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Research article

A novel study on the structure of left almost hypermodules

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ABSTRACT

The concept of left almost hypermodule evolves as a novel extension in the field of abstract algebra, specifically within the broader framework of hypermodules. The left almost hypermodule is characterized by a set endowed with two operations, evincing properties that extends across traditional module theory and hypermodules. This abstract intents to provide a succinct overview of salient attributes and prospective implications of the left almost hypermodule, stimulating further exploration of its properties and applications. The paper provides a new definition of hypermodule that acts on the left almost hyperring, referred to as left almost hypermodule (abbreviated as LA-hypermodule), and provides some examples of this new structure. We further examine the variations between hypermodules and left almost hypermodules. By using the concept of left almost polygroups, we explore the transition from left almost polygroup to a left almost hypermodule over left almost hyperring. Lastly, we observe the outcomes in connection to homomorphism and regular relations on left almost hypermodules.

1. Introduction

Frédéric Marty instigated an algebraic structure called "hypergroup" at the 8th Scandinavian Mathematicians Congress in 1934 [1]. It was later discovered that these structures have numerous applications in all sciences, see [2]. Kazim and Naseeruddin introduced the concept of LA-semigroups [3] in 1972. The alternative name of this structure is Abel Grassmann-groupoid, also referred to as AG-groupoid. The semigroups and left almost semigroups are both quasigroups but the main difference between the two is that a semigroup is an associative structure whilst the left almost semigroup is a non-associative structure. Following this, Mushtaq and Kamran [4] proposed the idea of left almost group (LA-group). Hila and Dine then gave the idea of LA-hypercompositional structures [5], which formed basis for the conception of LA-semihypergroups and were investigated afterwards by Yaqoob et al. [6] and Amjad et al. [7].

A non empty set S is said to be a ring, if $(S,+)$ is a commutative group, $(S,\times)$ is a semigroup and the distributive law holds in relation to multiplication over addition [8]. A vector space F is an abelian group that satisfies some axioms. A module is a generalization of vector space over a ring. The primary difference between the two is that the vector space is defined over a field whilst the module is defined over a ring.

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A set F is a field if it is a commutative group in relation to both $+$ and $\times$, and the distributive property with respect to multiplication over addition also holds. The system $(S,+,\times)$ is a ring if $(S,+)$ is a commutative group, $(S,\times)$ is a semigroup and multiplication is also distributive over addition. Let S be a ring, then the additive abelian group M becomes a left S-module if the mapping $\cdot : S\times M \to M$ whose value on a pair (k,x), for $k\in S$, $x\in M$, written $k\cdot x$, satisfies the following axioms:
(1) $k\cdot\left(x_1+x_2\right)=k\cdot x_1+k\cdot x_2$, for all $k\in S$ and $x_1,x_2\in M$,
(2) $\left(k_1+k_2\right)\cdot x=k_1\cdot x+k_2\cdot x$, for all $k_1,k_2\in S$ and $x\in M$,
(3) $k_1\cdot\left(k_2\cdot x\right)=\left(k_1\times k_2\right)\cdot x$, for all $k_1,k_2\in S$ and $x\in M$.
A semigroup $(S,\times)$ having a multiplicative identity $1\in S$, is called a semigroup with identity. If S is a ring having a multiplicative identity 1, then for any $m\in M$, $1.m=m$. If a module is defined over a ring with identity, then it is said to be unitary or unital (cf. Unitary module). The right S-module is defined in a similar way, only axiom 3 is replaced by $(m.k_1).k_2=m.(k_1\times k_2)$. Any right S-module can be considered as left S^{opp}-module over the opposite ring S^{opp} anti-isomorphic to S; hence, corresponding to any result about right S-modules there is a result about left S^{opp}-modules, and conversely. If the commutative law with respect to multiplication holds in S then every left S-module can be considered as a right S-module.

The special kind of a hyperring in which the hyperoperation is addition, but in its semigroup the hyperoperation is multiplication is called the Krasner hyperring [9]. In [10–13], some authors have elongated the idea of Krasner hyperring. The hypothesis of hypermodules that acts over Krasner hyperrings has been initiated and explored by Massouros [14]. Furthermore, Zhan et al. [15] illustrated the isomorphism theorems of hypermodules. The hypermodule theory has been further explored by various mathematicians, like Ameri [16], Fotea [17], Yin et al. [18], Anvariyeh et al. [19], Shojaei & Ameri [20], Zhan & Cristea [21], Ostadhadi-Dehkordi & Davvaz [22], Madanshekaf [23], Davvaz & Cristea [24] Ameri et al. [25] and Norouzi [26].

