}=1.807$
$S_{x1}=30{,}486.751$	$S_{x2}=15{,}180.769$	$S_{x3}=27{,}549.697$
$S_{x4}=18{,}218.931$	$S_{x5}=8497.776$	$S_{x6}=23{,}094.141$
$\overline{X}_1=20804.59$	$\overline{X}_2=9211.79$	$\overline{X}_3=14309.30$
$\overline{X}_4=9478.85$	$\overline{X}_5=5569.95$	$\overline{X}_6=12997.59$
$C_{X1}=1.465$	$C_{X2}=1.648$	$C_{X3}=1.925$
$C_{X4}=1.922$	$C_{X5}=1.526$	$C_{X6}=1.777$
$S_{xy1}=25{,}237{,}153.52$	$S_{xy2}=9{,}747{,}942.85$	$S_{xy3}=28{,}294{,}397.04$
$S_{xy4}=14{,}523{,}885.53$	$S_{xy5}=3{,}393{,}591.75$	$S_{xy6}=15{,}864{,}573.97$
$\rho_1=0.936$	$\rho_2=0.996$	$\rho_3=0.994$
$\rho_4=0.983$	$\rho_5=0.989$	$\rho_6=0.965$
$w_1=0.138$	$w_2=0.127$	$w_3=0.112$
$w_4=0.184$	$w_5=0.222$	$w_6=0.218$

Table 8. Summary statistics of six strata.

SE of t_1	SE of t_2	SE of t_3	SE of usual variance estimator	SE of Isaki[14]	SE of Bahl and Tuteja[15]
1,381,510,618	2,416,506.084	1,403,861.639	8,063,656,399	107,431,592,564	1,381,510,618

Table 9. Comparison of the SEs of different estimators.

they exhibit the lowest standard errors (SEs). This indicates that the calibration technique enhances the overall performance of the estimators in estimating population variation.

Conclusion

In this research, we introduced three new estimators inspired by Jabeen et al.[13] to estimate population variation. We employed three calibration constraints and used the chi-square distance measure to minimize the discrepancy between the original and calibrated weights of strata. The three estimators developed are: the calibration variance estimator, the calibration ratio estimator, and the calibration exponential estimator. To assess the performance of these estimators, we analyzed the mean squared error (MSE) of both the proposed and existing estimators through simulation studies and real-life data. Our findings indicate that the proposed calibrated estimators outperform the existing ones, such as those by Isaki[14] and Bahl and Tuteja[15], by achieving a lower MSE.

Data availability

The data used to support the findings of this study are included in the article.

Appendix 1

Proof for the calibrated variance estimator

$$t_1 = \sum_{h}^{l} \Omega_h s_{hy}^{2*}$$

The calibration constraints

$$\sum_{h}^{l}\Omega_h\overline{x}_h=\sum_{h}^{l}W_h\overline{X}_h$$
$$\sum_{h}^{l}\Omega_h s_{hx}^2=\sum_{h}^{l}W_h S_{hx}^2$$
$$\sum_{h}^{l}\Omega_h=\sum_{h}^{l}W_h$$

The following langrage's function is given according to the distance measure and calibration constraints

$$\Delta=\frac{\sum_{h}^{l}\left(W_h-\Omega_h\right)^2}{Q_hW_h}-2\lambda_1\left(\sum_{h}^{l}\Omega_h\overline{x}_h-\sum_{h}^{l}W_h\overline{X}_h\right)-2\lambda_2\left(\sum_{h}^{l}\Omega_h s_{hx}^2-\sum_{h}^{l}W_hS_{hx}^2\right)-2\lambda_3\left(\sum_{h}^{l}\Omega_h-\sum_{h}^{l}W_h\right)$$

Differentiate w.r.t. Ω_h

$$\frac{\partial\Delta}{\partial\Omega_h}=\frac{2\left(\Omega_h-W_h\right)}{Q_hW_h}-2\lambda_1\left(\overline{x}_h\right)-2\lambda_2 s_{hx}^2-2\lambda_3$$
$$\frac{\partial\Delta}{\partial\Omega_h}=\frac{2\left(\Omega_h-W_h\right)-2\lambda_1\left(\overline{x}_hQ_hW_h\right)-2\lambda_2\left(s_{hx}^2Q_hW_h\right)-2\lambda_3\left(Q_hW_h\right)}{Q_hW_h}$$
$$\frac{\partial\Delta}{\partial\Omega_h}=0$$
$$\left(\Omega_h-W_h\right)-\lambda_1\left(\overline{x}_hQ_hW_h\right)-\lambda_2 s_{hx}^2Q_hW_h-\lambda_3\left(Q_hW_h\right)=0$$
$$\Omega_h=W_h+\lambda_1\overline{x}_hQ_hW_h+\lambda_2\left(s_{hx}^2Q_hW_h\right)+\lambda_3\left(Q_hW_h\right)$$

Substituting Ω_h in constraints

$$\lambda_1\left(\sum_{h}^{l}\overline{x}_h^2Q_hW_h\right)+\lambda_2\left(\sum_{h}^{l}\overline{x}_hs_{hx}^2Q_hW_h\right)+\lambda_3\left(\sum_{h}^{l}\overline{x}_hQ_hW_h\right)=\sum_{h}^{l}W_h\overline{X}_h-\sum_{h}^{l}W_h\overline{x}_h$$
$$\lambda_1\left(\sum_{h}^{l}s_{hx}^2\overline{x}_hQ_hW_h\right)+\lambda_2\left(\sum_{h}^{l}s_{hx}^4Q_hW_h\right)+\lambda_3\left(\sum_{h}^{l}s_{hx}^2Q_hW_h\right)=\sum_{h}^{l}W_hS_{hx}^2-\sum_{h}^{l}W_hs_{hx}^2$$
$$\lambda_1\left(\sum_{h}^{l}\overline{x}_hQ_hW_h\right)+\lambda_2\left(\sum_{h}^{l}s_{hx}^2Q_hW_h\right)+\lambda_3\left(Q_hW_h\right)=0$$

