scientific reports

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Some properties of subclass of multivalent functions associated with a generalized differential operator

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In this paper, the new subclass $\mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$ of a linear differential operator's $\mathcal{N}^n_{\lambda,\delta,p}f(\zeta)$ associated with multivalent analytical function has been introduced. Further, the coefficient inequalities, extreme points for the extremal function, sharpness of the growth and distortion bounds, partial sums, starlikeness, and convexity of the subclass is investigated.

Keywords Multivalent functions, Convolution, Derivative operators

Assume that $\mathcal{U} : |\zeta| < 1$ is the unit circle and that $f(\zeta)$ is an analytical function, as exhibited by the power series.

$$w = f(\zeta) = \sum_{\nu=0}^{\infty} b_n \zeta^n = b_0 + b_1\zeta + b_2\zeta^2 + \cdots, \tag{1}$$

The sequence $\{b_n\}$ of coefficients in (1) is the basis for the function $f(\zeta)$, which maps $\mathcal{U}$ onto a sub-domain $\mathcal{S}$ of a Riemann surface.

An attribute of geometry of $\mathcal{S}$ is described by the statement that the univalent function $f(\zeta)$ is in $\mathcal{U}$. By definition, $f(\zeta)$ has the property of being univalent in $\mathcal{U}$ if

$$f(\zeta_1) = f(\zeta_2), \quad \zeta_1, \zeta_2 \in \mathcal{U} \implies \zeta_1 = \zeta_2. \tag{2}$$

Briefly, $f(\zeta)$ is said to be univalent in $\mathcal{U}$ if it does not take any value more than once for ζ in $\mathcal{U}$.

The image of $\mathcal{U}$ creates a simple domain in the w-plane, provided $f(\zeta)$ is univalent.

The multivalent function is a logical consequence of the idea of the univalent function. Assume that $p \in \mathcal{N}$. It is said that $f(\zeta) = w_0$ has p roots in $\mathcal{U}$ and that the function $f(\zeta)$ denotes p-valent in $\mathcal{U}$. Meanwhile, the constraints

$$f(\zeta_1) = f(\zeta_2) = \cdots, = f(\zeta_{p+1}), \quad \zeta_1, \zeta_2, \ldots \zeta_{p+1} \in \mathcal{U} \implies \zeta_i \neq \zeta_j. \tag{3}$$

for a certain pair, ensure that $i \neq j$. To put it simply, $f(\zeta)$ is p-valent in $\mathcal{U}$ assuming that some value but no value exceeds p times.

Typically, 1907 the work of Koebe[20] was considered the earliest stages of the concept of univalent functions. In 1933 by Montel[21] and in 1938 by Biernacki[7] were given two credible evaluations of the research on univalent and multivalent functions. After that, the volume of information grew rapidly, as usual, making it challenging for researchers to ascertain the current situation. Books from Schaeffer and Spencer[26], Jenkins[19], and others explore specialized parts of the topic in great detail. The writings by Hayman[18] and Goluzin[15] provided a thorough overview, and it contained enough unresolved issues for a while. The study of fragments by Bernardi[6] and Hayman[17] offered additional direction in the field.

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The differential and integral operators of normalized analytic functions have recently gained a lot of popularity. Numerous articles covered the operators and generalizations made by various authors. In 1975, Ruscheweyh[24] introduced the differential operator and it is generalized by Salagean[25] in 1985. For a long time, these two operators were utilized to investigate various subclasses of univalent function by researchers. In the year of 2004 Al-Oboudi's[2] generalized of the Salagean operator, followed by Shaqsi and Darus[3,4] generalized the Ruscheweyh and Salagean differential operators in 2008. Following that, several authors began to develop new operators based on the Salagean and Ruscheweyh in their own distinctive style. For example, see[5,8–10,12,13,16,22,23,27–31]. By the help of this survey, in this current work, certain properties of subclass of new linear differential operator of multivalent functions have been investigated.

Let $\mathcal{A}_p$ be called a class of multivalent analytic functions

$$f(\zeta) = \zeta^p + \sum_{\nu=p+1}^{\infty} a_\nu \zeta^\nu, \tag{4}$$

belongs to $\mathcal{U} = \{\zeta : |\zeta| < 1\}$.

For $f(\zeta) \in \mathcal{A}_p$, Aghalary et al.[1] studied the following multiplier transformation operator

$$\mathcal{I}_p(n,\lambda) = \zeta^p + \sum_{\nu=p+1}^{\infty} \left(\frac{\nu+\lambda}{p+\lambda}\right)^n a_\nu \zeta^\nu, \quad (\lambda \geq 0). \tag{5}$$

For $f(\zeta) \in \mathcal{A}_p$, a new differential operator has defined $\mathcal{N}^n_{\lambda,\delta,p}(f(\zeta)) = \mathcal{I}_p(n,\lambda) * f(\zeta)$ by
$\mathcal{N}^0_{\lambda,\delta,p} = \zeta^p + \sum_{\nu=p+1}^{\infty} a_\nu \zeta^\nu$
$\mathcal{N}^1_{\lambda,\delta,p} = (1-\delta)\mathcal{I}_p(1,\lambda) + \frac{\delta\zeta}{p}\left(\mathcal{I}_p(1,\lambda)\right)' = \zeta^p + \sum_{\nu=p+1}^{\infty}\left[\frac{p(\lambda+\nu)+(\nu-p)(\nu+\lambda)\delta}{p(p+\lambda)}\right] a_\nu \zeta^\nu$
$\mathcal{N}^2_{\lambda,\delta,p} = \mathcal{N}_{\lambda,\delta,p}\left(\mathcal{N}^1_{\lambda,\delta,p}\right)$ Similarly,

$$\mathcal{N}^n_{\lambda,\delta,p} = \mathcal{N}_{\lambda,\delta,p}\left(\mathcal{N}^{n-1}_{\lambda,\delta,p}\right) = \zeta^p + \sum_{\nu=p+1}^{\infty}\left[\frac{p(\lambda+\nu)+(\nu-p)(\lambda+\nu)\delta}{p(p+\lambda)}\right]^n a_\nu \zeta^\nu, (\lambda,\delta \geq 0, n \in N_0). \tag{6}$$

Remark 1.1 For $\delta = 0$ in (6), the multiplier transformations $I_p(n,\lambda)$ are obtained. It was stated by Aghalary et al.[1].

For $\delta = 0, p = 1$ in (6), the operator $\mathcal{I}^n_\lambda$ is obtained. It was presented by Cho and Srivastava[11].

For $\delta = 0, p = 1, \lambda = 1$ in (6), the differential oprator $\mathcal{I}^n$ was introduced by Uralegaddi et al.[32].

The operator $\mathcal{D}^n$ is stated by Salagean[25] for $\lambda = 0, \delta = 0, p = 1$ in (6).

For $\lambda = 0, \delta = 0, p = 1, n = -n$ in (6), the multiplier transformation I^{-n} is obtained; it was introduced by Flett[14].

