On Compactness of Products of Toeplitz Operators

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Abstract

We study compactness of product of Toeplitz operators with symbols continuous on the closure of the polydisc in terms of behavior of the symbols on the boundary. For certain classes of symbols f and g, we show that $T_f T_g$ is compact if and only if fg vanishes on the boundary. We provide examples to show that for more general symbols, the vanishing of fg on the whole polydisc might not imply the compactness of $T_f T_g$. On the other hand, the reverse direction is closely related to the zero product problem for Toeplitz operators on the unit disc, which is still open.

Keywords Toeplitz operator · Compact · Bergman space · Polydisc

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1 Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$. The Bergman space $A^2(\Omega)$ consists of all holomorphic functions on Ω that are square integrable with respect to the Lebesgue volume measure dV. The orthogonal projection $P : L^2(\Omega) \to A^2(\Omega)$ is known as the Bergman projection. For a bounded measurable function f on Ω, the Toeplitz operator $T_f : A^2(\Omega) \to A^2(\Omega)$ is defined as

$$T_f h = P(fh)$$

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for $h \in A^2(\Omega)$. We call f the symbol of T_f.

There is an extensive literature on the study of Toeplitz operators on various domains. In this paper, we are particularly interested in the case the domain is the polydisc and compactness of product of Toeplitz operators whose symbols are continuous up to the boundary.

A classical approach to compactness of Toeplitz operators involves the Berezin transform. For finite sum of finite products of Toeplitz operators on the Bergman space of the unit disc, the Axler–Zheng Theorem [1, Theorem 2.2] characterizes compactness in terms of the behavior of the Berezin transform of the operator. In higher dimensions, the Axler–Zheng Theorem is extended to the case of the polydisc as seen in [2] and [3, p. 232], and the unit ball as shown in [4, Theorem 9.5]. Recently, there have been a few generalizations of this result in different directions. See, for instance, [5–8].

In this paper, we study compactness of products of Toeplitz operators in terms of the behavior of the symbols on the boundary. More specifically, we would like to characterize functions f, g that are continuous on $\overline{\mathbb{D}^n}$ such that $T_f T_g$ is compact.

Coburn [9, Lemma 2] showed that on the Bergman space over unit ball $\mathbb{B}$, for f a continuous function on $\overline{\mathbb{B}}$, the Toeplitz operator T_f is compact if and only if $f = 0$ on $b\mathbb{B}$. Furthermore, [9, Theorem 1] established a $*$-isomorphism $\sigma : \tau(\mathbb{B})/\mathscr{K} \to C(b\mathbb{B})$ satisfying

$$\sigma(T_f + \mathscr{K}) = f|_{b\mathbb{B}},$$

where $\tau(\mathbb{B})$ is the Toeplitz algebra generated by $\{T_\varphi : \varphi \in C(\overline{\mathbb{B}})\}$ and $\mathscr{K}$ is the ideal of compact operators on $A^2(\mathbb{B})$. As a consequence, we see that for $f_1, \ldots, f_N \in C(\overline{\mathbb{B}})$, the product $T_{f_1} \cdots T_{f_N}$ is compact if and only if the product $f_1 \cdots f_N = 0$ on $b\mathbb{B}$.

On the polydisc $\mathbb{D}^n$, the first author [10] showed that, in the context of weighted Bergman spaces, for $f \in C(\overline{\mathbb{D}^n})$, the Toeplitz operator T_f is compact if and only if f vanishes on $b\mathbb{D}^n$, the (topological) boundary of $\mathbb{D}^n$. Generalizing this result, the second and the third authors in [11] proved that compactness of the Toeplitz operator with a symbol continuous on the closure of a bounded pseudoconvex domain in $\mathbb{C}^n$ with Lipschitz boundary is equivalent to the symbol vanishing on the boundary of the domain.

Motivated by Coburn's aforementioned result, one may expect that the necessary and sufficient condition for $T_f T_g$ to be compact is that fg vanishes on $b\mathbb{D}^n$. However, we shall see in our results and examples that while the above statement holds for a certain class of symbols, sufficiency is false in general. On the other hand, necessity is closely related with the famous "zero product problem" in the theory of Toeplitz operators on the unit disc, which is still wide open.

2 Main Result

Let $T = \sum_{j=1}^N T_{f_{j,1}} \cdots T_{f_{j,m_j}}$ be a finite sum of finite products of Toeplitz operators with $f_{j,k} \in C(\overline{\mathbb{D}})$. Coburn's aforementioned result implies that compactness of the operator T on $A^2(\mathbb{D})$ is equivalent to $\sum_{j=1}^N f_{j,1} \cdots f_{j,m_j} = 0$ on the circle. Therefore, throughout the paper we will assume that $n \geq 2$ as the case $n = 1$ is well understood.