$$\begin{bmatrix}\sum_{h}^{l}\overline{X}_h^2Q_hW_h & \sum_{h}^{l}s_{hx}^2\overline{X}_hQ_hW_h & \sum_{h}^{l}\overline{X}_hQ_hW_h\\ \sum_{h}^{l}\overline{X}_hs_{hx}^2Q_hW_h & \sum_{h}^{l}s_{hx}^4Q_hW_h & \sum_{h}^{l}s_{hx}^2Q_hW_h\\ \sum_{h}^{l}\overline{X}_hQ_hW_h & \sum_{h}^{l}s_{hx}^2Q_hW_h & Q_hW_h\end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix}=\begin{bmatrix}\sum_{h}^{l}W_h\overline{X}_h-\sum_{h}^{l}W_h\overline{X}_h\\ \sum_{h}^{l}W_hS_{hx}^2-\sum_{h}^{l}W_hs_{hx}^2\\ 0\end{bmatrix}$$

$$\begin{aligned}D=&\,3\left[\sum\overline{x}_hQ_hW_h\cdot\sum s_{hx}^2Q_hW_h\cdot\sum Q_hW_h\right]-3\left[\sum\overline{x}_h^2Q_hW_h\left(\sum s_{hx}^2Q_hW_h\right)^2\right]\\&-3\left[\left(\sum s_{hx}^2\overline{x}_hQ_hW_h\right)^2\sum Q_hW_h\right]+6\left[\sum s_{hx}^2Q_hW_h\sum\overline{x}Q_hW_h\sum s_{hx}^2\overline{x}_hQ_hW_h\right]\\&-3\left[\sum\overline{x}_h^2Q_hW_h\left(\sum s_{hx}^4Q_hW_h\right)^2\right]\end{aligned}$$

$$\begin{aligned}D=&\left[\sum\overline{x}_hQ_hW_h\cdot\sum s_{hx}^2Q_hW_h\cdot\sum Q_hW_h\right]-\left[\sum\overline{x}_h^2Q_hW_h\left(\sum s_{hx}^2Q_hW_h\right)^2\right]\\&-\left[\left(\sum s_{hx}^2\overline{x}_hQ_hW_h\right)^2\sum Q_hW_h\right]+2\left[\sum s_{hx}^2Q_hW_h\sum\overline{x}Q_hW_h\sum s_{hx}^2\overline{x}_hQ_hW_h\right]\\&-\left[\sum\overline{x}_h^2Q_hW_h\left(\sum s_{hx}^4Q_hW_h\right)^2\right]\end{aligned}$$

$$\begin{bmatrix} \sum_h^l W_h\overline{X}_h - \sum_h^l \overline{X}_h W_h & \sum_h^l s_{hx}^2\overline{X}_h Q_h W_h & \sum_h^l \overline{X}_h Q_h W_h \\ \sum_h^l W_h S_{hx}^2 - \sum_h^l s_{hx}^2 W_h & \sum_h^l s_{hx}^4 Q_h W_h & \sum_h^l s_{hx}^2 Q_h W_h \\ 0 & \sum_h^l s_{hx}^2 Q_h W_h & Q_h W_h \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \sum_h^l W_h\overline{X}_h - \sum_h^l W_h\overline{X}_h \\ \sum_h^l W_h S_{hx}^2 - \sum_h^l s_{hx}^2 W_h \\ 0 \end{bmatrix}$$

$$A = \left[\sum_h^l W_h\overline{X}_h - \sum_h^l \overline{X}_h W_h\right]\left[\left(\sum_h^l s_{hx}^2 Q_h W_h\right)^2 - \left(\sum_h^l s_{hx}^2 Q_h W_h\right)^2\right] - \left[\sum_h^l W_h S_{hx}^2 + \sum_h^l s_{hx}^2 W_h\right]\left[\left(\sum_h^l s_{hx}^2\overline{X}_h Q_h W_h\right)(Q_h W_h) + \left(\sum_h^l \overline{X}_h Q_h W_h\right)\left(\sum_h^l s_{hx}^2 Q_h W_h\right)\right]$$

$$\begin{bmatrix} \sum_h^l \overline{X}_h^2 Q_h W_h & \sum_h^l W_h\overline{X}_h - \sum_h^l \overline{X}_h W_h & \sum_h^l \overline{X}_h Q_h W_h \\ \sum_h^l \overline{X}_h s_{hx}^2 Q_h W_h & \sum_h^l W_h S_{hx}^2 - \sum_h^l s_{hx}^2 W_h & \sum_h^l s_{hx}^2 Q_h W_h \\ \sum_h^l \overline{X}_h Q_h W_h & 0 & Q_h W_h \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \sum_h^l W_h\overline{X}_h - \sum_h^l W_h\overline{X}_h \\ \sum_h^l W_h S_{hx}^2 - \sum_h^l s_{hx}^2 W_h \\ 0 \end{bmatrix}$$

$$B = -\left[\sum_h^l W_h\overline{X}_h - \sum_h^l \overline{X}_h W_h\right]\left[\left(\sum_h^l \overline{X}_h s_{hx}^2 Q_h W_h\right)(Q_h W_h) - \left(\sum_h^l s_{hx}^2 Q_h W_h\right)\left(\sum_h^l \overline{X}_h Q_h W_h\right)\right] + \left[\sum_h^l W_h S_{hx}^2 - \sum_h^l s_{hx}^2 W_h\right]\left[\left(\sum_h^l \overline{X}_h^2 Q_h W_h\right)(Q_h W_h) - \left(\sum_h^l \overline{X}_h Q_h W_h\right)\left(\sum_h^l \overline{X}_h Q_h W_h\right)\right]$$

$$\begin{bmatrix} \sum_h^l \overline{X}_h^2 Q_h W_h & \sum_h^l s_{hx}^2\overline{X}_h Q_h W_h & \sum_h^l W_h\overline{X}_h - \sum_h^l \overline{X}_h W_h \\ \sum_h^l \overline{X}_h s_{hx}^2 Q_h W_h & \sum_h^l s_{hx}^4 Q_h W_h & \sum_h^l W_h S_{hx}^2 - \sum_h^l s_{hx}^2 W_h \\ \sum_h^l \overline{X}_h Q_h W_h & \sum_h^l s_{hx}^2 Q_h W_h & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \sum_h^l W_h\overline{X}_h - \sum_h^l \overline{X}_h W_h \\ \sum_h^l W_h S_{hx}^2 - \sum_h^l s_{hx}^2 W_h \\ 0 \end{bmatrix}$$