The abstraction of left almost hyperring was initiated by Rehman, Yaqoob and Nawaz [27]. They gave some relevant basic results and characterized the LA-hyperrings based on their hyperideals and hypersystems. Massouros & Yaqoob [28] studied some results on the theory of left and right almost groups and hypergroups with their relevant enumerations. The concept of left almost polygroups was introduced by Yaqoob et al. [29]. Muftirridha [30] then introduced partial ordering relation on LA-hyperrings.

This paper focuses on the new notion of generalized hypermodules, called left almost hypermodules (abbreviated as LA-hypermodules). A module is an abelian group that acts on a ring and satisfies some properties. Hypermodules emerge as a result when the concept of hyperoperation is applied to modules. In a hypermodule, a canonical hypergroup acts on a hyperring and satisfies the properties of a module. This paper provides the theory of left almost hypermodules. An LA-hypermodule is an LA-polygroup that acts on an LA-hyperring and satisfies the axioms of a module. In particular, we study some fundamental results of this hyperstructure. We also discuss the properties related to subhyperstructures and provide new results on these hypermodules.

2. Preliminaries and basic definitions

In this section, we discuss some basic concepts related to left almost hyperrings (abbreviated LA-hyperrings) and left almost polygroups (abbreviated LA-polygroups). Let H be a set such that $H\neq\emptyset$, $P^*(H)=P(H)\backslash\emptyset$ and $P(H)$ is the collection of all proper and improper subsets of H and $\circ : H\times H\to P^*(H)$ be a hyperoperation. Then H becomes a hypergroupoid with respect to "$\circ$". Let $A,B\in P^*(H)$ and $x\in H$, then we define the hyperoperation "$\circ$" as follows:

$$A\circ B=\bigcup_{a\in A,b\in B} a\circ b \text{ and } a\circ B=\{a\}\circ B,\quad A\circ b=A\circ\{b\}.$$

The hyperproduct of elements $t_1,t_2,t_3,\dots,t_n$ of H is denoted by $\Pi_{i=1}^{n}t_i$ and is equal to $t_1\circ\Pi_{i=2}^{n}t_i$. An algebraic system $(H,\circ)$ endowed with a hyperoperation is called a hypergroupoid. A hypergroupoid becomes a quasihypergroup if for every $t\in H$, $t\circ H=H\circ t=H$, (this condition is known as reproductive law).

Definition 1. [29] A hypergroupoid $(H,\circ)$, is called an LA-semihypergroup, if the left invertive law is satisfied in H with respect to "$\circ$", i.e. $(g\circ h)\circ i=(i\circ h)\circ g$, $\forall g,h,i\in H$.

Example 1. [29] Let $H=\mathbb{Z}$. We define the hyperoperation "$\circ$" on H by $t\circ w=w-t+3\mathbb{Z}$, then $(H,\circ)$ is an LA-semihypergroup.

Definition 2. [27] A hypergroupoid $(H,\circ)$ becomes an LA-hypergroup if it satisfies the following two properties:
(i) $\forall g,h,i\in H$, $(g\circ h)\circ i=(i\circ h)\circ g$,
(ii) $\forall g\in H$, $g\circ H=H\circ g=H$.

Example 2. [27] Let $H=\{g,h,i\}$ be a set. We define the hyperoperations "$\circ_1$" and "$\circ_2$" as follows:

$\circ_1$	g	h	i
g	g	H	H
h	H	$\{h,i\}$	$\{h,i\}$
i	H	h	h

$\circ_2$	g	h	i
g	g	H	H
h	H	$\{h,i\}$	$\{h,i\}$
i	H	$\{g,h\}$	$\{g,h\}$

then $\left(H,\circ_1\right)$ and $\left(H,\circ_2\right)$ are LA-hypergroups.

Definition 3. [27] An algebraic system $(S,+,\times)$ becomes an LA-hyperring, if the following three conditions are satisfied:
(i) S is an LA-hypergroup with respect to "+",
(ii) S is an LA-semihypergroup with respect to "·",
(iii) $f(w+t)=(fw)+(ft)$, $\forall f,w,t\in S$.

Example 3. [27] Let $S=\{j,k,l\}$. Then w.r.t. the hyperoperations "+" and "×" defined in the following tables, $(S,+,\times)$ is an LA-hyperring.

$+$	j	k	l
j	j	S	S
k	S	$\{k,l\}$	$\{k,l\}$
l	S	S	S

$\times$	j	k	l
j	j	j	j
k	j	S	l
l	j	S	S

.