Before we state our results, we define the restriction operator $R_{k,\xi} : C(\overline{\mathbb{D}^n}) \to C(\overline{\mathbb{D}^{n-1}})$ for $\xi \in \mathbb{T}$ and $k = 1, \ldots, n$ as follows.

$$R_{1,\xi} f(z_1, \ldots, z_{n-1}) = f(\xi, z_1, \ldots, z_{n-1}),$$
$$R_{n,\xi} f(z_1, \ldots, z_{n-1}) = f(z_1, \ldots, z_{n-1}, \xi),$$

and

$$R_{k,\xi} f(z_1, \ldots, z_{n-1}) = f(z_1, \ldots, z_{k-1}, \xi, z_k, \ldots, z_{n-1})$$

for $2 \leq k \leq n - 1$ and $f \in C(\overline{\mathbb{D}^n})$.

In our main result, we give a characterization of compactness of the finite sum of finite products of Toeplitz operators in terms of the vanishing of the operator restricted to the polydiscs in the boundary. We recall that $b\mathbb{D}^n$ consists of all $z = (z_1, \ldots, z_n) \in \overline{\mathbb{D}^n}$ such that $|z_j| = 1$ for some j.

Theorem 1 *Let $T = \sum_{j=1}^N T_{f_{j,1}} \cdots T_{f_{j,m_j}}$ be a finite sum of finite products of Toeplitz operators on $A^2(\mathbb{D}^n)$ for $f_{j,k} \in C(\overline{\mathbb{D}^n})$ with $n \geq 2$. Then T is compact on $A^2(\mathbb{D}^n)$ if and only if*

$$\sum_{j=1}^N T_{R_{k,\xi} f_{j,1}} \cdots T_{R_{k,\xi} f_{j,m_j}} = 0$$

on $A^2(\mathbb{D}^{n-1})$ for all $\xi \in \mathbb{T}$ and $1 \leq k \leq n$.

As an immediate corollary we get the following.

Corollary 1 *Let $f_j \in C(\overline{\mathbb{D}^n})$ for $1 \leq j \leq m$. Assume that for each $\xi \in \mathbb{T}$ and $1 \leq k \leq n$ there exists j such that $R_{k,\xi} f_j = 0$ on $\mathbb{D}^{n-1}$. Then $T_{f_m} \cdots T_{f_1}$ is compact on $A^2(\mathbb{D}^n)$.*

3 Applications

Let φ and ψ be two functions in $C(\overline{\mathbb{D}})$. We define $f(z, w) = \varphi(w)$ and $g(z, w) = \psi(w)$ for $z, w \in \overline{\mathbb{D}}$. Then for any $\xi \in \mathbb{T}$,

$$R_{1,\xi} f(w) = f(\xi, w) = \varphi(w), \quad R_{1,\xi} g(w) = g(\xi, w) = \psi(w) \quad \text{for } w \in \mathbb{D}$$

and

$$R_{2,\xi} f(z) = \varphi(\xi), \quad R_{2,\xi} g(z) = \psi(\xi) \quad \text{for } z \in \mathbb{D}.$$

By Theorem 1, the product $T_f T_g$ is a compact operator on $A^2(\mathbb{D}^2)$ if and only if $T_\varphi T_\psi = 0$ on $A^2(\mathbb{D})$ and $\varphi(\xi)\psi(\xi) = 0$ for all $\xi \in \mathbb{T}$. Since the second condition is actually a consequence of the first, we conclude that for such f and g, the product $T_f T_g$ is compact on $A^2(\mathbb{D}^2)$ if and only if $T_\varphi T_\psi = 0$ on $A^2(\mathbb{D})$, which is equivalent to $T_f T_g = 0$ on $A^2(\mathbb{D}^2)$.

Example 1 Let

$$\varphi(w) = \begin{cases} 1 - 2|w| & \text{for } 0 \le |w| \le \frac{1}{2} \\ 0 & \text{for } |w| > \frac{1}{2}, \end{cases}$$

and

$$\psi(w) = \begin{cases} 0 & \text{for } 0 \le |w| \le \frac{1}{2} \\ 2|w| - 1 & \text{for } |w| > \frac{1}{2}. \end{cases}$$

Using polar coordinates, one can check that both operators T_φ and T_ψ are diagonalizable with respect to the standard orthonormal basis and their eigenvalues are all strictly positive. Hence $T_\varphi T_\psi \not\equiv 0$ on $A^2(\mathbb{D})$. On the other hand, $\varphi\psi = 0$ on $\overline{\mathbb{D}}$. Then for $f(z, w) = \varphi(w)$ and $g(z, w) = \psi(w)$, we have $fg = 0$ on $\overline{\mathbb{D}^2}$ but $T_f T_g$ is not compact on $A^2(\mathbb{D}^2)$ as $T_\varphi T_\psi \not\equiv 0$. This example shows that the vanishing of fg on $b\mathbb{D}^2$ (or even on $\overline{\mathbb{D}^2}$) does not imply the compactness of $T_f T_g$.