$$C = \left[\sum_h^l W_h\overline{X}_h - \sum_h^l \overline{X}_h W_h\right]\left[\left(\sum_h^l \overline{X}_h s_{hx}^2 Q_h W_h\right)\left(\sum_h^l s_{hx}^2 Q_h W_h\right) - \left(\sum_h^l s_{hx}^4 Q_h W_h\right)\left(\sum_h^l \overline{X}_h Q_h W_h\right)\right] - \left[\sum_h^l W_h S_{hx}^2 - \sum_h^l s_{hx}^2 W_h\right]\left[\left(\sum_h^l \overline{X}_h^2 Q_h W_h\right)\left(\sum_h^l s_{hx}^2 Q_h W_h\right) - \left(\sum_h^l s_{hx}^2\overline{X}_h Q_h W_h\right)\left(\sum_h^l \overline{X}_h Q_h W_h\right)\right]$$

$$\lambda_1 = \frac{A}{D},\ \lambda_2 = \frac{B}{D},\ \lambda_3 = \frac{C}{D}$$

Now put the $\lambda_1, \lambda_2, \lambda_3$ values in Ω_h

$$\begin{aligned}\Omega_h = W_h + Q_h W_h \Bigg[&\left[\sum_h^l W_h\overline{X}_h - \sum_h^l \overline{X}_h W_h\right]\left[\left(\sum_h^l s_{hx}^2 Q_h W_h\right)^2 - \left(\sum_h^l s_{hx}^2 Q_h W_h\right)^2\right] \\ &- \left[\sum_h^l W_h S_{hx}^2 + \sum_h^l s_{hx}^2 W_h\right]\left[\left(\sum_h^l s_{hx}^2\overline{X}_h Q_h W_h\right)(Q_h W_h) + \left(\sum_h^l \overline{X}_h Q_h W_h\right)\left(\sum_h^l s_{hx}^2 Q_h W_h\right)\right]\left(\sum_h^l \overline{X}_h\right) \\ &+ - \left[\sum_h^l W_h\overline{X}_h - \sum_h^l \overline{X}_h W_h\right]\left[\left(\sum_h^l \overline{X}_h s_{hx}^2 Q_h W_h\right)(Q_h W_h) - \left(\sum_h^l s_{hx}^2 Q_h W_h\right)\left(\sum_h^l \overline{X}_h Q_h W_h\right)\right] \\ &+ \left[\sum_h^l W_h S_{hx}^2 - \sum_h^l s_{hx}^2 W_h\right]\left[\left(\sum_h^l \overline{X}_h^2 Q_h W_h\right)(Q_h W_h) - \left(\sum_h^l \overline{X}_h Q_h W_h\right)\left(\sum_h^l \overline{X}_h Q_h W_h\right)\right]\left(\sum_h^l s_{hx}^2\right) \\ &+ \left[\sum_h^l W_h\overline{X}_h - \sum_h^l \overline{X}_h W_h\right]\left[\left(\sum_h^l \overline{X}_h s_{hx}^2 Q_h W_h\right)\left(\sum_h^l s_{hx}^2 Q_h W_h\right) - \left(\sum_h^l s_{hx}^4 Q_h W_h\right)\left(\sum_h^l \overline{X}_h Q_h W_h\right)\right] \\ &- \left[\sum_h^l W_h S_{hx}^2 - \sum_h^l s_{hx}^2 W_h\right]\left[\left(\sum_h^l \overline{X}_h^2 Q_h W_h\right)\left(\sum_h^l s_{hx}^2 Q_h W_h\right) - \left(\sum_h^l s_{hx}^2\overline{X}_h Q_h W_h\right)\left(\sum_h^l \overline{X}_h Q_h W_h\right)\right]\Bigg]\end{aligned}$$

By putting the value of Ω_h, we get

The calibrated variance estimator

$$\begin{aligned}
t_1 = \sum_h^l \Bigg\{ & W_h + Q_h W_h \left[\left[\sum_h^l W_h \overline{X}_h - \sum_h^l \overline{X}_h W_h\right]\left[\left(\sum_h^l s_{hx}^2 Q_h W_h\right)^2 - \left(\sum_h^l s_{hx}^2 Q_h W_h\right)^2\right]\right. \\
& - \left[\sum_h^l W_h S_{hx}^2 + \sum_h^l s_{hx}^2 W_h\right]\left[\left(\sum_h^l s_{hx}^2 \overline{X}_h Q_h W_h\right)(Q_h W_h) + \left(\sum_h^l \overline{X}_h Q_h W_h\right)\left(\sum_h^l s_{hx}^2 Q_h W_h\right)\right]\left(\sum_h^l \overline{X}_h\right) \\
& + - \left[\sum_h^l W_h \overline{X}_h - \sum_h^l \overline{X}_h W_h\right]\left[\left(\sum_h^l \overline{X}_h s_{hx}^2 Q_h W_h\right)(Q_h W_h) - \left(\sum_h^l s_{hx}^2 Q_h W_h\right)\left(\sum_h^l \overline{X}_h Q_h W_h\right)\right] \\
& + \left[\sum_h^l W_h S_{hx}^2 - \sum_h^l s_{hx}^2 W_h\right]\left[\left(\sum_h^l \overline{X}_h^2 Q_h W_h\right)(Q_h W_h) - \left(\sum_h^l \overline{X}_h Q_h W_h\right)\left(\sum_h^l \overline{X}_h Q_h W_h\right)\right]\left(\sum_h^l s_{hx}^2\right) \\
& + \left[\sum_h^l W_h \overline{X}_h - \sum_h^l \overline{X}_h W_h\right]\left[\left(\sum_h^l \overline{X}_h s_{hx}^2 Q_h W_h\right)\left(\sum_h^l s_{hx}^2 Q_h W_h\right) - \left(\sum_h^l s_{hx}^4 Q_h W_h\right)\left(\sum_h^l \overline{X}_h Q_h W_h\right)\right] \\
& - \left.\left[\sum_h^l W_h S_{hx}^2 - \sum_h^l s_{hx}^2 W_h\right]\left[\left(\sum_h^l \overline{X}_h^2 Q_h W_h\right)\left(\sum_h^l s_{hx}^2 Q_h W_h\right) - \left(\sum_h^l s_{hx}^2 \overline{X}_h Q_h W_h\right)\left(\sum_h^l \overline{X}_h Q_h W_h\right)\right]\right]\Bigg\} s_{hy}^{2*}
\end{aligned}$$

where $s_{hy}^{*} = s_{hy}^2$.