Definition 4. [29] An algebraic system $\left(H,\circ,e,^{-1}\right)$, where e is identity element of H, $^{-1}:H\rightarrow H$ is a unitary operation and $\circ: H\times H\rightarrow P^{*}(H)$ is a hyperoperation on H, is called an LA-polygroup, if for all $l,s,g\in H$, the following axioms are satisfied:
(i) $(l\circ s)\circ g=(g\circ s)\circ l$,
(ii) $g\circ H=H\circ g=H$,
(iii) there is an element $e\in H$, such that $e\circ g=g$, $\forall g\in H$, this element e is called left identity,
(iv) inverse of each $g\in H$ exists in H (i.e. $e\in g\circ g^{-1}\cap g^{-1}\circ g$),
(v) $l\in s\circ g$ implies that $s\in l\circ g^{-1}$.

In this definition e is an element which is identity from left side. From the above properties we see that the following results hold in an LA-polygroup:

$$e^{-1}=e \text{ and } \left(l^{-1}\right)^{-1}=l.$$

Example 4. [29] Let $H=\left\{v_1,v_2,v_3\right\}$ be a set. Then $\left(H,\circ,u,^{-1}\right)$ is an LA-polygroup where "$\circ$" in H is defined as follows

$\circ$	v_1	v_2	v_3
v_1	v_1	v_2	v_3
v_2	v_3	$\left\{v_1,v_2,v_3\right\}$	$\left\{v_2,v_3\right\}$
v_3	v_2	$\left\{v_1,v_3\right\}$	$\left\{v_1,v_2,v_3\right\}$

,

the element v_1 is the left identity, $^{-1}:H\rightarrow H$ is taken as:

	v_1	v_2	v_3
-1	v_1	v_2	v_3

.

Definition 5. [29] Let $\left(H,\circ,u,^{-1}\right)$ be an LA-polygroup and U be a subset of H, such that $U\neq\emptyset$, then U becomes an LA-subpolygroup of H if $\left(U,\circ,u,^{-1}\right)$ is an LA-polygroup.

3. Left almost hypermodules

In this section, we explain the basic concept of left almost hypermodule (abbreviated LA-hypermodule). We also discuss the characteristics of an LA-hypermodule and provide some examples on how to construct new hypercompositional structure. We also discuss the useful properties concerning with LA-hypermodules.

Definition 6. Let M be a set that contains at least one element. Then M becomes a left almost hypermodule over the left almost hyperring S if $(M,\circ)$ is a left almost polygroup and there exists a mapping $\diamond: S\times M\rightarrow P^{*}(M)$ by $(s,t)\rightarrow s\diamond t$ such that, $\forall s_1, s_2\in S$ and $\forall t_1, t_2\in M$, the following axioms are satisfied:
(1) $s_1\diamond\left(t_1\circ t_2\right)=\left(s_1\diamond t_1\right)\circ\left(s_1\diamond t_2\right)$,
(2) $\left(s_1+s_2\right)\diamond t_1=\left(s_1\diamond t_1\right)\circ\left(s_2\diamond t_1\right)$,
(3) $\left(s_1\times s_2\right)\diamond t_1=s_1\diamond\left(s_2\diamond t_1\right)$.

Example 5. Let $S=\{j,k,l\}$ be a set with the hyperoperation $+$ and $\times$ defined in the following tables:

$+$	j	k	l
j	j	S	S
k	S	$\{k,l\}$	$\{k,l\}$
l	S	S	S

$\times$	j	k	l
j	j	j	j
k	j	S	l
l	j	S	S

.

Then $(S,+,\times)$ is an LA-hyperring. Let $M=\{l_1,l_2,l_3\}$ be a set with the hyperoperation "$\circ$" defined as follows:

$\circ$	l_1	l_2	l_3
l_1	l_1	l_2	l_3
l_2	l_3	M	$\{l_2,l_3\}$
l_3	l_2	$\{l_2,l_3\}$	M

Then $(M,\circ)$ is an LA-polygroup. Now we define the external product $\diamond : S\times M\to P^*(M)$ as follows:

$\diamond$	l_1	l_2	l_3
j	l_1	l_1	l_1
k	l_1	M	M
l	l_1	M	M

Then M is an LA-hypermodule over the LA-hyperring S.

Example 6. Let $S=\{r,f,s\}$ be a set. The hyperoperation $+$ and $\times$ are defined in S as follows:

$+$	r	f	s
r	S	S	S
f	$\{r,f\}$	$\{f,s\}$	$\{f,s\}$
s	$\{r,s\}$	$\{f,s\}$	$\{f,s\}$

$\times$	r	f	s
r	S	$\{f,s\}$	$\{f,s\}$
f	$\{f,s\}$	$\{f,s\}$	s
s	$\{f,s\}$	f	$\{f,s\}$

Then $(S,+,\times)$ is an LA-hyperring. Let $M=\{g,h,i,t,m\}$ be a set with the hyperoperation $\circ$ defined as follows:

$\circ$	g	h	i	t	m
g	g	h	i	t	m
h	i	$\{h,i\}$	$\{g,h,i\}$	t	m
i	h	$\{g,h,i\}$	$\{h,i\}$	t	m
t	t	t	t	$\{g,h,i\}$	m
m	m	m	m	m	$\{g,h,i,t\}$

Then $(M,\circ)$ is an LA-polygroup. Now we define the external product $\diamond : S\times M\to P^*(M)$ as follows:

$\diamond$	g	h	i	t	m
r	g	$\{g,h,i\}$	$\{g,h,i\}$	$\{g,h,i\}$	$\{g,h,i\}$
f	g	$\{g,h,i\}$	$\{g,h,i\}$	$\{g,h,i\}$	$\{g,h,i\}$
s	g	$\{g,h,i\}$	$\{g,h,i\}$	$\{g,h,i\}$	$\{g,h,i\}$

Then M is an LA-hypermodule over the LA-hyperring S.

Example 7. Consider a finite set M that contains at least 3 elements. Define a hyperoperation $\circ$ on M as given below:

$$t_p \circ t_q = \begin{cases} t_q & \text{for } p=1, \\ t_k & \text{for } q=1 \text{ and } k\equiv 2-p \bmod |M|, \\ M & \text{for } p=q \text{ and } p\neq 1, q\neq 1, \\ M^*=M\setminus\{t_1\} & \text{for } p\neq q \text{ and } p\neq 1, q\neq 1. \end{cases}$$

Then $(M,\circ)$ becomes an LA-polygroup [29] and $t_1\circ t_q=t_q$, for $q=1,2,3,...,n$ and $^{-1} : M\to M$ shows the inverses of all elements of M. This inverse operation is explained in table given below:

	t_1	t_2	t_3	.	.	.	$t_{\|M\|}$
$^{-1}$	t_1	t_2	t_3	.	.	.	$t_{\|M\|}$

Let $S=\{s_1,s_2,s_3\}$ be a set. The hyperoperation $+$ and $\times$ is defined as given below:

$+$	s_1	s_2	s_3
s_1	s_1	S	S
s_2	S	$\{s_2,s_3\}$	$\{s_2,s_3\}$
s_3	S	$\{s_1,s_2\}$	$\{s_1,s_2\}$

$\times$	s_1	s_2	s_3
s_1	s_1	s_1	s_1
s_2	s_1	S	S
s_3	s_1	$\{s_2,s_3\}$	S

Then $(S,+,\times)$ become an LA-hyperring. Now we define the external product $\diamond : S \times M \to P^*(M)$ as given below:

$$s_p \diamond t_q = \begin{cases} t_1 & \text{for } q = 1, \\ M & \text{for } q \neq 1, p = 1, \\ \{t_1, t_{q+1}, t_{q+1} \circ t_1\} & \text{for } q \neq 1, p = 2, \text{ and } t_q \circ t_1 = t_{|M}, t_{|M|} \circ t_1 = t_q, \\ M & \text{for } q \neq 1, p = 2, \text{ and } t_q \circ t_1 \neq t_{|M|}, \\ M & \text{for } q \neq 1, p = 3, \text{ and } t_q \circ t_1 = t_{|M|}, t_{|M|} \circ t_1 = t_q, \\ M^* = M \setminus \{t_1\} & \text{for } q \neq 1, p = 3, \text{ and } t_q \circ t_1 \neq t_{|M|}. \end{cases}$$

Then M is an LA-hypermodule over the LA-hyperring S.

We explain the above general form of LA-hypermodule by an example.

Example 8. Consider a set $M = \{t_1, t_2, t_3, t_4, t_5, t_6\}$ and define a hyperoperation $\circ$ on M as given below:

$\circ$	t_1	t_2	t_3	t_4	t_5	t_6
t_1	t_1	t_2	t_3	t_4	t_5	t_6
t_2	t_6	M	M^*	M^*	M^*	M^*
t_3	t_5	M^*	M	M^*	M^*	M^*
t_4	t_4	M^*	M^*	M	M^*	M^*
t_5	t_3	M^*	M^*	M^*	M	M^*
t_6	t_2	M^*	M^*	M^*	M^*	M

where $M^* = \{t_2, t_3, t_4, t_5, t_6\}$. Then $(M, \circ)$ is an LA-polygroup. Now the external product $\diamond : S \times M \to P^*(M)$ is defined in the following table:

$\diamond$	t_1	t_2	t_3	t_4	t_5	t_6
s_1	t_1	M	M	M	M	M
s_2	t_1	$\{t_1, t_3, t_5\}$	M	M	M	$\{t_1, t_3, t_5\}$
s_3	t_1	M	M^*	M^*	M^*	M

Then M is an LA-hypermodule over the LA-hyperring S.