Example 2 Take f as in Example 1 and define

$$g(z, w) = \varphi(z) + \psi(w).$$

Then fg is not identically zero on $\mathbb{D}^2$ because $f(0, 0) = g(0, 0) = 1$ and $fg = 0$ on $b\mathbb{D}^2$. Yet, by Theorem 1, the product $T_f T_g$ is not compact since for $\xi \in \mathbb{T}$,

$$T_{R_{1,\xi}f} T_{R_{1,\xi}g} = T_\varphi T_\psi$$

is not the zero operator on $A^2(\mathbb{D})$.

Remark 1 From the previous examples we see that $fg = 0$ on $b\mathbb{D}^2$ is not a sufficient condition for the compactness of $T_f T_g$. Is it a necessary condition? It turns out this question is related to the zero product problem for Toeplitz operators on the disc. More specifically, as in Example 1, we see that with $f(z, w) = \varphi(w)$ and $g(z, w) = \psi(w)$, if the product $T_f T_g$ is compact on $A^2(\mathbb{D}^2)$, then $T_\varphi T_\psi = 0$ on $A^2(\mathbb{D})$ (which gives $\varphi\psi = 0$ on $\mathbb{T}$). However, it is not known if this condition implies that $\varphi\psi = 0$ on $\mathbb{D}$. For $\xi \in \mathbb{T}$ and $z, w \in \mathbb{D}$, we have $f(\xi, w)g(\xi, w) = \varphi(w)\psi(w)$ and $f(z, \xi)g(z, \xi) = \varphi(\xi)\psi(\xi)$. So $fg = 0$ on $b\mathbb{D}^2$ if and only if $\varphi\psi = 0$ on $\mathbb{D}$.

In Proposition 1 below, we show that if the symbols are harmonic along the discs in the boundary, then we have necessary and sufficient conditions for the compactness of the product of two Toeplitz operators. A function $f \in C^2(\mathbb{D}^n)$ is said to be n-harmonic if

$$\Delta_j f = 4 \frac{\partial^2 f}{\partial z_j \partial \overline{z}_j} = 0,$$

for all $j = 1, 2, \ldots, n$. That is, f is harmonic in each variable separately [12, pg. 16].

Proposition 1 *Let $f, g \in C(\overline{\mathbb{D}^n})$ (with $n \ge 2$) such that for $\xi \in \mathbb{T}$, and $1 \le k \le n$, the functions $R_{k,\xi}f$ and $R_{k,\xi}g$ are $(n-1)$-harmonic on $\mathbb{D}^{n-1}$. Then $T_f T_g$ is compact if and only if $fg = 0$ on $b\mathbb{D}^n$.*

We note that in Example 1, both f and g depend on the same single variable. In Proposition 2 below, we give a characterization when the symbols are product of single-variable functions.

Proposition 2 *Let* $T = \prod_{k=1}^M T_{f_k}$ *be a finite product of Toeplitz operators on* $A^2(\mathbb{D}^n)$ *such that* $f_k(z) = \prod_{j=1}^n f_{j,k}(z_j)$ *for* $f_{j,k} \in C(\overline{\mathbb{D}})$ *and* $z = (z_1, \ldots, z_n) \in \mathbb{D}^n$. *Let* $F = \prod_{k=1}^M f_k$. *Then the following statements hold.*

   (i) *If* T *is a nonzero compact operator, then* $F = 0$ *on* $b\mathbb{D}^n$.
   (ii) *If* $F = 0$ *on* $b\mathbb{D}^n$ *and* F *is not identically zero on* $\mathbb{D}^n$, *then* T *is compact.*

Remark 2 We do not know whether (i) in Proposition 2 still holds in the case T is the zero operator. This is closely related to the zero product problem. More specifically, consider $f(z, w) = \varphi(w)$ and $g(z, w) = \psi(w)$, where $\varphi, \psi \in C(\overline{\mathbb{D}})$. Then $T = T_f T_g = 0$ on $A^2(\mathbb{D}^2)$ if and only if $T_\varphi T_\psi = 0$ on $A^2(\mathbb{D})$. On the other hand, $F = fg = 0$ on $b\mathbb{D}^n$ if and only if $\varphi\psi = 0$ on $\mathbb{D}$. It is still an open problem whether $T_\varphi T_\psi = 0$ on $A^2(\mathbb{D})$ implies that $\varphi\psi = 0$ on $\mathbb{D}$.

Remark 3 The conclusion of (ii) in Proposition 2 does not hold if F is identically zero on $\mathbb{D}^n$. Indeed, the functions f and g in Example 1 are of the type considered here and $F = fg = 0$ on $\overline{\mathbb{D}^2}$ but $T_f T_g$ is not compact on $A^2(\mathbb{D}^2)$.