Appendix 2

Proof for the calibrated variance ratio estimator

$$t_2 = \sum_h^l \Omega_h S_{hy}^{2*}$$

The calibration constraints

$$\begin{aligned}
\sum_h^l \Omega_h \overline{x}_h &= \sum_h^l W_h \overline{X}_h \\
\sum_h^l \Omega_h s_{hx}^2 &= \sum_h^l W_h S_{hx}^2 \\
\sum_h^l \Omega_h &= \sum_h^l W_h
\end{aligned}$$

The following langrage's function is given according to the distance measure and calibration constraints

$$\Delta = \frac{\sum_h^l (W_h - \Omega_h)^2}{Q_h W_h} - 2\lambda_1\left(\sum_h^l \Omega_h \overline{X}_h - \sum_h^l W_h \overline{X}_h\right) - 2\lambda_2\left(\sum_h^l \Omega_h s_{hx}^2 - \sum_h^l W_h S_{hx}^2\right) - 2\lambda_3\left(\sum_h^l \Omega_h - \sum_h^l W_h\right)$$

Differentiate w.r.t. Ω_h

$$\begin{aligned}
&\frac{\partial \Delta}{\partial \Omega_h} = \frac{2(\Omega_h - W_h)}{Q_h W_h} - 2\lambda_1(\overline{x}_h) - 2\lambda_2 s_{hx}^2 - 2\lambda_3 \\
&\frac{\partial \Delta}{\partial \Omega_h} = \frac{2(\Omega_h - W_h) - 2\lambda_1(\overline{x}_h Q_h W_h) - 2\lambda_2\left(s_{hx}^2 Q_h W_h\right) - 2\lambda_3(Q_h W_h)}{Q_h W_h} \\
&\frac{\partial \Delta}{\partial \Omega_h} = 0 \\
&(\Omega_h - W_h) - \lambda_1(\overline{x}_h Q_h W_h) - \lambda_2 s_{hx}^2 Q_h W_h - \lambda_3(Q_h W_h) = 0 \\
&\Omega_h = W_h + \lambda_1 \overline{x}_h Q_h W_h + \lambda_2\left(s_{hx}^2 Q_h W_h\right) + \lambda_3(Q_h W_h)
\end{aligned}$$

Substituting Ω_h in constraints

$$\lambda_1\left(\sum_h^l \overline{X}_h^2 Q_h W_h\right)+\lambda_2\left(\sum_h^l \overline{X}_h s_{hx}^2 Q_h W_h\right)+\lambda_3\left(\sum_h^l \overline{X}_h Q_h W_h\right)=\sum_h^l W_h\overline{X}_h-\sum_h^l W_h\overline{X}_h$$

$$\lambda_1\left(\sum_h^l s_{hx}^2\overline{X}_h Q_h W_h\right)+\lambda_2\left(\sum_h^l s_{hx}^4 Q_h W_h\right)+\lambda_3\left(\sum_h^l s_{hx}^2 Q_h W_h\right)=\sum_h^l W_h S_{hx}^2-\sum_h^l W_h s_{hx}^2$$

$$\lambda_1\left(\sum_h^l \overline{X}_h Q_h W_h\right)+\lambda_2\left(\sum_h^l s_{hx}^2 Q_h W_h\right)+\lambda_3\left(Q_h W_h\right)=0$$

$$\begin{bmatrix} \sum_h^l \overline{X}_h^2 Q_h W_h & \sum_h^l s_{hx}^2\overline{X}_h Q_h W_h & \sum_h^l \overline{X}_h Q_h W_h \\ \sum_h^l \overline{X}_h s_{hx}^2 Q_h W_h & \sum_h^l s_{hx}^4 Q_h W_h & \sum_h^l s_{hx}^2 Q_h W_h \\ \sum_h^l \overline{X}_h Q_h W_h & \sum_h^l s_{hx}^2 Q_h W_h & Q_h W_h \end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix}=\begin{bmatrix} \sum_h^l W_h\overline{X}_h-\sum_h^l W_h\overline{X}_h \\ \sum_h^l W_h S_{hx}^2-\sum_h^l W_h s_{hx}^2 \\ 0\end{bmatrix}$$

$$\begin{aligned} D={}&3\left[\sum \overline{X}_h Q_h W_h.\sum s_{hx}^{2} Q_h W_h.\sum Q_h W_h\right]-3\left[\sum \overline{X}_h^2 Q_h W_h\left(\sum s_{hx}^2 Q_h W_h\right)^2\right]\\ &-3\left[\left(\sum s_{hx}^2\overline{X}_h Q_h W_h\right)^2\sum Q_h W_h\right]+6\left[\sum s_{hx}^2 Q_h W_h\sum \overline{x} Q_h W_h\sum s_{hx}^2\overline{X}_h Q_h W_h\right]\\ &-3\left[\sum \overline{X}_h^2 Q_h W_h\left(\sum s_{hx}^4 Q_h W_h\right)^2\right]\end{aligned}$$

$$\begin{aligned} D={}&\left[\sum \overline{X}_h Q_h W_h.\sum s_{hx}^{2} Q_h W_h\cdot\sum Q_h W_h\right]-\left[\sum \overline{X}_h^2 Q_h W_h\left(\sum s_{hx}^2 Q_h W_h\right)^2\right]\\ &-\left[\left(\sum s_{hx}^2\overline{X}_h Q_h W_h\right)^2\sum Q_h W_h\right]+2\left[\sum s_{hx}^2 Q_h W_h\sum \overline{x} Q_h W_h\sum s_{hx}^2\overline{X}_h Q_h W_h\right]\\ &-\left[\sum \overline{X}_h^2 Q_h W_h\left(\sum s_{hx}^4 Q_h W_h\right)^2\right]\end{aligned}$$

$$\begin{bmatrix} \sum_h^l W_h\overline{X}_h-\sum_h^l \overline{X}_h W_h & \sum_h^l s_{hx}^2\overline{X}_h Q_h W_h & \sum_h^l \overline{X}_h Q_h W_h \\ \sum_h^l W_h S_{hx}^2-\sum_h^l s_{hx}^2 W_h & \sum_h^l s_{hx}^4 Q_h W_h & \sum_h^l s_{hx}^2 Q_h W_h \\ 0 & \sum_h^l s_{hx}^2 Q_h W_h & Q_h W_h \end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix}=\begin{bmatrix} \sum_h^l W_h\overline{X}_h-\sum_h^l W_h\overline{X}_h \\ \sum_h^l W_h S_{hx}^2-\sum_h^l s_{hx}^2 W_h \\ 0\end{bmatrix}$$