The class $\mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$

Definition 2.1 Let $\mathcal{S}^n_{b,\lambda,\delta,p}(\phi(\zeta))$ denote the subclass of $f(\zeta) \in \mathcal{A}_p$, in which

$$1 + \frac{1}{b}\left(\frac{\frac{\zeta}{p}(\mathcal{N}^n_{\lambda,\delta,p})'}{\mathcal{N}^n_{\lambda,\delta,p}} - 1\right) \prec \phi(\zeta). \tag{7}$$

Definition 2.2 Let $\mathcal{S}^n_{b,\lambda,\delta,p}(\phi(\zeta)) \equiv \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$ represents a subclass belonging to $f(\zeta) \in \mathcal{A}_p$, then

$$Re\left(1 + \frac{1}{b}\left(\frac{\frac{\zeta}{p}(\mathcal{N}^n_{\lambda,\delta,p})'}{\mathcal{N}^n_{\lambda,\delta,p}} - 1\right)\right) > \alpha. \tag{8}$$

where $\phi(\zeta) = \frac{1+(1-2\alpha)\zeta}{(1-\zeta)}, n \in N_0, 0 \leq \alpha < 1, \lambda, \delta \geq 0, b \in C - \{0\}$ and all $\zeta \in \mathcal{U}$.

Estimate the coefficient inequality

The concepts of univalent and multivalent functions are crucial while studying complex analysis. They are usually defined on the complex plane. It is customary in this context to estimate the coefficients of these functions, more precisely, their inequalities. We will gain insight into the branching structure of multivalent functions by estimating their coefficients. The coefficient inequalities provide information about how branch points behave over the complex plane of the function. In both cases, understanding the coefficients and their inequalities in univalent and multivalent functions are essential for various applications in complex analysis, particularly in the fields of conformal mapping, complex geometry, and Riemann surfaces. The coefficient estimation provides valuable information about the behavior of functions and its geometric properties, helping mathematicians and scientists work with them effectively in various contexts.

Theorem 2.1 *Let* $f(\zeta) \in \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$, *then*

$$\sum_{\nu=p+1}^{\infty}\left|\frac{\alpha bp-\nu+p-pb}{p}\right|\left[\frac{p(\lambda+\nu)+(\nu-p)(\lambda+\nu)\delta}{p(p+\lambda)}\right]^n|a_\nu| \le (1-\alpha)|b|. \tag{9}$$

Proof Let

$$F(\zeta)=1+\frac{1}{b}\left(\frac{\frac{\zeta}{p}(\mathcal{N}^n_{\lambda,\delta,p})'}{\mathcal{N}^n_{\lambda,\delta,p}}-1\right)-\alpha$$
$$=1+\frac{\frac{\zeta}{p}(\mathcal{N}^n_{\lambda,\delta,p})'-(1+\alpha b)\mathcal{N}^n_{\lambda,\delta,p}}{b\mathcal{N}^n_{\lambda,\delta,p}}$$

By the condition of the class,

$$F(\zeta) \prec \frac{1+\zeta}{1-\zeta}.$$

There exist a schwarz function $w(\zeta)$, with $w(0)=0$ and $|w|<1$, such that

$$F(\zeta)=\frac{1+w(\zeta)}{1-w(\zeta)}.$$

This implies that

$$w(\zeta)=\frac{F(\zeta)-1}{F(\zeta)+1}.$$

We know that

$$|w(\zeta)|=\left|\frac{F(\zeta)-1}{F(\zeta)+1}\right|<1.$$

Then

$$\left|\frac{F(\zeta)-1}{F(\zeta)+1}\right|=\left|\frac{\frac{\zeta}{p}(\mathcal{N}^n_{\lambda,\delta,p})'-(1+\alpha b)\mathcal{N}^n_{\lambda,\delta,p}}{\frac{\zeta}{p}(\mathcal{N}^n_{\lambda,\delta,p})'-(1+\alpha b-2b)\mathcal{N}^n_{\lambda,\delta,p}}\right|$$
$$=\left|\frac{\zeta^p+\sum_{\nu=p+1}^{\infty}\frac{\nu}{p}c_\nu a_\nu\zeta^\nu-(1+\alpha b)\zeta^p-\sum_{\nu=p+1}^{\infty}(1+\alpha b)c_\nu a_\nu\zeta^\nu}{\zeta^p+\sum_{\nu=p+1}^{\infty}\frac{\nu}{p}c_\nu a_\nu\zeta^\nu-(1+\alpha b-2b)\zeta^p-\sum_{\nu=p+1}^{\infty}(1+\alpha b-2b)c_\nu a_\nu\zeta^\nu}\right|$$
$$=\left|\frac{-\alpha b-\sum_{\nu=p+1}^{\infty}(1+\alpha b-\frac{\nu}{p})c_\nu a_\nu\zeta^{\nu-p}}{(2-\alpha)b-\sum_{\nu=p+1}^{\infty}(1+\alpha b-2b-\frac{\nu}{p})c_\nu a_\nu\zeta^{\nu-p}}\right|$$
$$\le\frac{\alpha|b|+\sum_{\nu=p+1}^{\infty}\left|(1+\alpha b-\frac{\nu}{p})\right|c_\nu|a_\nu|\left|\zeta^{\nu-p}\right|}{(2-\alpha)|b|-\sum_{\nu=p+1}^{\infty}\left|(1+\alpha b-2b-\frac{\nu}{p})\right|c_\nu|a_\nu|\left|\zeta^{\nu-p}\right|}.$$

The last expression is bounded by 1, if

$$\alpha|b|+\sum_{\nu=p+1}^{\infty}\left|(1+\alpha b-\frac{\nu}{p})\right|c_\nu|a_\nu|\le(2-\alpha)|b|-\sum_{\nu=p+1}^{\infty}\left|(1+\alpha b-2b-\frac{\nu}{p})\right|c_\nu|a_\nu|.$$

Which implies that,

$$\sum_{\nu=p+1}^{\infty}\left|\left(1+\alpha b-b-\frac{\nu}{p}\right)\right|c_\nu|a_\nu|\le(1-\alpha)|b|,$$

where Hence the equation (9) is hold. □

Corollary 2.1 *Let* $f \in \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$, *then*

$$|a_\nu|\le\frac{(1-\alpha)|b|}{\left|\frac{\alpha bp-\nu+p-pb}{p}\right|\left[\frac{p(\lambda+\nu)+(\nu-p)(\lambda+\nu)\delta}{p(p+\lambda)}\right]^n}, \tag{10}$$

and the equality is concluded for the function $f(\zeta)$ *is given by*

$$f(\zeta) = \zeta^p + \frac{(1-\alpha)|b|}{\left|\frac{\alpha bp-\nu+p-pb}{p}\right|\left[\frac{p(\lambda+\nu)+(\nu-p)(\lambda+\nu)\delta}{p(p+\lambda)}\right]^n}\zeta^\nu, \quad \nu \geq p+1. \tag{11}$$

Extreme points

Extremal points are analyses in the framework of multivalent functions in order to comprehend branch cuts, singularities, and branching behavior. It is essential to comprehending the function of complex structure and Riemann surface.