In the proposition below, we show that when all but at most one of the symbols are polynomials, compactness of a Toeplitz product on $A^2(\mathbb{D}^2)$ is equivalent to the vanishing of the product of the symbols on $b\mathbb{D}^2$. For this result, we need to restrict to dimension two. It would be interesting to extend the result to all $n \geq 2$. See Remark 4.

Proposition 3 *Let* $f_1, \ldots, f_M$ *and* $g_1, \ldots, g_N$ *be polynomials in* z, w *and* $\overline{z}, \overline{w}$, *and* $h \in C(\overline{\mathbb{D}^2})$. *Then* $T_{f_1} \cdots T_{f_M} T_h T_{g_1} \cdots T_{g_N}$ *is compact on* $A^2(\mathbb{D}^2)$ *if and only if*

$$f_1 \cdots f_M h g_1 \cdots g_N = 0 \text{ on } b\mathbb{D}^2.$$

4 Proofs

Let $BT(p)$ denote the Berezin transform of a bounded linear operator $T : A^2(\mathbb{D}^n) \to A^2(\mathbb{D}^n)$ at $p \in \mathbb{D}^n$. That is,

$$BT(p) = \langle Tk_p, k_p \rangle$$

where

$$k_p(z) = \frac{K(z, p)}{\sqrt{K(p, p)}}$$

is the normalized Bergman kernel of $\mathbb{D}^n$.

We will need the following lemma whose proof is contained in the proof of Theorem 1 in [13]. We provide a sketch of the proof here for the convenience of the reader. We note that Bf denotes BT_f whenever f is a bounded function and we use the following notation: $z' = (z_2, \ldots, z_n) \in \mathbb{C}^{n-1}$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. For functions h_1

defined on $\mathbb{D}$ and h_2 defined on $\mathbb{D}^{n-1}$, we use $h_1 h_2$ to denote the function $h_1(z_1)h_2(z')$ on $\mathbb{D}^n$.

Lemma 1 *Suppose $n \geq 2$ and $\psi \in C(\overline{\mathbb{D}^n})$. Let $q = (\zeta, q') \in \mathbb{T} \times \overline{\mathbb{D}^{n-1}}$ and define $\psi_\zeta(z) = \psi(\zeta, z')$ for $z \in \mathbb{D}^n$.*

(i) If $\{h_p : p \in \mathbb{D}^n\}$ is a bounded set in $L^2(\mathbb{D}^{n-1})$, then

$$\lim_{p \to q} \left\| (\psi - \psi_\zeta) k_{p_1}^{\mathbb{D}} h_p \right\| = 0.$$

(ii) If $\psi_1, \ldots, \psi_v \in C(\overline{\mathbb{D}^n})$ are functions independent of z_1 and W is any bounded operator on $L^2(\mathbb{D}^n)$, then

$$\lim_{p \to q} \left\| W T_{\psi - \psi_\zeta} T_{\psi_1} \cdots T_{\psi_v} k_p \right\| = 0.$$

Proof (i) Let $\epsilon > 0$ be given. By the uniform continuity of ψ, there exists $\delta > 0$ such that for all $z' \in \mathbb{D}^{n-1}$,

$$|\psi(z_1, z') - \psi_\xi(z_1, z')| < \frac{\epsilon}{\sup\{\|h_p\|_{L^2(\mathbb{D}^{n-1})}\} + 1} \quad \text{whenever } |z_1 - \xi| < \delta.$$

Then,

$$\begin{aligned}
\|(\psi - \psi_\xi) k_{p_1}^{\mathbb{D}} h_p\|^2 &= \|(\psi - \psi_\xi) k_{p_1}^{\mathbb{D}} h_p\|^2_{L^2(\{z \in \mathbb{D}^n : |z_1 - \xi| < \delta\})} \\
&\quad + \|(\psi - \psi_\xi) k_{p_1}^{\mathbb{D}} h_p\|^2_{L^2(\{z \in \mathbb{D}^n : |z_1 - \xi| \geq \delta\})} \\
&\leq \epsilon^2 + \pi \|h_p\|^2_{L^2(\mathbb{D}^{n-1})} \|(\psi - \psi_\xi) k_{p_1}^{\mathbb{D}}\|^2_{L^\infty(\{z \in \mathbb{D}^n : |z_1 - \xi| \geq \delta\})}.
\end{aligned}$$

However,

$$\sup \left\{ \left| k_{p_1}^{\mathbb{D}}(z_1) \right| : |z_1 - \xi| \geq \delta \right\} \to 0 \text{ as } p_1 \to \xi.$$

Then, $\limsup_{p \to q} \|(\psi - \psi_\xi) k_{p_1}^{\mathbb{D}} h_p\| \leq \epsilon$. Since $\epsilon > 0$ was arbitrary, we conclude that

$$\lim_{p \to q} \left\| (\psi - \psi_\zeta) k_{p_1}^{\mathbb{D}} h_p \right\| = 0.$$