Proposition 1. *Let S be an LA-hyperring and M be an LA-polygroup, such that for all $t \in M$, $t \circ t^{-1}$ contains at least one element of M other than the left identity e. If we define $\diamond : S \times M \to P^*(M)$ as follows:*

$$s \diamond t = \begin{cases} e & \text{if } t = e, \\ M & \text{if } t \neq e. \end{cases}$$

Then M is an LA-hypermodule over the LA-hyperring S.

4. LA-subhypermodules

Definition 7. Let M be an LA-hypermodule over the LA-hyperring S and $\emptyset \neq A \subseteq M$, then A is said to be an LA-subhypermodule of M, if A is itself an LA-hypermodule over the LA-hyperring S.

Example 9. Let $S = \{p, v, r, d\}$ be a set and the hyperoperations $+$ and $\times$ are defined as follows:

$+$	p	v	r	d
p	p	$\{p, v\}$	S	S
v	$\{p, v\}$	$\{p, v\}$	S	$\{r, d\}$
r	S	$\{r, d\}$	$\{r, d\}$	$\{r, d\}$
d	S	S	S	S

$\times$	p	v	r	d
p	p	p	p	p
v	p	$\{p, v\}$	S	S
r	p	S	S	d
d	p	S	$\{r, d\}$	S

Then $(S,+,\times)$ is an LA-hyperring [30]. Let $M = \{y, a, q, o, b\}$ be a set and the hyperoperation "$\circ$" defined in M as follows:

$\circ$	y	a	q	o	b
y	y	a	q	o	b
a	q	$\{a, q\}$	$\{y, a\}$	o	b
q	a	$\{y, q\}$	$\{a, q\}$	o	b
o	o	o	o	M	$\{o, b\}$
b	b	b	b	$\{o, b\}$	M

Then $(M, \circ)$ is an LA-polygroup. Now we define the external product $\diamond : R \times M \to P^*(M)$ as follows:

$\diamond$	y	a	q	o	b
p	y	$\{y,a,q\}$	$\{y,a,q\}$	$\{y,a,q\}$	$\{y,a,q\}$
v	y	$\{y,a,q\}$	$\{y,a,q\}$	$\{y,a,q,o,b\}$	$\{y,a,q,o,b\}$
r	y	$\{y,a,q\}$	$\{y,a,q\}$	$\{y,a,q,o,b\}$	$\{y,a,q,o,b\}$
d	y	$\{y,a,q\}$	$\{y,a,q\}$	$\{y,a,q,o,b\}$	$\{y,a,q,o,b\}$

.

Then M becomes an LA-hypermodule over the LA-hyperring S. If we consider $A = \{y, a, q\}$, then clearly A is an LA-subhypermodule of M.

Lemma 1. *Let A be a subset of an LA-hypermodule M such that $A \neq \emptyset$, then A is an LA-subhypermodule of M iff:*
(i) $x \circ y \subseteq A$, $\forall x, y \in A$,
(ii) $x^{-1} \in A$, $\forall x \in A$,
(iii) $r \diamond x \subseteq A$, $\forall r \in S$, and $x \in A$.

Proof. Straightforward. □

Lemma 2. *Let M be an LA-hypermodule then the followings properties are satisfied for all $p, q, r, s \in M$:*
(i) $(p \circ q) \circ (r \circ s) = (p \circ r) \circ (q \circ s)$,
(ii) $p \circ (q \circ r) = q \circ (p \circ r)$,
(iii) $(p \circ q) \circ (r \circ s) = (s \circ r) \circ (q \circ p)$.

Proof. Straightforward. □

Lemma 3. *Let M be an LA-hypermodule and U be an LA-subhypermodule of M. Then for all $a, b \in M$, the following results are true:*
(i) $U = U \circ U$,
(ii) $e \circ U = U \circ e = U$,
(iii) $a \circ U = (U \circ a) \circ e$,
(iv) $(a \circ b) \circ U = U \circ (b \circ a)$.

Proof. Straightforward. □

Remark 1. We can partition the LA-hypermodule only into right coset (or left coset) and there is no requirement of two side decomposition.

Definition 8. Let M be an LA-hypermodule over the LA-hyperring S and A be an LA-subhypermodule of M. Then the quotient LA-polygroup $M/A = \{A \circ t | t \in M\}$, with the external product $\circledast : S \times M/A \to P^*(M/A)$ defined by $(r, A \circ t) \to A \circ r \diamond t$ is an LA-hypermodule and is called quotient LA-hypermodule of M by A.

Definition 9. If A is an LA-subhypermodule of an LA-hypermodule M, then we define the relation $t_1 \equiv t_2$ iff $t_1 \circ A = t_2 \circ A$, for every $t_1, t_2 \in M$. This relation is denoted by $t_1 A^* t_2$.