$$\begin{aligned} A={}&\left[\sum_h^l W_h\overline{X}_h-\sum_h^l \overline{X}_h W_h\right]\left[\left(\sum_h^l s_{hx}^2 Q_h W_h\right)^2-\left(\sum_h^l s_{hx}^2 Q_h W_h\right)^2\right]-\left[\sum_h^l W_h S_{hx}^2\right.\\ &\left.+\sum_h^l s_{hx}^2 W_h\right]\left[\left(\sum_h^l s_{hx}^2\overline{X}_h Q_h W_h\right)(Q_h W_h)+\left(\sum_h^l \overline{X}_h Q_h W_h\right)\left(\sum_h^l s_{hx}^2 Q_h W_h\right)\right]\end{aligned}$$

$$\begin{bmatrix} \sum_h^l \overline{X}_h^2 Q_h W_h & \sum_h^l W_h\overline{X}_h-\sum_h^l \overline{X}_h W_h & \sum_h^l \overline{X}_h Q_h W_h \\ \sum_h^l \overline{X}_h s_{hx}^2 Q_h W_h & \sum_h^l W_h S_{hx}^2-\sum_h^l s_{hx}^2 W_h & \sum_h^l s_{hx}^2 Q_h W_h \\ \sum_h^l \overline{X}_h Q_h W_h & 0 & Q_h W_h \end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix}=\begin{bmatrix} \sum_h^l W_h\overline{X}_h-\sum_h^l W_h\overline{X}_h \\ \sum_h^l W_h S_{hx}^2-\sum_h^l s_{hx}^2 W_h \\ 0\end{bmatrix}$$

$$\begin{aligned} B={}&-\left[\sum_h^l W_h\overline{X}_h-\sum_h^l \overline{X}_h W_h\right]\left[\left(\sum_h^l \overline{X}_h s_{hx}^2 Q_h W_h\right)(Q_h W_h)-\left(\sum_h^l s_{hx}^2 Q_h W_h\right)\left(\sum_h^l \overline{X}_h Q_h W_h\right)\right]\\ &+\left[\sum_h^l W_h S_{hx}^2-\sum_h^l s_{hx}^2 W_h\right]\left[\left(\sum_h^l \overline{X}_h^2 Q_h W_h\right)(Q_h W_h)-\left(\sum_h^l \overline{X}_h Q_h W_h\right)\left(\sum_h^l \overline{X}_h Q_h W_h\right)\right]\end{aligned}$$

$$\begin{bmatrix} \sum_{h}^{l}\overline{X}_h^2 Q_h W_h & \sum_{h}^{l} s_{hx}^2 \overline{X}_h Q_h W_h & \sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l}\overline{X}_h W_h \\ \sum_{h}^{l}\overline{X}_h s_{hx}^2 Q_h W_h & \sum_{h}^{l} s_{hx}^4 Q_h W_h & \sum_{h}^{l} W_h S_{hx}^2 - \sum_{h}^{l} s_{hx}^2 W_h \\ \sum_{h}^{l}\overline{X}_h Q_h W_h & \sum_{h}^{l} s_{hx}^2 Q_h W_h & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l}\overline{X}_h W_h \\ \sum_{h}^{l} W_h S_{hx}^2 - \sum_{h}^{l} s_{hx}^2 W_h \\ 0 \end{bmatrix}$$

$$\begin{aligned} C = & \left[\sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l}\overline{X}_h W_h\right]\left[\left(\sum_{h}^{l}\overline{X}_h s_{hx}^2 Q_h W_h\right)\left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right) - \left(\sum_{h}^{l} s_{hx}^4 Q_h W_h\right)\left(\sum_{h}^{l}\overline{X}_h Q_h W_h\right)\right] \\ & - \left[\sum_{h}^{l} W_h S_{hx}^2 - \sum_{h}^{l} s_{hx}^2 W_h\right]\left[\left(\sum_{h}^{l}\overline{X}_h^2 Q_h W_h\right)\left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right) - \left(\sum_{h}^{l} s_{hx}^2\overline{X}_h Q_h W_h\right)\left(\sum_{h}^{l}\overline{X}_h Q_h W_h\right)\right] \end{aligned}$$

$$\lambda_1 = \frac{A}{D},\ \lambda_2 = \frac{B}{D},\ \lambda_3 = \frac{C}{D}$$

Now put the $\lambda_1, \lambda_2, \lambda_3$ values in Ω_h

$$\begin{aligned} \Omega_h = & W_h + Q_h W_h \left[\left[\sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l}\overline{X}_h W_h\right]\left[\left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right)^2 - \left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right)^2\right] - \left[\sum_{h}^{l} W_h S_{hx}^2\right.\right. \\ & \left. + \sum_{h}^{l} s_{hx}^2 W_h\right]\left[\left(\sum_{h}^{l} s_{hx}^2\overline{X}_h Q_h W_h\right)(Q_h W_h) + \left(\sum_{h}^{l}\overline{X}_h Q_h W_h\right)\left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right)\right]\left(\sum_{h}^{l}\overline{X}_h\right) \\ & + - \left[\sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l}\overline{X}_h W_h\right]\left[\left(\sum_{h}^{l}\overline{X}_h s_{hx}^2 Q_h W_h\right)(Q_h W_h) - \left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right)\left(\sum_{h}^{l}\overline{X}_h Q_h W_h\right)\right] \\ & + \left[\sum_{h}^{l} W_h S_{hx}^2 - \sum_{h}^{l} s_{hx}^2 W_h\right]\left[\left(\sum_{h}^{l}\overline{X}_h^2 Q_h W_h\right)(Q_h W_h) - \left(\sum_{h}^{l}\overline{X}_h Q_h W_h\right)\left(\sum_{h}^{l}\overline{X}_h Q_h W_h\right)\right]\left(\sum_{h}^{l} s_{hx}^2\right) \\ & + \left[\sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l}\overline{X}_h W_h\right]\left[\left(\sum_{h}^{l}\overline{X}_h s_{hx}^2 Q_h W_h\right)\left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right) - \left(\sum_{h}^{l} s_{hx}^4 Q_h W_h\right)\left(\sum_{h}^{l}\overline{X}_h Q_h W_h\right)\right] \\ & \left. - \left[\sum_{h}^{l} W_h S_{hx}^2 - \sum_{h}^{l} s_{hx}^2 W_h\right]\left[\left(\sum_{h}^{l}\overline{X}_h^2 Q_h W_h\right)\left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right) - \left(\sum_{h}^{l} s_{hx}^2\overline{X}_h Q_h W_h\right)\left(\sum_{h}^{l}\overline{X}_h Q_h W_h\right)\right]\right] \end{aligned}$$