(ii) We note that $k_p = k_{p_1}^{\mathbb{D}} k_{p'}^{\mathbb{D}^{n-1}}$ for $p = (p_1, p')$. We define

$$h_p = T_{\psi_1} \cdots T_{\psi_v} k_{p'}^{\mathbb{D}^{n-1}} \quad \text{for } p \in \mathbb{D}^n.$$

Since each ψ_j is independent of z_1, h_p is independent of z_1 and hence it can be considered as an element of $L^2(\mathbb{D}^{n-1})$. Note that the set $\{h_p : p \in \mathbb{D}^n\}$ is bounded by $\|T_{\psi_1} \cdots T_{\psi_v}\|$. Furthermore, we have $T_{\psi_1} \cdots T_{\psi_v} k_p = k_{p_1}^{\mathbb{D}} h_p$. It follows that

$$\left\| W T_{\psi - \psi_\zeta} T_{\psi_1} \cdots T_{\psi_v} k_p \right\| \leq \|W\| \cdot \left\| (\psi - \psi_\zeta) k_{p_1}^{\mathbb{D}} h_p \right\|,$$

which, by (i), converges to zero as $p \to q$. $\square$

Proof of Theorem 1 We first make an observation. If φ is a bounded function on $\mathbb{D}^{n-1}$, then T_φ, while initially defined on $A^2(\mathbb{D}^{n-1})$, can be naturally considered as a Toeplitz operator with symbol $E_1\varphi(z_1, z') = \varphi(z')$ acting on $A^2(\mathbb{D}^n)$. This will not create any confusion due to the fact that for $h \in A^2(\mathbb{D}^n)$ independent of z_1, the function $T_{E_1\varphi}h$ is also independent of z_1 and $(T_{E_1\varphi}h)(z) = (T_\varphi h)(z')$ for all $z = (z_1, z') \in \mathbb{D}^n$.

Let $\xi \in \mathbb{T}$. For each j and m_j, the function f_{j,m_j} can be written as $f_{j,m_j} = (f_{j,m_j} - R_{1,\xi}f_{j,m_j}) + R_{1,\xi}f_{j,m_j}$. We then expand $T = \sum_{j=1}^{N} T_{f_{j,1}} \cdots T_{f_{j,m_j}}$ as

$$
\begin{aligned}
T = \sum_{j=1}^{N} &\left(T_{R_{1,\xi}f_{j,1}} \cdots T_{R_{1,\xi}f_{j,m_j}} + T_{f_{j,1}-R_{1,\xi}f_{j,1}} T_{R_{1,\xi}f_{j,2}} \cdots T_{R_{1,\xi}f_{j,m_j}} + T_{f_{j,1}} \right. \\
&\left. \cdot T_{f_{j,2}-R_{1,\xi}f_{j,2}} T_{R_{1,\xi}f_{j,3}} \cdots T_{R_{1,\xi}f_{j,m_j}} + \cdots + T_{f_{j,1}} T_{f_{j,2}} \cdots T_{f_{j,m_j-1}} T_{f_{j,m_j}-R_{1,\xi}f_{j,m_j}} \right) \\
= \sum_{j=1}^{N} &T_{R_{1,\xi}f_{j,1}} \cdots T_{R_{1,\xi}f_{j,m_j}} + \sum_{j=1}^{N} \left(T_{f_{j,1}-R_{1,\xi}f_{j,1}} T_{R_{1,\xi}f_{j,2}} \cdots T_{R_{1,\xi}f_{j,m_j}} + T_{f_{j,1}} \right. \\
&\left. \cdot T_{f_{j,2}-R_{1,\xi}f_{j,2}} T_{R_{1,\xi}f_{j,3}} \cdots T_{R_{1,\xi}f_{j,m_j}} + \cdots + T_{f_{j,1}} T_{f_{j,2}} \cdots T_{f_{j,m_j-1}} T_{f_{j,m_j}-R_{1,\xi}f_{j,m_j}} \right).
\end{aligned}
$$

Note that in the second sum, each summand has the form considered in Lemma 1(ii). We then conclude that for any $q = (\xi, q') \in \mathbb{T} \times \overline{\mathbb{D}^{n-1}}$,

$$
\lim_{p \to q} \left\| Tk_p - \sum_{j=1}^{N} T_{R_{1,\xi}f_{j,1}} \cdots T_{R_{1,\xi}f_{j,m_j}} k_p \right\| = 0. \tag{1}
$$

Now suppose that T is compact. Fix $p' \in \mathbb{D}^{n-1}$. Since $k_{(p_1,p')} \to 0$ weakly as $p_1 \to \xi$, the compactness of T implies that $\|Tk_{(p_1,p')}\| \to 0$ as $p_1 \to \xi$. Equation (1) then gives

$$
\lim_{p_1 \to \xi} \left\| \sum_{j=1}^{N} T_{R_{1,\xi}f_{j,1}} \cdots T_{R_{1,\xi}f_{j,m_j}} k_{(p_1,p')} \right\| = 0. \tag{2}
$$