Theorem 2.2 *Let*

$$f_p(\zeta) = \zeta^p, f_\nu(\zeta) = \zeta^p + \eta_\nu \frac{(1-\alpha)|b|}{C(\lambda)}\zeta^\nu, \nu = p+1, p+2, \ldots,$$

where

$$C(\lambda) = \sum_{\nu=p+1}^{\infty}\left|\frac{\alpha bp-\nu+p-pb}{p}\right|\left[\frac{p(\lambda+\nu)+(\nu-p)(\lambda+\nu)\delta}{p(p+\lambda)}\right]^n.$$

Then $f \in \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$ *only when it is in the form*

$$f(\zeta) = \eta_p f_p(\zeta) + \sum_{\nu=p+1}^{\infty}\eta_\nu f_\nu(\zeta),$$

where $\eta_\nu \geq 0$ *and* $\eta_p = 1 - \sum_{\nu=p+1}^{\infty}\eta_\nu$.

Proof Let assume that

$$f(\zeta) = \eta_p f_p(\zeta) + \sum_{\nu=p+1}^{\infty}\eta_\nu f_\nu(\zeta).$$

Then

$$\begin{aligned} f(\zeta) &= \left(1 - \sum_{\nu=p+1}^{\infty}\eta_\nu\right)\zeta^p + \sum_{\nu=p+1}^{\infty}\eta_\nu\left(\zeta^p + \frac{(1-\alpha)|b|}{C(\lambda)}\zeta^\nu\right) \\ &= \zeta^p + \sum_{\nu=p+1}^{\infty}\eta_\nu\frac{(1-\alpha)|b|}{C(\lambda)}\zeta^\nu \\ &= \zeta^p + \sum_{\nu=p+1}^{\infty}a_\nu\zeta^\nu \end{aligned}$$

Thus,

$$\begin{aligned} &\sum_{\nu=p+1}^{\infty}C(\lambda)|a_\nu| \\ &= \sum_{\nu=p+1}^{\infty}C(\lambda)\eta_\nu\frac{(1-\alpha)|b|}{C(\lambda)} \\ &= (1-\alpha)|b|\sum_{\nu=p+1}^{\infty}\eta_\nu \\ &= (1-\alpha)|b|(1-\eta_p) \\ &\quad < (1-\alpha)|b|, \end{aligned}$$

which demonstrates

$$f \in \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha).$$

Conversely,
Consider this

$$f \in \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha).$$

While

$$|a_\nu| \leq \frac{(1-\alpha)|b|}{C(\lambda)}, \nu = p+1, p+2, \dots$$

Let

$$\eta_\nu \leq \frac{C(\lambda)}{(1-\alpha)|b|} a_\nu, \eta_p = 1 - \sum_{\nu=p+1}^{\infty} \eta_\nu.$$

Thus,

$$\begin{aligned}
f(\zeta) &= \zeta^p + \sum_{\nu=p+1}^{\infty} a_\nu \zeta^\nu \\
f(\zeta) &= (\eta_p + \sum_{\nu=p+1}^{\infty} \eta_\nu)\zeta^p + \sum_{\nu=p+1}^{\infty} \eta_\nu \frac{(1-\alpha)|b|}{C(\lambda)} \zeta^\nu \\
&= \eta_p f_p(\zeta) + \sum_{\nu=p+1}^{\infty} \eta_\nu \{\zeta^\nu + \frac{(1-\alpha)|b|}{C(\lambda)} \zeta^\nu\} \\
&= \eta_p f_p(\zeta) + \sum_{\nu=p+1}^{\infty} \eta_\nu f_\nu(\zeta).
\end{aligned}$$

□

Growth and distortion theorems

Growth and distortion theorems are useful tools in the study of univalent and multivalent functions because they help to characterize and comprehend the behavior of these functions and how they relate to the geometry of the complex plane. According to the growth theorem, a complex-valued function is inherently constant if it is entire and bounded. The geometry of curves and regions in the complex plane is influenced by analytic functions, as revealed by the distortion theorem. It sets limits on the maximum amount of stretching or distortion that can happen when a function transfers a region or curve from one domain to another. By using these theorems, mathematicians and researchers can study the behavior of complex analytic functions and how it impacts the sizes and shapes of curves and regions in the complex plane.

Theorem 2.3 *If $f \in \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$,then*

$$\rho^p - \frac{(1-\alpha)|b|}{\left|\frac{\alpha bp-1-bp}{p}\right|\left(\frac{(p+1+\lambda)(p+\delta)}{p(p+\lambda)}\right)^n} \rho^{p+1} \leq \left|f(\zeta)\right| \leq \rho^p + \frac{(1-\alpha)|b|}{\left|\frac{\alpha bp-1-bp}{p}\right|\left(\frac{(p+1+\lambda)(p+\delta)}{p(p+\lambda)}\right)^n} \rho^{p+1},$$

$|\zeta| = \rho < 1$, provided $\nu \geq p+1$. The result called as sharp for

$$f(\zeta) = \zeta^p + \frac{(1-\alpha)|b|}{\left|\frac{\alpha bp-1-bp}{p}\right|\left(\frac{(p+1+\lambda)(p+\delta)}{p(p+\lambda)}\right)^n} \zeta^{p+1}.$$

Proof By making use of the inequality (9) for $f \in \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$ together with

$$\left|\frac{\alpha bp-1-bp}{p}\right|\left(\frac{(p+1+\lambda)(p+\delta)}{p(p+\lambda)}\right)^n \leq \left|\frac{\alpha bp-\nu+p-pb}{p}\right|\left(\frac{p(\lambda+\nu)+(\nu-p)(\lambda+\nu)\delta}{p(p+\lambda)}\right)^n,$$

then

$$\begin{aligned}
&\left|\frac{\alpha bp-1-bp}{p}\right|\left(\frac{(p+1+\lambda)(p+\delta)}{p(p+\lambda)}\right)^n \sum_{\nu=p+1}^{\infty} a_\nu \\
&\leq \left|\frac{\alpha bp-\nu+p-pb}{p}\right|\left(\frac{p(\lambda+\nu)+(\nu-p)(\lambda+\nu)\delta}{p(p+\lambda)}\right)^n \sum_{\nu=p+1}^{\infty} a_\nu \leq (1-\alpha)|b|.
\end{aligned}$$

$$\sum_{\nu=p+1}^{\infty} a_\nu \leq \frac{(1-\alpha)|b|}{\left|\frac{\alpha bp-1-bp}{p}\right|\left(\frac{(p+1+\lambda)(p+\delta)}{p(p+\lambda)}\right)^n}. \tag{12}$$