Lemma 4. *Let A be an LA-subhypermodule of an LA-hypermodule M. Then, A^* is an equivalence relation.*

Proof. Straightforward. □

Definition 10. [27] Let $(S, +, \times)$ be an LA-hyperring and A is a subset of S. Then A is called an LA-subhyperring of S if $(A, +, \times)$ is itself an LA-hyperring.

Definition 11. [27] If A is an LA-subhyperring of an LA-hyperring $(S, +, \times)$, then A is called a left hyperideal if $R \times A \subseteq A$ and A is called right hyperideal if $A \times R \subseteq A$. An LA-subhyperring A is called a hyperideal if A is both the left hyperideal and right hyperideal.

Remark 2. If J is a hyperideal of an LA-hyperring S, then we define the relation $t \equiv u$ iff $t + J = u + J$. We denote this relation by tJ^*u.

Let A be an LA-subhypermodule of an LA-hypermodule M. Here, we construct quotient LA-polygroup $[M : A^*]$, and prove that when A is an LA-subhypermodule, $[M : A^*]$ is an abelian group. Let $\emptyset \neq X \subseteq M$ and $\{M_i : i \in I\}$ be the set of all LA-subhypermodules of M, such that this family of LA-subhypermodules contain X. Then, $\cap_{i \in I} M_i$ is called the LA-hypermodule generated by X and is denoted by $\langle X\rangle$. If $X=\{t_1,t_2,t_3,...,t_n\}$, then the LA-hypermodule $\langle X\rangle$ is denoted by $\langle t_1,t_2,t_3,...,t_n\rangle$. Let M be an LA-hypermodule, S_1 and M_1, M_2 be nonempty subsets of S and M, respectively. We define:

$$S_1\diamond M_1=\{x\in M : x\in\sum_{i=1}^{n} r_i\diamond t_i, r_i\in S_1, t_i\in M_1, n\in\mathbb{N}\},$$
$$M_1\circ M_2=\{x\in M : x\in t_1\circ t_2, t_1\in M_1, t_2\in M_2\},$$
$$\mathbb{Z}X=\{t\in M : t\in\sum_{i=1}^{n} n_i x_i, n_i\in\mathbb{Z}, x_i\in X\}.$$

Proposition 2. *Let J be a hyperideal of an LA-hyperring S. Then, $[S : J^*]$ is an LA-hyperring with the hyperoperations defined below:*

$$J^*(x)\boxplus J^*(y)=\left\{J^*(z)\,|\,z\in J^*(x)+J^*(y)\right\},$$
$$J^*(x)\circledcirc J^*(y)=\left\{J^*(z)\,|\,z\in J^*(x)\times J^*(y)\right\}.$$

Proof. We have to prove that $([S : J^*],\boxplus,\circledcirc)$ is an LA-hyperring, so we prove that:
(1) $[S : J^*],\boxplus)$ is an LA-hypergroup,
(2) $([S : J^*],\circledcirc)$ is an LA-semihypergroup,
(3) $\circledcirc$ is distributive with respect to $\boxplus$.
(1) $[S : J^*],\boxplus)$ is an LA-hypergroup.
(i) for all $J^*(x),J^*(y),J^*(z)\in[S : J^*]$, $(J^*(x)\boxplus J^*(y))\boxplus J^*(z)=(J^*(z)\boxplus J^*(y))\boxplus J^*(x)$.
Consider,

$$\begin{aligned}\left(J^*(x)\boxplus J^*(y)\right)\boxplus J^*(z)&=\left\{J^*(t)\,|\,t\in J^*(x)+J^*(y)\right\}\boxplus J^*(z)\\&=\left\{J^*(u)\,|\,u\in J^*(t)+J^*(z),t\in J^*(x)+J^*(y)\right\}\\&=\left\{J^*(u)\,|\,u\in\left(J^*(x)+J^*(y)\right)+J^*(z)\right\}\\&=\left\{J^*(u)\,|\,u\in\left(J^*(z)+J^*(y)\right)+J^*(x)\right\}\\&=\left\{J^*(p)\,|\,p\in J^*(z)+J^*(y)\right\}\boxplus J^*(x)\\&=\left(J^*(z)\boxplus J^*(y)\right)\boxplus J^*(x).\end{aligned}$$

(ii) for every $J^*(x)\in[S : J^*]$, $J^*(x)\boxplus[S : J^*]=[S : J^*]\boxplus J^*(x)=[S : J^*]$. Consider,

$$\begin{aligned}J^*(x)\boxplus\left[S : J^*\right]&=J^*(x)\boxplus\left\{J^*(t)\,|\,t\in R\right\}\\&=\left\{J^*(q)\,|\,q\in J^*(x)+J^*(t),t\in S\right\}\\&=\left\{J^*(q)\,|\,q\in J^*(x)+J^*(S)\right\}\\&=\left[S : J^*\right].\end{aligned}$$