By putting the value of Ω_h, we get
The calibrated ratio variance estimator

$$\begin{aligned} t_2 = \sum_{h}^{l} \Bigg\{ & W_h + Q_h W_h \left[\left[\sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l}\overline{X}_h W_h\right]\left[\left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right)^2 - \left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right)^2\right]\right. \\ & - \left[\sum_{h}^{l} W_h S_{hx}^2 + \sum_{h}^{l} s_{hx}^2 W_h\right]\left[\left(\sum_{h}^{l} s_{hx}^2\overline{X}_h Q_h W_h\right)(Q_h W_h) + \left(\sum_{h}^{l}\overline{X}_h Q_h W_h\right)\left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right)\right]\left(\sum_{h}^{l}\overline{X}_h\right) \\ & + - \left[\sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l}\overline{X}_h W_h\right]\left[\left(\sum_{h}^{l}\overline{X}_h s_{hx}^2 Q_h W_h\right)(Q_h W_h) - \left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right)\left(\sum_{h}^{l}\overline{X}_h Q_h W_h\right)\right] \\ & + \left[\sum_{h}^{l} W_h S_{hx}^2 - \sum_{h}^{l} s_{hx}^2 W_h\right]\left[\left(\sum_{h}^{l}\overline{X}_h^2 Q_h W_h\right)(Q_h W_h) - \left(\sum_{h}^{l}\overline{X}_h Q_h W_h\right)\left(\sum_{h}^{l}\overline{X}_h Q_h W_h\right)\right]\left(\sum_{h}^{l} s_{hx}^2\right) \\ & + \left[\sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l}\overline{X}_h W_h\right]\left[\left(\sum_{h}^{l}\overline{X}_h s_{hx}^2 Q_h W_h\right)\left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right) - \left(\sum_{h}^{l} s_{hx}^4 Q_h W_h\right)\left(\sum_{h}^{l}\overline{X}_h Q_h W_h\right)\right] \\ & \left. - \left[\sum_{h}^{l} W_h S_{hx}^2 - \sum_{h}^{l} s_{hx}^2 W_h\right]\left[\left(\sum_{h}^{l}\overline{X}_h^2 Q_h W_h\right)\left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right) - \left(\sum_{h}^{l} s_{hx}^2\overline{X}_h Q_h W_h\right)\left(\sum_{h}^{l}\overline{X}_h Q_h W_h\right)\right]\right]\Bigg\} S_{hy}^{2*} \end{aligned}$$

where $S_{hy}^{2*} = S_{yh}^2 \cdot \frac{S_{xh}^2}{s_{xh}^2}$.

Appendix 3
Proof for the calibrated variance exponential estimator

$$t_3 = \sum_{h}^{l} \Omega_h S_{hy}^{2*}$$

The calibration constraints

$$\sum_{h}^{l}\Omega_h\overline{x}_h=\sum_{h}^{l}W_h\overline{X}_h$$
$$\sum_{h}^{l}\Omega_h s_{hx}^2=\sum_{h}^{l}W_h S_{hx}^2$$
$$\sum_{h}^{l}\Omega_h=\sum_{h}^{l}W_h$$

The following langrage's function is given according to the distance measure and calibration constraints

$$\Delta=\frac{\sum_{h}^{l}(W_h-\Omega_h)^2}{Q_hW_h}-2\lambda_1\left(\sum_{h}^{l}\Omega_h\overline{X}_h-\sum_{h}^{l}W_h\overline{X}_h\right)-2\lambda_2\left(\sum_{h}^{l}\Omega_h s_{hx}^2-\sum_{h}^{l}W_h S_{hx}^2\right)-2\lambda_3\left(\sum_{h}^{l}\Omega_h-\sum_{h}^{l}W_h\right)$$

Differentiate w.r.t. Ω_h

$$\frac{\partial\Delta}{\partial\Omega_h}=\frac{2(\Omega_h-W_h)}{Q_hW_h}-2\lambda_1(\overline{x}_h)-2\lambda_2 s_{hx}^2-2\lambda_3$$
$$\frac{\partial\Delta}{\partial\Omega_h}=\frac{2(\Omega_h-W_h)-2\lambda_1(\overline{x}_hQ_hW_h)-2\lambda_2\left(s_{hx}^2Q_hW_h\right)-2\lambda_3(Q_hW_h)}{Q_hW_h}$$
$$\frac{\partial\Delta}{\partial\Omega_h}=0$$
$$(\Omega_h-W_h)-\lambda_1(\overline{x}_hQ_hW_h)-\lambda_2 s_{hx}^2Q_hW_h-\lambda_3(Q_hW_h)=0$$
$$\Omega_h=W_h+\lambda_1\overline{x}_hQ_hW_h+\lambda_2\left(s_{hx}^2Q_hW_h\right)+\lambda_3(Q_hW_h)$$

Substituting Ω_h in constraints

$$\lambda_1\left(\sum_{h}^{l}\overline{X}_h^2Q_hW_h\right)+\lambda_2\left(\sum_{h}^{l}\overline{X}_h s_{hx}^2Q_hW_h\right)+\lambda_3\left(\sum_{h}^{l}\overline{X}_hQ_hW_h\right)=\sum_{h}^{l}W_h\overline{X}_h-\sum_{h}^{l}W_h\overline{X}_h$$
$$\lambda_1\left(\sum_{h}^{l}s_{hx}^2\overline{X}_hQ_hW_h\right)+\lambda_2\left(\sum_{h}^{l}s_{hx}^4Q_hW_h\right)+\lambda_3\left(\sum_{h}^{l}s_{hx}^2Q_hW_h\right)=\sum_{h}^{l}W_hS_{hx}^2-\sum_{h}^{l}W_hs_{hx}^2$$
$$\lambda_1\left(\sum_{h}^{l}\overline{X}_hQ_hW_h\right)+\lambda_2\left(\sum_{h}^{l}s_{hx}^2Q_hW_h\right)+\lambda_3(Q_hW_h)=0$$

$$\begin{bmatrix}\sum_{h}^{l}\overline{X}_h^2Q_hW_h & \sum_{h}^{l}s_{hx}^2\overline{X}_hQ_hW_h & \sum_{h}^{l}\overline{X}_hQ_hW_h\\ \sum_{h}^{l}\overline{X}_hs_{hx}^2Q_hW_h & \sum_{h}^{l}s_{hx}^4Q_hW_h & \sum_{h}^{l}s_{hx}^2Q_hW_h\\ \sum_{h}^{l}\overline{X}_hQ_hW_h & \sum_{h}^{l}s_{hx}^2Q_hW_h & Q_hW_h\end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix}=\begin{bmatrix}\sum_{h}^{l}W_h\overline{X}_h-\sum_{h}^{l}W_h\overline{X}_h\\ \sum_{h}^{l}W_hS_{hx}^2-\sum_{h}^{l}W_hs_{hx}^2\\ 0\end{bmatrix}$$