By using (12) for the function $f(\zeta) = \zeta^p + \sum_{\nu=p+1}^{\infty} a_\nu \zeta^\nu \in \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$, since $|\zeta| = \rho$,

$$\begin{aligned}|f(\zeta)| &= \rho^p + \sum_{\nu=p+1}^{\infty} a_\nu \rho^\nu \\ &\le \rho^p + \rho^{p+1} \sum_{\nu=p+1}^{\infty} a_\nu \\ &\le \rho^p + \frac{(1-\alpha)|b|}{\left|\frac{\alpha bp-1-bp}{p}\right| \left(\frac{(p+1+\lambda)(p+\delta)}{p(p+\lambda)}\right)^n} \rho^{p+1},\end{aligned}$$

and similarly,

$$|f(\zeta)| \ge \rho^p - \frac{(1-\alpha)|b|}{\left|\frac{\alpha bp-1-bp}{p}\right| \left(\frac{(p+1+\lambda)(p+\delta)}{p(p+\lambda)}\right)^n} \rho^{p+1}.$$

□

Theorem 2.4 *If* $f \in \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$,*then*

$$p\rho^{p-1} - \frac{(p+1)(1-\alpha)|b|}{\left|\frac{\alpha bp-1-bp}{p}\right| \left(\frac{(p+1+\lambda)(p+\delta)}{p(p+\lambda)}\right)^n} \rho^p \le \left|f'(\zeta)\right| \le p\rho^{p-1} + \frac{(p+1)(1-\alpha)|b|}{\left|\frac{\alpha bp-1-bp}{p}\right| \left(\frac{(p+1+\lambda)(p+\delta)}{p(p+\lambda)}\right)^n} \rho^p,$$

$|\zeta| = \rho < 1$, *provided* $\nu \ge p+1$. *Clearly, the outcome is sharp for*

$$f(\zeta) = \zeta^p + \frac{(1-\alpha)|b|}{\left|\frac{\alpha bp-1-bp}{p}\right| \left(\frac{(p+1+\lambda)(p+\delta)}{p(p+\lambda)}\right)^n} \zeta^{p+1}.$$

Proof By using the inequality (9) for $f \in \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$, then

$$\sum_{\nu=p+1}^{\infty} a_\nu \le \frac{(1-\alpha)|b|}{\left|\frac{\alpha bp-1-bp}{p}\right| \left(\frac{(p+1+\lambda)(p+\delta)}{p(p+\lambda)}\right)^n}.$$

By using (12), then

$$\sum_{\nu=p+1}^{\infty} \nu a_\nu \le \frac{(p+1)(1-\alpha)|b|}{\left|\frac{\alpha bp-1-bp}{p}\right| \left(\frac{(p+1+\lambda)(p+\delta)}{p(p+\lambda)}\right)^n}.$$

For the function $f(\zeta) = \zeta^p + \sum_{\nu=p+1}^{\infty} a_\nu \zeta^\nu \in \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$, then

$$\begin{aligned}|f'(\zeta)| &= p\rho^{p-1} + \sum_{\nu=p+1}^{\infty} \nu a_\nu \rho^{\nu-1} \\ &\le p\rho^{p-1} + \rho^p \sum_{\nu=p+1}^{\infty} \nu a_\nu \\ &\le p\rho^{p-1} + \frac{(p+1)(1-\alpha)|b|}{\left|\frac{\alpha bp-1-bp}{p}\right| \left(\frac{(p+1+\lambda)(p+\delta)}{p(p+\lambda)}\right)^n} \rho^p,\end{aligned}$$

and similarly,

$$|f'(\zeta)| \ge p\rho^{p-1} - \frac{(p+1)(1-\alpha)|b|}{\left|\frac{\alpha bp-1-bp}{p}\right| \left(\frac{(p+1+\lambda)(p+\delta)}{p(p+\lambda)}\right)^n} \rho^p.$$

□

Convexity and starlikeness

The coefficient inequalities of power series functions are frequently caused by starlikeness and convexity. Starlike functions fulfill the well-known Bieberbach conjecture, which gives restriction on the coefficients of starlike function. The geometric shapes can be preserved by mapping functions that are starlike or convex. The starlikeness and convexity of multivalent functions maintain specific structures, these qualities are crucial.

Theorem 2.5 *Let* $f \in \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$, *then the subclass claimed as convex .*

Proof Let

$$f_j(\zeta) = \zeta^p + \sum_{\nu=p+1}^{\infty} a_{\nu,j}\zeta^{\nu}, a_{\nu,j} \geq 0, j = 1, 2,$$

contains $f \in \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$.
it is necessary to show that

$$h(\zeta) = (\tau+1)f_1(\zeta) - \tau f_2(\zeta), 0 \leq \tau \leq 1.$$

while

$$h(\zeta) = \zeta^p + \sum_{\nu=p+1}^{\infty} \left[(1+\tau)a_{\nu,1} - \tau a_{\nu,2}\right]\zeta^{\nu},$$

which implies that

$$\begin{aligned}\sum_{\nu=p+1}^{\infty} &\left|\frac{\alpha bp - \nu + p - pb}{p}\right| \left[\frac{p(\lambda+\nu) + (\nu-p)(\lambda+\nu)\delta}{p(p+\lambda)}\right]^n (1+\tau)a_{\nu,1} \\ &- \left|\frac{\alpha bp - \nu + p - pb}{p}\right| \left[\frac{p(\lambda+\nu) + (\nu-p)(\lambda+\nu)\delta}{p(p+\lambda)}\right]^n \tau a_{\nu,2} \\ &\leq (1+\tau)(1-\alpha)|b| - \tau(1-\alpha)|b| \\ &\leq (1-\alpha)|b|,\end{aligned}$$

Thus

$$h \in \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha).$$

Hence $\mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$ called convex. □

Theorem 2.6 *If* $f \in \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$, *then according to order* ς *f is p-valently convex in the disc* $|\zeta| < \rho_2$, *where*

$$\rho_2 := inf\left(\frac{p(\varsigma-p)\left(\frac{pb\alpha-\nu+p-pb}{p}\right)\left[\frac{p(\lambda+\nu)+(\nu-p)(\lambda+\nu)\delta}{p(p+\lambda)}\right]^n}{\nu(\nu-\varsigma)(1-\alpha)|b|}\right)^{\frac{1}{\nu-p}}, (\nu \geq p+1).$$

The bound for $|\zeta|$ *is sharp for each* ν*,with the form* (11) *serving as the extreme function.*

Proof If $f \in \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$, and *f* is claimed orderly convex of ς, then it is required to prove that

$$\left|1 + \frac{\zeta f''(\zeta)}{f'(\zeta)} - p\right| < p - \varsigma \ \ for|\zeta| < \rho_2. \tag{13}$$