Since

$$
\begin{aligned}
\sum_{j=1}^{N} T_{R_{1,\xi}f_{j,1}} \cdots T_{R_{1,\xi}f_{j,m_j}} k_{(p_1,p')} &= \sum_{j=1}^{N} T_{R_{1,\xi}f_{j,1}} \cdots T_{R_{1,\xi}f_{j,m_j}} (k_{p_1}^{\mathbb{D}} k_{p'}^{\mathbb{D}^{n-1}}) \\
&= k_{p_1}^{\mathbb{D}} \cdot \sum_{j=1}^{N} T_{R_{1,\xi}f_{j,1}} \cdots T_{R_{1,\xi}f_{j,m_j}} k_{p'}^{\mathbb{D}^{n-1}}
\end{aligned}
$$

and $\|k_{p_1}^{\mathbb{D}}\| = 1$ for all p_1, (2) implies that

$$
\sum_{j=1}^{N} T_{R_{1,\xi}f_{j,1}} \cdots T_{R_{1,\xi}f_{j,m_j}} k_{p'}^{\mathbb{D}^{n-1}} = 0.
$$

Because p' was arbitrary, it follows that $\sum_{j=1}^{N} T_{R_{1,\xi} f_{j,1}} \cdots T_{R_{1,\xi} f_{j,m_j}}$ is the zero operator on $A^2(\mathbb{D}^{n-1})$. Applying the same method for other values of k, we have

$$\sum_{j=1}^{N} T_{R_{k,\xi} f_{j,1}} \cdots T_{R_{k,\xi} f_{j,m_j}} = 0$$

on $A^2(\mathbb{D}^{n-1})$ for $1 \leq k \leq n$ and all $\xi \in \mathbb{T}$.

We now prove the converse. Let $q = (\xi, q') \in b\mathbb{D}^n$ with $\xi \in \mathbb{T}$ and $q' \in \overline{\mathbb{D}^{n-1}}$. Since it is assumed that $\sum_{j=1}^{N} T_{R_{1,\xi} f_{j,1}} \cdots T_{R_{1,\xi} f_{j,m_j}} = 0$, equation (1) implies that $\lim_{p \to q} \|Tk_p\| = 0$. As a consequence,

$$\lim_{p \to q} BT(p) = \lim_{p \to q} \langle Tk_p, k_p \rangle = 0.$$

The same argument is applicable for all $q \in b\mathbb{D}^n$. By Axler–Zheng Theorem for $\mathbb{D}^n$ ([2] and [3, p. 232]), we conclude that T is compact on $A^2(\mathbb{D}^n)$.

Proof of Corollary 1 We assume that for each $\xi \in \mathbb{T}$ and $1 \leq k \leq n$ there exists j such that $R_{k,\xi} f_j = 0$. Then $T_{R_{k,\xi} f_m} \cdots T_{R_{k,\xi} f_1} = 0$ on $A^2(\mathbb{D}^{n-1})$. Hence, Theorem 1 implies that $T_{f_m} \cdots T_{f_1}$ is compact on $A^2(\mathbb{D}^n)$.

Proof of Proposition 1 To prove the forward direction, we first use Theorem 1 to conclude that the operator $T_{R_{k,\xi} g} T_{R_{k,\xi} f}$ is zero on $A^2(\mathbb{D}^{n-1})$ for all $\xi \in \mathbb{T}$ and $1 \leq k \leq n$. Since the symbols $R_{k,\xi} f$ and $R_{k,\xi} g$ are $(n-1)$-harmonic on $\mathbb{D}^{n-1}$, we apply [14, Theorem 1.1] (or [15, Corollary 2] in the case $n = 2$) to conclude that either $R_{k,\xi} f = 0$ or $R_{k,\xi} g = 0$. Then $fg = 0$ on $b\mathbb{D}^n$ as desired.

To prove the converse we argue as follows. For each $1 \leq k \leq n$ and $\xi \in \mathbb{T}$, since both $R_{k,\xi} f$ and $R_{k,\xi} g$ are $(n-1)$-harmonic and their product is zero on $\mathbb{D}^{n-1}$, either $R_{k,\xi} f = 0$ or $R_{k,\xi} g = 0$. Then $T_{R_{k,\xi} g} T_{R_{k,\xi} f} = 0$ on $A^2(\mathbb{D}^{n-1})$ for all $\xi \in \mathbb{T}$ and $1 \leq k \leq n$. Theorem 1 now implies that $T_g T_f$ is compact.