$$\begin{aligned}D=&\,3\left[\sum\overline{X}_hQ_hW_h.\sum s_{hx}^{2}Q_hW_h.\sum Q_hW_h\right]-3\left[\sum\overline{X}_h^2Q_hW_h\left(\sum s_{hx}^2Q_hW_h\right)^2\right]\\&-3\left[\left(\sum s_{hx}^2\overline{X}_hQ_hW_h\right)^2\sum Q_hW_h\right]+6\left[\sum s_{hx}^2Q_hW_h\sum\overline{x}Q_hW_h\sum s_{hx}^2\overline{X}_hQ_hW_h\right]\\&-3\left[\sum\overline{X}_h^2Q_hW_h\left(\sum s_{hx}^4Q_hW_h\right)^2\right]\end{aligned}$$

$$\begin{aligned}D=&\left[\sum\overline{X}_hQ_hW_h\cdot\sum s_{hx}^{2}Q_hW_h\cdot\sum Q_hW_h\right]-\left[\sum\overline{X}_h^2Q_hW_h\left(\sum s_{hx}^2Q_hW_h\right)^2\right]\\&-\left[\left(\sum s_{hx}^2\overline{X}_hQ_hW_h\right)^2\sum Q_hW_h\right]+2\left[\sum s_{hx}^2Q_hW_h\sum\overline{x}Q_hW_h\sum s_{hx}^2\overline{X}_hQ_hW_h\right]\\&-\left[\sum\overline{X}_h^2Q_hW_h\left(\sum s_{hx}^4Q_hW_h\right)^2\right]\end{aligned}$$

$$\begin{bmatrix} \sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l} \overline{X}_h W_h & \sum_{h}^{l} s_{hx}^2\overline{X}_h Q_h W_h & \sum_{h}^{l} \overline{X}_h Q_h W_h \\ \sum_{h}^{l} W_h S_{hx}^2 - \sum_{h}^{l} s_{hx}^2 W_h & \sum_{h}^{l} s_{hx}^4 Q_h W_h & \sum_{h}^{l} s_{hx}^2 Q_h W_h \\ 0 & \sum_{h}^{l} s_{hx}^2 Q_h W_h & Q_h W_h \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l} W_h\overline{X}_h \\ \sum_{h}^{l} W_h S_{hx}^2 - \sum_{h}^{l} s_{hx}^2 W_h \\ 0 \end{bmatrix}$$

$$A = \left[\sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l} \overline{X}_h W_h\right]\left[\left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right)^2 - \left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right)^2\right] - \left[\sum_{h}^{l} W_h S_{hx}^2 + \sum_{h}^{l} s_{hx}^2 W_h\right]\left[\left(\sum_{h}^{l} s_{hx}^2\overline{X}_h Q_h W_h\right)(Q_h W_h) + \left(\sum_{h}^{l} \overline{X}_h Q_h W_h\right)\left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right)\right]$$

$$\begin{bmatrix} \sum_{h}^{l} \overline{X}_h^2 Q_h W_h & \sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l} \overline{X}_h W_h & \sum_{h}^{l} \overline{X}_h Q_h W_h \\ \sum_{h}^{l} \overline{X}_h s_{hx}^2 Q_h W_h & \sum_{h}^{l} W_h S_{hx}^2 - \sum_{h}^{l} s_{hx}^2 W_h & \sum_{h}^{l} s_{hx}^2 Q_h W_h \\ \sum_{h}^{l} \overline{X}_h Q_h W_h & 0 & Q_h W_h \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l} W_h\overline{X}_h \\ \sum_{h}^{l} W_h S_{hx}^2 - \sum_{h}^{l} s_{hx}^2 W_h \\ 0 \end{bmatrix}$$

$$B = -\left[\sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l} \overline{X}_h W_h\right]\left[\left(\sum_{h}^{l} \overline{X}_h s_{hx}^2 Q_h W_h\right)(Q_h W_h) - \left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right)\left(\sum_{h}^{l} \overline{X}_h Q_h W_h\right)\right] + \left[\sum_{h}^{l} W_h S_{hx}^2 - \sum_{h}^{l} s_{hx}^2 W_h\right]\left[\left(\sum_{h}^{l} \overline{X}_h^2 Q_h W_h\right)(Q_h W_h) - \left(\sum_{h}^{l} \overline{X}_h Q_h W_h\right)\left(\sum_{h}^{l} \overline{X}_h Q_h W_h\right)\right]$$

$$\begin{bmatrix} \sum_{h}^{l} \overline{X}_h^2 Q_h W_h & \sum_{h}^{l} s_{hx}^2\overline{X}_h Q_h W_h & \sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l} \overline{X}_h W_h \\ \sum_{h}^{l} \overline{X}_h s_{hx}^2 Q_h W_h & \sum_{h}^{l} s_{hx}^4 Q_h W_h & \sum_{h}^{l} W_h S_{hx}^2 - \sum_{h}^{l} s_{hx}^2 W_h \\ \sum_{h}^{l} \overline{X}_h Q_h W_h & \sum_{h}^{l} s_{hx}^2 Q_h W_h & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l} \overline{X}_h W_h \\ \sum_{h}^{l} W_h S_{hx}^2 - \sum_{h}^{l} s_{hx}^2 W_h \\ 0 \end{bmatrix}$$

$$C = \left[\sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l} \overline{X}_h W_h\right]\left[\left(\sum_{h}^{l} \overline{X}_h s_{hx}^2 Q_h W_h\right)\left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right) - \left(\sum_{h}^{l} s_{hx}^4 Q_h W_h\right)\left(\sum_{h}^{l} \overline{X}_h Q_h W_h\right)\right] - \left[\sum_{h}^{l} W_h S_{hx}^2 - \sum_{h}^{l} s_{hx}^2 W_h\right]\left[\left(\sum_{h}^{l} \overline{X}_h^2 Q_h W_h\right)\left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right) - \left(\sum_{h}^{l} s_{hx}^2\overline{X}_h Q_h W_h\right)\left(\sum_{h}^{l} \overline{X}_h Q_h W_h\right)\right]$$