Now, the equation (13) gives

$$\left|1 + \frac{\zeta f''(\zeta)}{f'(\zeta)} - p\right| = \left|\frac{f'(\zeta) + \zeta f''(\zeta) - pf'(\zeta)}{f'(\zeta)}\right| \leq \frac{\sum_{\nu=p+1}^{\infty}\nu(\nu-p)a_\nu|\zeta|^{\nu-p}}{p + \sum_{\nu=p+1}^{\infty}\nu a_\nu|\zeta|^{\nu-p}}. \tag{14}$$

From (13) and (14), derive

$$\sum_{\nu=p+1}^{\infty}\frac{\nu(\nu-\varsigma)}{p(\varsigma-p)}a_\nu|\zeta|^{\nu-p} \leq 1. \tag{15}$$

In the view of (13), it follows that (15) is true if

$$|\zeta| \leq \left(\frac{p(\varsigma-p)\left|\frac{\alpha bp-\nu+p-pb}{p}\right|\left[\frac{p(\lambda+\nu)+(\nu-p)(\lambda+\nu)\delta}{p(p+\lambda)}\right]^n}{\nu(\nu-\varsigma)(1-\alpha)|b|}\right)^{\frac{1}{\nu-p}}, (\nu \geq p+1). \tag{16}$$

Setting $|\zeta| = \rho_2$ in (16), the result follows. The sharpness can be verified. □

Theorem 2.7 *If* $f \in \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$, *then according to order* ς*,f is p-valently starlike* $(0 \leq \varsigma < p)$ *in the disc* $|\zeta| < \rho_3$, *where*

$$\rho_3 := inf\left(\frac{(\varsigma-p)\left(\frac{pb\alpha-\nu+p-pb}{p}\right)\left[\frac{p(\lambda+\nu)+(\nu-p)(\lambda+\nu)\delta}{p(p+\lambda)}\right]^n}{(\nu-\varsigma)(1-\alpha)|b|}\right)^{\frac{1}{\nu-p}}, (\nu \geq p+1).$$

The bound for $|\zeta|$ *is sharp for each* ν, *with the form* (11) *serving as the extreme function.*

Proof If $f \in \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$, and f is claimed orderly starlike of ς, then it is required to demonstrate that

$$\left|\frac{\zeta f'(\zeta)}{f(\zeta)} - p\right| < p - \varsigma \ for |\zeta| < \rho_3. \tag{17}$$

Now, the equation (17) gives

$$\left|\frac{\zeta f'(\zeta)}{f(\zeta)} - p\right| = \left|\frac{\zeta f'(\zeta) - pf(\zeta)}{f(\zeta)}\right| \leq \frac{\sum_{\nu=p+1}^{\infty}(\nu-p)a_\nu|\zeta|^{\nu-p}}{1+\sum_{\nu=p+1}^{\infty}a_\nu|\zeta|^{\nu-p}}. \tag{18}$$

From (17) and (18), the following equation obtain

$$\sum_{\nu=p+1}^{\infty}\frac{(\nu-\varsigma)}{(\varsigma-p)}a_\nu|\zeta|^{\nu-p} \leq 1. \tag{19}$$

In the view of (17), it follows that (19) is true if

$$|\zeta| \leq \left(\frac{(\varsigma-p)\left|\frac{\alpha bp-\nu+p-pb}{p}\right|\left[\frac{p(\lambda+\nu)+(\nu-p)(\lambda+\nu)\delta}{p(p+\lambda)}\right]^n}{(\nu-\varsigma)(1-\alpha)|b|}\right)^{\frac{1}{\nu-p}}, (\nu \geq p+1). \tag{20}$$

Setting $|\zeta| = \rho_3$ in (20), the result follows. The sharpness can be verified. □

Partial sums

The concept of partial sums is one that is commonly used in the study of infinite series. On the other hand, partial sums are useful in complicated analysis and can be used in many other mathematical situations, including function analysis. This section looks into the relationship between form (4) and its series of partial sums.

$$f(\zeta) = \zeta^p$$

and

$$f_\nu(\zeta) = \zeta^p + \sum_{\nu=p+1}^{n} a_\nu\zeta^\nu, \nu = p+1, p+2, p+3, \ldots,$$

when the coefficients are small enough to satisfy the analogous condition

$$\sum_{\nu=p+1}^{\infty}\left|\frac{\alpha bp-\nu+p-pb}{p}\right|\left[\frac{p(\lambda+\nu)+(\nu-p)(\lambda+\nu)\delta}{p(p+\lambda)}\right]^n|a_\nu| \leq (1-\alpha)|b|.$$

It can be written as

$$\sum_{\nu=p+1}^{\infty}\mathcal{X}_\nu|a_\nu| \leq 1,$$

where

$$\mathcal{X}_\nu = \frac{\left(\left|\frac{\alpha bp-\nu+p-pb}{p}\right|\left[\frac{p(\lambda+\nu)+(\nu-p)(\lambda+\nu)\delta}{p(p+\lambda)}\right]^n\right)}{(1-\alpha)|b|}.$$

Then $f \in \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$.

Theorem 2.8 *If* $f \in \mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$, *satisfying* (7),*then*

$$Re\left(\frac{f(\zeta)}{f_\nu(\zeta)}\right) \geq 1 - \frac{1}{\mathcal{X}_{n+1}}.$$

Proof Clearly $\mathcal{X}_{\nu+1} > \mathcal{X}_\nu > 1, \nu = p+1, p+2, p+3, \ldots$,
Utilising (4), to get

$$\sum_{\nu=p+1}^{n}|a_\nu|+\mathcal{X}_{n+1}\sum_{\nu=n+1}^{\infty}|a_\nu|\leq\sum_{\nu=p+1}^{\infty}\mathcal{X}_\nu|a_\nu|\leq 1.$$

Let

$$\Phi_1(\zeta)=\mathcal{X}_{n+1}\left[\frac{f(\zeta)}{f_\nu(\zeta)}-\left(1-\frac{1}{\mathcal{X}_{n+1}}\right)\right]$$

$$=1+\frac{\mathcal{X}_{n+1}\sum_{\nu=n+1}^{\infty}a_\nu\zeta^{\nu-1}}{1+\sum_{\nu=p+1}^{n}a_\nu\zeta^{\nu-1}}.$$

Through basic computations, there is

$$\left|\frac{\Phi_1(\zeta)-1}{\Phi_1(\zeta)+1}\right|\leq\frac{\mathcal{X}_{n+1}\sum_{\nu=n+1}^{\infty}|a_\nu|}{2+2\sum_{\nu=p+1}^{n}|a_\nu|+\mathcal{X}_{n+1}\sum_{\nu=n+1}^{\infty}|a_\nu|}\leq 1,$$

which gives,

$$Re\left(\frac{f(\zeta)}{f_\nu(\zeta)}\right)\geq 1-\frac{1}{\mathcal{X}_{n+1}}.$$

Hence $f(\zeta)=\zeta+\frac{\zeta^{n+1}}{\mathcal{X}_{n+1}}\Big)$ will give the sharp result. □