Proof of Proposition 2 We first prove (i). Assume that T is a nonzero compact operator. Then by Theorem 1 when restricted on the first coordinate, for any $\xi \in \mathbb{T}$,

$$0 = \prod_{k=1}^{M} T_{R_{1,\xi} f_k} = \left(\prod_{k=1}^{M} f_{1,k}(\xi) \right) \prod_{k=1}^{M} T_{\widetilde{f_k}}$$

on $A^2(\mathbb{D}^{n-1})$, where $\widetilde{f_k}(z_2, \ldots, z_n) = f_{2,k}(z_2) \cdots f_{n,k}(z_n)$. Since T is not the zero operator, the second factor on the right hand side above is a nonzero operator. This follows from the fact that T can be written as the product

$$\left(\prod_{k=1}^{M} T_{f_{1,k}} \right) \cdot \left(\prod_{k=1}^{M} T_{\widetilde{f_k}} \right)$$

where the first factor acts on functions in z_1 and the second factor acts on functions in $z' = (z_2, \ldots, z_n)$. Hence, $\prod_{k=1}^M f_{1,k}(\xi) = 0$. It follows that

$$F(\xi, z_2, \ldots, z_n) = \prod_{k=1}^M f_k(\xi, z_2, \ldots, z_n) = \left(\prod_{k=1}^M f_{1,k}(\xi) \right) \left(\prod_{j=2}^n \prod_{k=1}^M f_{j,k}(z_j) \right) = 0$$

on $\mathbb{T} \times \mathbb{D}^{n-1}$. The same argument applies to other coordinates and we have $F = 0$ on $b\mathbb{D}^n$.

Next we prove (ii). Assume that $F = \prod_{k=1}^M f_k = 0$ on $b\mathbb{D}^n$ and F is not identically zero on $\mathbb{D}^n$. Choose $q = (q_1, \ldots, q_n) \in \mathbb{D}^n$ such that $f_k(q) \neq 0$ for all k, which implies that $f_{j,k}(q_j) \neq 0$ for all j and k. For any $\xi \in \mathbb{T}$, since $z = (\xi, q_2, \ldots, q_n) \in b\mathbb{D}^n$, we have

$$0 = F(z) = \left(\prod_{k=1}^M f_{1,k}(\xi) \right) \cdot \prod_{j=2}^n \prod_{k=1}^M f_{j,k}(q_j).$$

Because the second factor is nonzero, it follows that $\prod_{k=1}^M f_{1,k}(\xi) = 0$. As a result,

$$\prod_{k=1}^M T_{R_{1,\xi} f_k} = \left(\prod_{k=1}^M f_{1,k}(\xi) \right) \prod_{k=1}^M T_{\widetilde{f_k}} = 0$$

on $A^2(\mathbb{D}^{n-1})$, where, as before, $\widetilde{f_k}(z_2, \ldots, z_n) = f_{2,k}(z_2) \cdots f_{n,k}(z_n)$. The same argument applies to other parts of $b\mathbb{D}^n$. Then Theorem 1 implies that $T = \prod_{k=1}^M T_{f_k}$ is compact on $A^2(\mathbb{D}^n)$.

The proof of Proposition 3 hinges on several elementary facts about polynomials that we describe below. We use $\mathbb{C}[z, \bar{z}]$ to denote the vector space of all polynomials in z and $\bar{z}$.

The following lemma is well known. The proof follows from the fact that if a real analytic function vanishes on a non-empty open set, it must be identically zero.

Lemma 2 *Let $f \in \mathbb{C}[z, \bar{z}]$ be not identically zero. Then the set*

$$\{z \in \mathbb{C} : f(z) = 0\}$$

has an empty interior.

Lemma 3 *Let $f \in \mathbb{C}[z, \bar{z}]$. Assume that there exist infinitely many $\xi \in \mathbb{T}$ such that $f(\xi) = 0$. Then there is a polynomial $g \in \mathbb{C}[z, \bar{z}]$ such that $f(z) = (1 - |z|^2) g(z)$. In particular, $f(\xi) = 0$ for all $\xi \in \mathbb{T}$.*

Proof For non-negative integers s, t, we write

$$\bar{z}^s z^t = \begin{cases} |z|^{2s} z^{t-s} & \text{if } t \geq s, \\ |z|^{2t} \bar{z}^{s-t} & \text{if } t < s. \end{cases}$$

As a result, there are integers m, $M \geq 0$ and polynomials p_j (for $0 \leq j \leq M$) and q_j (for $0 \leq j \leq m$) of a single variable such that

$$f(z) = \sum_{j=0}^{M} p_j(|z|^2)z^j + \sum_{j=0}^{m} q_j(|z|^2)\overline{z}^j.$$

By the hypothesis, there exists infinitely many $\xi \in \mathbb{T}$ such that

$$\sum_{j=0}^{M} p_j(1)\xi^j + \sum_{j=0}^{m} q_j(1)\overline{\xi}^j = f(\xi) = 0.$$

This implies that $p_j(1) = q_j(1) = 0$ for each j. As a consequence, all $p_j(r)$ and $q_j(r)$ are divisible by $1 - r$. We then conclude that $f(z)$ is divisible by $1 - |z|^2$, from which the conclusion of the lemma follows.