$$\lambda_1 = \frac{A}{D},\ \lambda_2 = \frac{B}{D},\ \lambda_3 = \frac{C}{D}$$

Now put the $\lambda_1,\ \lambda_2,\ \lambda_3$ values in Ω_h

$$\Omega_h = W_h + Q_h W_h \Bigg[\left[\sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l} \overline{X}_h W_h\right]\left[\left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right)^2 - \left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right)^2\right] - \left[\sum_{h}^{l} W_h S_{hx}^2 + \sum_{h}^{l} s_{hx}^2 W_h\right]\left[\left(\sum_{h}^{l} s_{hx}^2\overline{X}_h Q_h W_h\right)(Q_h W_h) + \left(\sum_{h}^{l} \overline{X}_h Q_h W_h\right)\left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right)\right]\left(\sum_{h}^{l} \overline{X}_h\right) + -\left[\sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l} \overline{X}_h W_h\right]\left[\left(\sum_{h}^{l} \overline{X}_h s_{hx}^2 Q_h W_h\right)(Q_h W_h) - \left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right)\left(\sum_{h}^{l} \overline{X}_h Q_h W_h\right)\right] + \left[\sum_{h}^{l} W_h S_{hx}^2 - \sum_{h}^{l} s_{hx}^2 W_h\right]\left[\left(\sum_{h}^{l} \overline{X}_h^2 Q_h W_h\right)(Q_h W_h) - \left(\sum_{h}^{l} \overline{X}_h Q_h W_h\right)\left(\sum_{h}^{l} \overline{X}_h Q_h W_h\right)\right]\left(\sum_{h}^{l} s_{hx}^2\right) + \left[\sum_{h}^{l} W_h\overline{X}_h - \sum_{h}^{l} \overline{X}_h W_h\right]\left[\left(\sum_{h}^{l} \overline{X}_h s_{hx}^2 Q_h W_h\right)\left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right) - \left(\sum_{h}^{l} s_{hx}^4 Q_h W_h\right)\left(\sum_{h}^{l} \overline{X}_h Q_h W_h\right)\right] - \left[\sum_{h}^{l} W_h S_{hx}^2 - \sum_{h}^{l} s_{hx}^2 W_h\right]\left[\left(\sum_{h}^{l} \overline{X}_h^2 Q_h W_h\right)\left(\sum_{h}^{l} s_{hx}^2 Q_h W_h\right) - \left(\sum_{h}^{l} s_{hx}^2\overline{X}_h Q_h W_h\right)\left(\sum_{h}^{l} \overline{X}_h Q_h W_h\right)\right]\Bigg]$$

By putting the value of Ω_h, we get the calibrated exponential variance estimator is given by

$$
\begin{aligned}
t_3 = \sum_{h}^{l} \Bigg\{ & W_h + Q_h W_h \Bigg[\Bigg[\sum_{h}^{l} W_h \overline{X}_h - \sum_{h}^{l} \overline{X}_h W_h \Bigg] \Bigg[\Bigg(\sum_{h}^{l} s_{hx}^2 Q_h W_h \Bigg)^2 - \Bigg(\sum_{h}^{l} s_{hx}^2 Q_h W_h \Bigg)^2 \Bigg] \\
& - \Bigg[\sum_{h}^{l} W_h S_{hx}^2 + \sum_{h}^{l} s_{hx}^2 W_h \Bigg] \Bigg[\Bigg(\sum_{h}^{l} s_{hx}^2 \overline{X}_h Q_h W_h \Bigg) (Q_h W_h) + \Bigg(\sum_{h}^{l} \overline{X}_h Q_h W_h \Bigg) \Bigg(\sum_{h}^{l} s_{hx}^2 Q_h W_h \Bigg) \Bigg] \Bigg(\sum_{h}^{l} \overline{X}_h \Bigg) \\
& + - \Bigg[\sum_{h}^{l} W_h \overline{X}_h - \sum_{h}^{l} \overline{X}_h W_h \Bigg] \Bigg[\Bigg(\sum_{h}^{l} \overline{X}_h s_{hx}^2 Q_h W_h \Bigg) (Q_h W_h) - \Bigg(\sum_{h}^{l} s_{hx}^2 Q_h W_h \Bigg) \Bigg(\sum_{h}^{l} \overline{X}_h Q_h W_h \Bigg) \Bigg] \\
& + \Bigg[\sum_{h}^{l} W_h S_{hx}^2 - \sum_{h}^{l} s_{hx}^2 W_h \Bigg] \Bigg[\Bigg(\sum_{h}^{l} \overline{X}_h^2 Q_h W_h \Bigg) (Q_h W_h) - \Bigg(\sum_{h}^{l} \overline{X}_h Q_h W_h \Bigg) \Bigg(\sum_{h}^{l} \overline{X}_h Q_h W_h \Bigg) \Bigg] \Bigg(\sum_{h}^{l} s_{hx}^2 \Bigg) \\
& + \Bigg[\sum_{h}^{l} W_h \overline{X}_h - \sum_{h}^{l} \overline{X}_h W_h \Bigg] \Bigg[\Bigg(\sum_{h}^{l} \overline{X}_h s_{hx}^2 Q_h W_h \Bigg) \Bigg(\sum_{h}^{l} s_{hx}^2 Q_h W_h \Bigg) - \Bigg(\sum_{h}^{l} s_{hx}^4 Q_h W_h \Bigg) \Bigg(\sum_{h}^{l} \overline{X}_h Q_h W_h \Bigg) \Bigg] \\
& - \Bigg[\sum_{h}^{l} W_h S_{hx}^2 - \sum_{h}^{l} s_{hx}^2 W_h \Bigg] \Bigg[\Bigg(\sum_{h}^{l} \overline{X}_h^2 Q_h W_h \Bigg) \Bigg(\sum_{h}^{l} s_{hx}^2 Q_h W_h \Bigg) - \Bigg(\sum_{h}^{l} s_{hx}^2 \overline{X}_h Q_h W_h \Bigg) \Bigg(\sum_{h}^{l} \overline{X}_h Q_h W_h \Bigg) \Bigg] \Bigg] \Bigg\} S_{hy}^{2*}
\end{aligned}
$$

where $S_{hy}^{2*} = s_{hy}^2 \exp\left[\frac{S_{hx}^2 - s_{hx}^2}{S_{hx}^2 + s_{hx}^2}\right]$.

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Author contributions

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Competing interests

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Additional information

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