Theorem 2.9 *If $f\in\mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$ and satisfies (7). Then*

$$Re\left(\frac{f_\nu(\zeta)}{f(\zeta)}\right)\geq\frac{\mathcal{X}_{n+1}}{1+\mathcal{X}_{n+1}}.$$

Proof Clearly $\mathcal{X}_{\nu+1}>\mathcal{X}_\nu>1, \nu=p+1,p+2,p+3,\ldots$.
Let

$$\Phi_2(\zeta)=(1+\mathcal{X}_{n+1})\left[\frac{f_\nu(\zeta)}{f(\zeta)}-\left(\frac{\mathcal{X}_{n+1}}{1+\mathcal{X}_{n+1}}\right)\right]$$
$$=1+\frac{(1+\mathcal{X}_{n+1})\sum_{\nu=n+1}^{\infty}a_\nu\zeta^{\nu-1}}{1+\sum_{\nu=p+1}^{n}a_\nu\zeta^{\nu-1}}.$$

Through basic computations, there is

$$\left|\frac{\Phi_2(\zeta)-1}{\Phi_2(\zeta)+1}\right|\leq\frac{(1+\mathcal{X}_{n+1})\sum_{\nu=n+1}^{\infty}|a_\nu|}{2+2\sum_{\nu=p+1}^{n}|a_\nu|+(1+\mathcal{X}_{n+1})\sum_{\nu=n+1}^{\infty}|a_\nu|}\leq 1.$$

Hence, the result

$$Re\left(\frac{f_\nu(\zeta)}{f(\zeta)}\right)\geq\frac{\mathcal{X}_{n+1}}{1+\mathcal{X}_{n+1}}$$

is sharp for all n. □

Theorem 2.10 *If $f\in\mathcal{S}^n_{b,\lambda,\delta,p}(\alpha)$, satisfying (7), then*

$$Re\left(\frac{f'(\zeta)}{f'_\nu(\zeta)}\right)\geq 1-\frac{n+1}{\mathcal{X}_{n+1}}.$$

Proof Clearly $\mathcal{X}_{\nu+1}>\mathcal{X}_\nu>1, \nu=p+1,p+2,p+3,\ldots$.
Let

$$\Phi_3(\zeta)=\mathcal{X}_{n+1}\left[\frac{f'(\zeta)}{f'_\nu(\zeta)}-\left(1-\frac{n+1}{\mathcal{X}_{n+1}}\right)\right]$$
$$=1+\frac{\frac{\mathcal{X}_{n+1}}{n+1}\sum_{\nu=n+1}^{\infty}\nu a_\nu\zeta^{\nu-1}}{1+\sum_{\nu=p+1}^{n}\nu a_\nu\zeta^{\nu-1}}.$$

Through basic computations, there is

$$\left|\frac{\Phi_3(\zeta)-1}{\Phi_3(\zeta)+1}\right|\leq\frac{\frac{\mathcal{X}_{n+1}}{n+1}\sum_{\nu=n+1}^{\infty}\nu|a_\nu|}{2+2\sum_{\nu=p+1}^{n}\nu|a_\nu|+\frac{\mathcal{X}_{n+1}}{n+1}\sum_{\nu=n+1}^{\infty}\nu|a_\nu|}\leq 1,$$

which gives,

$$Re\left(\frac{f'(\zeta)}{f'_{\nu}(\zeta)}\right) \geq 1 - \frac{n+1}{\mathcal{X}_{n+1}}.$$

Hence the result is sharp. □

Theorem 2.11 *If $f \in \mathcal{S}^{n}_{b,\lambda,\delta,p}(\alpha)$, satisfying (7), then*

$$Re\left(\frac{f'_{\nu}(\zeta)}{f'(\zeta)}\right) \geq \frac{\mathcal{X}_{n+1}}{n+1+\mathcal{X}_{n+1}}.$$

Proof Clearly $\mathcal{X}_{\nu+1} > \mathcal{X}_{\nu} > 1, \nu = p+1, p+2, p+3, \ldots$.
Let

$$\begin{aligned}\Phi_4(\zeta) &= ((n+1) + \mathcal{X}_{\nu})\left[\frac{f'_{\nu}(\zeta)}{f'(\zeta)} - \left(\frac{\mathcal{X}_{n+1}}{n+1+\mathcal{X}_{n+1}}\right)\right] \\ &= 1 + \frac{\left(1+\frac{\mathcal{X}_{n+1}}{n+1}\right)\sum_{\nu=n+1}^{\infty} \nu a_{\nu}\zeta^{\nu-1}}{1+\sum_{\nu=p+1}^{n} \nu a_{\nu}\zeta^{\nu-1}}.\end{aligned}$$

Through basic computations, there is

$$\left|\frac{\Phi_4(\zeta)-1}{\Phi_4(\zeta)+1}\right| \leq \frac{\left(1+\frac{\mathcal{X}_{n+1}}{n+1}\right)\sum_{\nu=n+1}^{\infty} \nu|a_{\nu}|}{2+2\sum_{\nu=p+1}^{n} \nu|a_{\nu}| + \left(1+\frac{\mathcal{X}_{n+1}}{n+1}\right)\sum_{\nu=n+1}^{\infty} \nu|a_{\nu}|} \leq 1,$$

which gives,

$$Re\left(\frac{f'_{\nu}(\zeta)}{f'(\zeta)}\right) \geq \frac{\mathcal{X}_{n+1}}{n+1+\mathcal{X}_{n+1}}.$$

Hence the result is sharp. □

Graphical representation for the function $f(\zeta)$

Functions that operate on Complex numbers are referred to as Complex functions. An extension of the complex functions that accepts a complex number as input and returns a complex number is output. Input has two dimensions of information and output another two, giving us a total of four dimensions to fit into our graph. It is challenging to draw the graph for complex functions. Even though the Complex functions are often used for mapping and transformation, such as conformal mapping in complex analysis. The phase and absolute value diagrams help visualize how these mappings and transformations alter the complex plane, preserving angles or shapes, which is a fundamental property of conformal mappings. The conformal mappings find applications in engineering and physics, where complex numbers describe electrical circuits, waves, and quantum mechanics, among other things. Understanding the phase and magnitude of complex functions is essential for solving problems in these domains.

Phase and absolute value diagrams, also known as Argand diagrams or complex plane diagrams, are useful tools for visualizing and analyzing complex functions, whether they are univalent or multivalent. These diagrams help us understand the behavior of complex functions in terms of their magnitude (absolute value) and phase (argument) at various points in the complex plane. The phase diagram can help identify singularities (such as poles and branch points) as they typically manifest as discontinuities or infinite slopes in the diagram. The absolute value diagram can show the behavior of the function near these points, indicating if it approaches infinity or remains bounded.