Lemma 4 *Let $f(z, w)$ be a polynomial in z, w, $\overline{z}$, $\overline{w}$ and let $h \in C(\overline{\mathbb{D}^2})$. Assume that $fh = 0$ on $b\mathbb{D}^2$. Then $f|_{\mathbb{T} \times \overline{\mathbb{D}}} = 0$ or $h|_{\mathbb{T} \times \overline{\mathbb{D}}} = 0$ and $f|_{\overline{\mathbb{D}} \times \mathbb{T}} = 0$ or $h|_{\overline{\mathbb{D}} \times \mathbb{T}} = 0$.*

Proof Assume that h does not vanish identically on $\mathbb{T} \times \overline{\mathbb{D}}$. By continuity, there exist a non-empty arc $J \subseteq \mathbb{T}$ and a non-empty open set $V \subseteq \mathbb{D}$ such that $h(\xi, w) \neq 0$ for all $\xi \in J$ and $w \in V$. It follows that $f(\xi, w) = 0$ for all such ξ and w. For each $\xi \in J$, applying Lemma 2, we conclude that $f(\xi, w) = 0$ for all $w \in \overline{\mathbb{D}}$. Then for each $w \in \overline{\mathbb{D}}$, since $f(\xi, w)$ vanishes on J (which is an infinite set), Lemma 3 implies that $f(\xi, w) = 0$ for all $\xi \in \mathbb{T}$. Therefore, f vanishes identically on $\mathbb{T} \times \overline{\mathbb{D}}$. The proof for $\overline{\mathbb{D}} \times \mathbb{T}$ is similar.

Lemma 5 [[16, Corollary 1.8]] *Suppose $\varphi_1, \ldots, \varphi_M$ and $\psi_1, \ldots, \psi_N$ are polynomials of $z, \overline{z}$ in $\mathbb{D}$ and $g \in L^2(\mathbb{D})$. If $T_{\varphi_1} \cdots T_{\varphi_M} T_g T_{\psi_1} \cdots T_{\psi_N} = 0$ on $A^2(\mathbb{D})$, then one of the symbols must be zero.*

Proof of Proposition 3 Assume that $T_{f_1} \cdots T_{f_M} T_h T_{g_1} \cdots T_{g_N}$ is compact on $A^2(\mathbb{D}^2)$, then by Theorem 1,

$$T_{R_{1,\xi} f_1} \cdots T_{R_{1,\xi} f_M} T_{R_{1,\xi} h} T_{R_{1,\xi} g_1} \cdots T_{R_{1,\xi} g_N} = 0$$

on $A^2(\mathbb{D})$ for all $\xi \in \mathbb{T}$. By Lemma 5, one of $R_{1,\xi} f_1, \ldots, R_{1,\xi} f_M$, $R_{1,\xi} h$, and $R_{1,\xi} g_1, \ldots, R_{1,\xi} g_N$ is a zero function on $\mathbb{D}$. Thus, $f_1 \cdots f_M h g_1 \cdots g_N = 0$ on $\mathbb{T} \times \overline{\mathbb{D}}$. Similar argument works for $\overline{\mathbb{D}} \times \mathbb{T}$. Therefore, $f_1 \cdots f_M h g_1 \cdots g_N = 0$ on $b\mathbb{D}^2$.

For the converse, by Lemma 4, one of the symbols is identically zero on $\mathbb{T} \times \overline{\mathbb{D}}$. It then follows that

$$T_{R_{1,\xi} f_1} \cdots T_{R_{1,\xi} f_M} T_{R_{1,\xi} h} T_{R_{1,\xi} g_1} \cdots T_{R_{1,\xi} g_N} = 0.$$

Similarly,

$$T_{R_{2,\xi} f_1} \cdots T_{R_{2,\xi} f_M} T_{R_{2,\xi} h} T_{R_{2,\xi} g_1} \cdots T_{R_{2,\xi} g_N} = 0.$$

Therefore, by Theorem 1, we conclude that $T_{f_1} \cdots T_{f_M} T_h T_{g_1} \cdots T_{g_N}$ is compact on $A^2(\mathbb{D}^2)$.

Remark 4 It is desirable to generalize Proposition 3 to $\mathbb{D}^n$ for all $n \geq 2$. While Lemmas 2, 3 and 4 remain true for all n, Lemma 5 has only been known for the disc. In order to extend Proposition 3 to all $n \geq 2$, one needs to prove a several-variable version of Lemma 5. Some partial results have been obtained in the literature. For example, the main results of [17] imply that Lemma 5 holds in several variables when $g = 1$ or when all φ_j, ψ_k are monomials. As a result, Proposition 3 holds on $\mathbb{D}^n$ for all $n \geq 2$ in the case $h = 1$, or in the case all f_j and g_k are monomials.

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Declarations

Conflict of Interest The authors have no conflict of interest to declare that are relevant to the content of this article.

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