Similarly, we can prove that, $[S : J^*]\boxplus J^*(x)=[S : J^*]$. Hence, $[S : J^*],\boxplus)$ is an LA-hypergroup.
(2) $([S : J^*],\circledcirc)$ is an LA-semihypergroup.
(i) for all $J^*(x),J^*(y),J^*(z)\in[S : J^*]$, $(J^*(x)\circledcirc J^*(y))\circledcirc J^*(z)=(J^*(z)\circledcirc J^*(y))\circledcirc J^*(x)$.
Consider,

$$\begin{aligned}\left(J^*(x)\circledcirc J^*(y)\right)\circledcirc J^*(z)&=\left\{J^*(t)\,|\,t\in J^*(x)\times J^*(y)\right\}\circledcirc J^*(z)\\&=\left\{J^*(u)\,|\,u\in J^*(t)\times J^*(z),t\in J^*(x)\circ J^*(y)\right\}\\&=\left\{J^*(u)\,|\,u\in\left(J^*(x)\times J^*(y)\right)\times J^*(z)\right\}\\&=\left\{J^*(u)\,|\,u\in\left(J^*(z)\times J^*(y)\right)\times J^*(x)\right\}\\&=\left\{J^*(p)\,|\,p\in J^*(z)\times J^*(y)\right\}\circledcirc J^*(x)\\&=\left(J^*(z)\circledcirc J^*(y)\right)\circledcirc J^*(x).\end{aligned}$$

Hence, $([S : J^*],\circledcirc)$ is an LA-semihypergroup.
(3) "$\circledcirc$" is distributive with respect to "$\boxplus$".
For all $J^*(x),J^*(y),J^*(z)\in[S : J^*]$, $J^*(x)\circledcirc(J^*(y)\boxplus J^*(z))=(J^*(x)\circledcirc J^*(y))\boxplus(J^*(x)\circledcirc J^*(z))$.
Consider,

$$\begin{aligned} J^*(x) \circledcirc \left(J^*(y) \boxplus J^*(z)\right) &= J^*(x) \circledcirc \left\{J^*(t) \,|\, t \in J^*(x) + J^*(y)\right\} \\ &= \left\{J^*(u) \,|\, u \in J^*(x) \times J^*(t), t \in J^*(x) + J^*(y)\right\} \\ &= \left\{J^*(u) \,|\, u \in J^*(x) \times (J^*(y) + J^*(z))\right\} \\ &= \left\{J^*(u) \,|\, u \in (J^*(x) \times J^*(y)) + (J^*(x) \times J^*(z)\right\} \\ &= \left\{J^*(p) \,|\, p \in J^*(x) \times J^*(y)\right\} \boxplus \left\{J^*(q) \,|\, q \in J^*(x) \times J^*(z)\right\} \\ &= \left(J^*(x) \circledcirc J^*(y)\right) \boxplus \left(J^*(x) \circledcirc J^*(z)\right). \end{aligned}$$

Hence, "$\circledcirc$" is distributive with respect to "$\boxplus$". Therefore, $([S : J^*], \boxplus, \circledcirc)$ is an LA-hyperring. □

Theorem 1. *Let M be an LA-hypermodule over an LA-hyperring S. Let I be a hyperideal of S and W be an LA-subhypermodule of M. Then, $[M : W^*]$ is an LA-hypermodule over the LA-hyperring $[S : I^*]$ with the following hyperoperations:*

$$\begin{aligned} W^*(m_1) \boxtimes W^*(m_2) &= \left\{W^*(m) \,|\, m \in W^*(m_1) \circ W^*(m_2)\right\}, \\ I^*(r) \boxdot W^*\left(m_1\right) &= \left\{W^*(m) \,|\, m \in I^*(r) \diamond W^*(m_1)\right\}. \end{aligned}$$

And $[M : W^*]$ is an LA-hypermodule over the LA-hyperring S, with the following hyperoperations:

$$\begin{aligned} W^*(m_1) \boxtimes W^*(m_2) &= \left\{W^*(m) \,|\, m \in W^*(m_1) \circ W^*(m_2)\right\} \\ r \boxdot W^*(m_1) &= \left\{W^*(m) \,|\, m \in r \diamond W^*(m_1)\right\}. \end{aligned}$$

Proof. First we have to prove that, $[M : W^*]$ is an LA-hypermodule over the LA-hyperring $[S : I^*]$ with the following hyperoperations:

$$\begin{aligned} W^*(m_1) \boxtimes W^*(m_2) &= \left\{W^*(m) \,|\, m \in W^*(m_1) \circ W^*(m_2)\right\}, \\ I^*(r) \boxdot W^*\left(m_1\right) &= \left\{W^*(m) \,|\, m \in I^*(r) \diamond W^*(m_1)\right\}. \end{aligned}$$