In this section, the phase and absolute values of the function $f(\zeta)$ from (11) have been examined (Figs. 1, 2, 3, 4 and 5) with various parameters and the following graphs (Figs. 1, 2, 3, 4 and 5) are drawn by using MATLAB. The phase and absolute values for the figures provide a geometric and intuitive way to understand the behavior of complex functions. They are particularly useful when dealing with univalent and multivalent functions, as they help identify key features, singularities, and transformations in the complex plane, making complex analysis more accessible and insightful.

Conclusions

In this article, the coefficient inequality, extreme points, growth and distortion, starlikeness and convexity, and partial sums for a new subclass by using the linear operator have been examined. Furthermore, the graphs of extremal functions are analyzed in terms of how it has been expressed while replacing the suitable values of the parameters. This work motivates the researchers to extend the results of this article into some new subclasses of meromorphic functions and q-calculus.

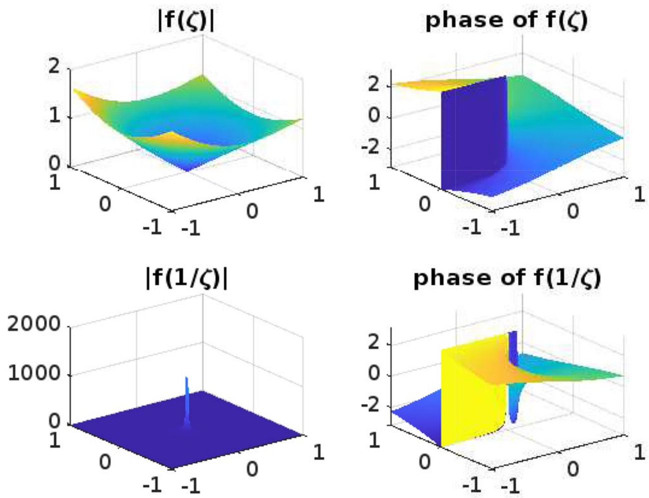

Figure 1. For $\alpha = 0.1$; $b = 1$; $\nu = 2$; $n = 1$; $\delta = 1$; $\lambda = 1$; $p = 1$;$-1 \leq Re(\zeta) \leq 1$; $-1 \leq Im(\zeta) \leq 1$.

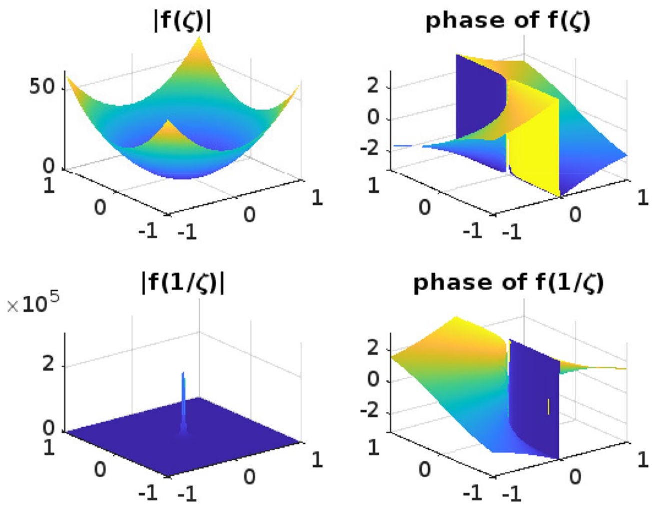

Figure 2. For $\alpha = 0.5$; $b = 1$; $\nu = 2$; $n = 1$; $\delta = 1$; $\lambda = 1$; $p = 10$;$-1 \leq Re(\zeta) \leq 1$; $-1 \leq Im(\zeta) \leq 1$.

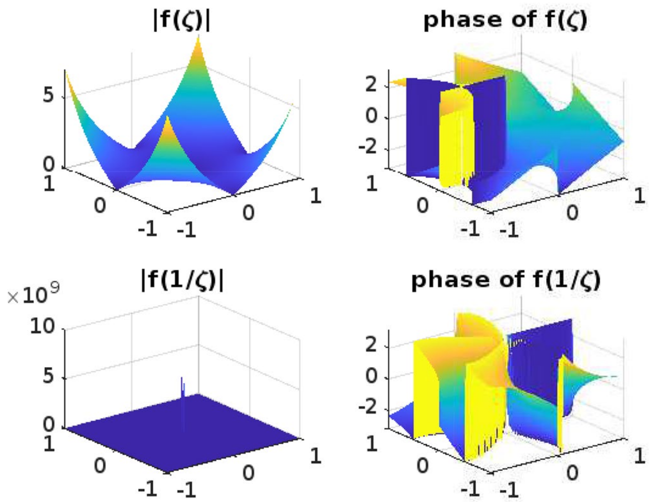

Figure 3. For $\alpha = 0.1$; $b = 1$; $\nu = 5$; $n = 1$; $\delta = 1$; $\lambda = 1$; $p = 5$;$-1 \leq Re(\zeta) \leq 1$; $-1 \leq Im(\zeta) \leq 1$.

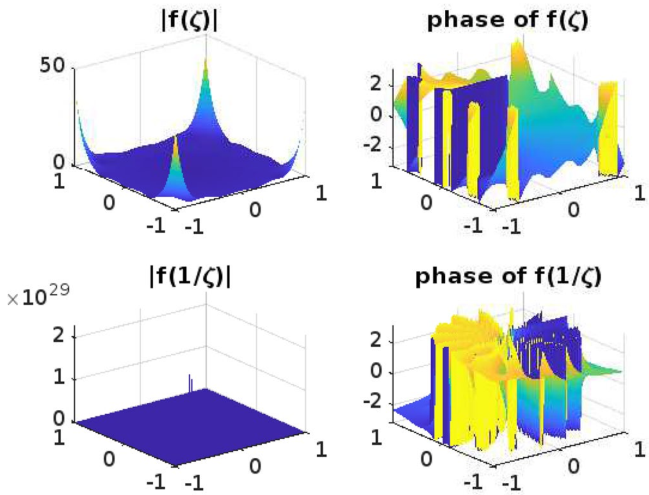

Figure 4. For $\alpha = 0.5$; $b = 1$; $\nu = 15$; $n = 1$; $\delta = 1$; $\lambda = 1$; $p = 10$;$-1 \leq Re(\zeta) \leq 1$; $-1 \leq Im(\zeta) \leq 1$.

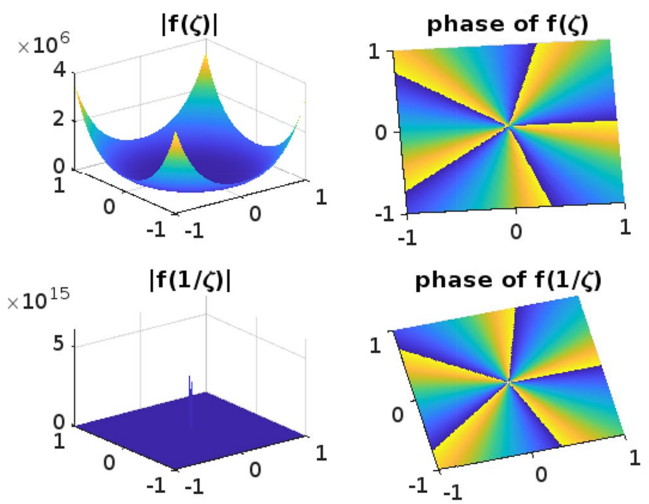

Figure 5. For $\alpha = 0.9$; $b = 4$; $\nu = 5$; $n = 5$; $\delta = 1$; $\lambda = 1$; $p = 20$;$-1 \leq Re(\zeta) \leq 1$; $-1 \leq Im(\zeta) \leq 1$.

Data availibility

No datasets were generated or analysed during the current study.

Received: 1 January 2024; Accepted: 3 April 2024
Published online: 16 April 2024

References

1. Aghalary, R., Rosihan, M. A., Joshi, S. B. & Ravichandran, V. Inequalities for analytic functions defined by certain linear operators. *Int. J. Math. Sci.* **4**(2), 267–274 (2005).
2. Al-Oboudi, F. M. On univalent functions defined by a generalized Salagean operator. *Int. J. Math. Math. Sci.* **27**, 1429–1436 (2004).
3. Al-Shaqsi, K. & Darus, M. An operator defined by convolution involving polylogarithms functions. *J. Math. Stat.* **4**(1), 46–50 (2008).
4. Al-Shaqsi, K. & Darus, M. Differential Subordination with generalised derivative operator. *Int. J. Comp. Math. Sci* **2**(2), 75–78 (2008).
5. Amourah, A. A. & Feras, Y. Some properties of a class of analytic functions involving a new generalized differential operator. *Bol. Soc. Paran. Mat.* **38**(6), 33–42 (2020).
6. Bernardi, S. D. A survey of the development of the theory of schlicht functions. *Duke Math. J.* **19**, 263–287 (1952).
7. Biernacki, M. *Les Fonctions Multivalentes* (Hermann, 1938).
8. Caglar, M., Deniz, E. & Orhan, H. New coefficient inequalities for certain subclasses of p-valent analytic functions. *J. Adv. Appl. Comput. Math.* **1**(2), 40–42 (2014).
9. Caglar, M., Orhan, H. & Deniz, E. Coefficient bounds for certain classes of multivalent functions. *Stud. Univ. Babeç-Bolyai Math.* **56**(4), 49–63 (2011).

10. Cho, N. E., Patel, J. & Mohapatra, G. P. Argument estimates of certain multivalent functions involving a linear operator. *Int. J. Math. Math. Sci.* **31**, 659–673 (2002).
11. Cho, N. E. & Srivastava, H. M. Argument estimates of certain analytic functions defined by a class of multiplier transformations. *Math. Comput. Model.* **203**, 39–49 (2003).
12. Deniz, E., Caglar, M. & Orhan, H. Some properties for certain subclasses of analytic functions defined by a general differential operator. *Asian-Eur. J. Math.* **13**(7), 1–12 (2020).
13. Eker, S., Sümer, H., Özlem, G. & Shigeyoshi, O. Integral means of certain multivalent functions. *Int. J. Math. Math. Sci.* **2006**, 145 (2006).
14. Flett, T. M. The dual of an inequality of Hardy and Littlewood and some related inequalities. *J. Math. Anal. Appl.* **38**, 746–765 (1972).
15. Goluzin, G. M. Geometric theory of functions of a complex variable, *GITTL, Moscow, 1952; 2nd ed., Nauka; Moscow, 1966; German transi., VEB Deutscher Verlag, Berlin* (1952).
16. Hadi, S. H., Maslina, D. & Jung, R. L. Some geometric properties of multivalent functions associated with a new generalized q-Mittag-Leffler function. *AIMS Math.* **7**(7), 11772–11783 (2022).
17. Hayman, G. Coefficient problems for univalent functions and related function classes. *J. Lond. Math. Soc.* **40**, 385–406 (1965).
18. Hayman, G. *Multivalent Functions* (Cambridge University Press, 1958).
19. Jenkins, J. A. *Univalent Functions and Conformal Mapping* (Springer, 1958) ((**Russian transi, IL, Moscow, 1962**)).
20. Koebe, P. Uber die Uniformisierung beliebiger analytischer Kurven. *Nachr Ges. Wiss. Gottingen* **1907**, 191–210 (1907).
21. Montel, P. *Leçons sur les Fonctions Univalentes ou Multivalentes* (Gauthier-Villars, 1933).
22. Murugusundaramoorthy, G. Multivalent β-uniformly starlike functions involving the Hurwitz-Lerch Zeta function. *Acta Univ. Sapientiae* **3**(2), 152 (2011).
23. Rashid, A. M., Abdul, R. S. J. & Sibel, Y. Subordination properties for classes of analytic univalent involving linear operator. *Kyungpook Math. J.* **63**(2), 225–234 (2023).
24. Ruscheweyh, S. New criteria for univalent functions. *Proc. Am. Math. Soc.* **49**, 109–115 (1975).
25. Salagean, G. S. Subclasses of univalent functions. *Lect. Notes Math.* **1013**, 362–372 (1983).
26. Schaeffer, A. C. & Spencer, D. C. Coefficient regions for schlicht functions. *Am. Math. Soc. Colloq. Publ. Am. Math. Soc. Providence R. I.* **35**, 145 (1950).
27. Bulut, Serap. Coefficient bounds for p-valently close-to-convex functions associated with vertical strip domain. *Korean J. Math.* **29**(2), 395–407 (2021).
28. Shanmugam, T. N., Sivasubramanian, S. & Shigeyoshi, O. Argument estimates of certain multivalent functions involving Dziok-Srivastava operator. *Gen. Math.* **1**, 15 (2007).
29. Shi, L., Khan, Q., Srivastava, G., Liu, J.-L. & Arif, M. A study of multivalent q-starlike functions connected with circular domain. *Mathematics* **7**(8), 670 (2019).
30. Stalin, T., Thirucheran, M. & Anand, A. Obtain subclass of analytic functions connected with convolution of polylogarithm function. *Adv. Math.: Sci. J.* **9**(11), 9639–9645 (2020).
31. Thirucheran, M. & Stalin, T. On a new subclass of multivalent functions defined by Al-Oboudi differential operator. *Glob. J. Pure Appl. Math.* **14**(5), 733–741 (2018).
32. Uralegaddi, B. A. & Somanatha, C. Certain classes of univalent functions. In *Current Topics in Analytic Function Theory* (Srivastava, H. M. & Owa, S. eds.) 371–374 (World Scientific Publishing Company, 1992).

Acknowledgements

The study was funded by Researchers Supporting Project number (RSPD2024R749), King Saud University, Riyadh, Saudi Arabia.

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All the equally are equally contributed.

Competing interests

The authors declare no competing interests.

Additional information

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