As $([M : W^*], \boxtimes)$ is an LA-polygroup, so we prove the following axioms:
(1): $I^*(r) \boxdot \left(W^*(m_1) \boxtimes W^*(m_2)\right) = \left(I^*(r) \boxdot W^*\left(m_1\right)\right) \boxtimes \left(I^*(r) \boxdot W^*\left(m_2\right)\right)$.
Consider,

$$\begin{aligned} I^*(r) \boxdot \left(W^*(m_1) \boxtimes W^*(m_2)\right) &= I^*(r) \boxdot \left\{W^*(m) \,|\, m \in W^*(m_1) \circ W^*(m_2)\right\} \\ &= \left\{W^*(t) \,|\, t \in I^*(r) \diamond W^*(m), m \in W^*(m_1) \circ W^*(m_2)\right\} \\ &= \left\{W^*(t) \,|\, t \in I^*(r) \diamond \left(W^*(m_1) \circ W^*(m_2)\right)\right\} \\ &= \left\{W^*(t) \,|\, t \in (I^*(r) \diamond W^*(m_1)) \circ (I^*(r) \diamond W^*(m_2))\right\} \\ &= \left\{W^*(p) \,|\, p \in I^*(r) \diamond W^*(m_1)\right\} \boxtimes \left\{W^*(q) \,|\, q \in I^*(r) \diamond W^*(m_2)\right\} \\ &= \left(I^*(r) \boxdot W^*\left(m_1\right)\right) \boxtimes \left(I^*(r) \boxdot W^*\left(m_2\right)\right). \end{aligned}$$

(2): $\left(I^*(r_1) \circ I^*(r_2)\right) \boxdot N^*(m) = \left(I^*(r_1) \boxdot N^*(m)\right) \boxtimes \left(I^*(r_2) \boxdot N^*(m)\right)$.
Consider,

$$\begin{aligned} \left(I^*(r_1) \circ I^*(r_2)\right) \boxdot N^*(m) &= \left\{I^*(t) \,|\, t \in I^*\left(r_1\right) + I^*\left(r_2\right)\right\} \boxdot N^*(m) \\ &= \left\{N^*(p) \,|\, p \in I^*(t) * N^*(m), t \in I^*\left(r_1\right) + I^*\left(r_2\right)\right\} \\ &= \left\{N^*(p) \,|\, p \in \left(I^*\left(r_1\right) + I^*\left(r_2\right)\right) * N^*(m)\right\} \\ &= \left\{N^*(p) \,|\, p \in (I^*\left(r_1\right) * N^*(m)) + (I^*\left(r_2\right) * N^*(m))\right\} \\ &= \left\{N^*(u) \,|\, u \in I^*\left(r_1\right) * N^*(m)\right\} \boxplus \left\{N^*(v) \,|\, v \in I^*\left(r_2\right) * N^*(m)\right\} \\ &= \left(I^*(r_1) \boxdot N^*(m)\right) \boxplus \left(I^*(r_2) \boxdot N^*(m)\right). \end{aligned}$$

(3): $\left(I^*(r_1) \circledcirc I^*(r_2)\right) \boxdot W^*(m) = I^*(r_1) \boxdot \left(I^*(r_2) \boxdot W^*(m)\right)$.
Consider,

$$\begin{aligned} (\left(I^*(r_1) \circledcirc I^*(r_2)\right) \boxdot W^*(m) &= \left\{I^*(t) \,|\, t \in I^*\left(r_1\right) \times I^*\left(r_2\right)\right\} \boxdot W^*(m) \\ &= \left\{W^*(p) \,|\, p \in I^*(t) \diamond W^*(m), t \in I^*\left(r_1\right) \times I^*\left(r_2\right)\right\} \\ &= \left\{W^*(p) \,|\, p \in (\left(I^*\left(r_1\right) \times I^*\left(r_2\right)\right) \diamond W^*(m)\right\} \end{aligned}$$

$$= \{W^*(p) \,|\, p \in I^*(r_1) \times ((I^*(r_2) \diamond W^*(m))\}$$
$$= I^*(r_1) \boxdot \{W^*(w) \,|\, w \in I^*(r_2) \diamond W^*(m)\}$$
$$= I^*(r_1) \boxdot (I^*(r_2) \boxdot W^*(m)).$$

Hence, $[M : W^*]$ is an LA-hypermodule over the LA-hyperring $[S : I^*]$. Similarly, we can prove that $[M : W^*]$ is an LA-hypermodule over the LA-hyperring S, with the following hyperoperations:

$$W^*(m_1) \boxtimes W^*(m_2) = \{W^*(m) \,|\, m \in W^*(m_1) \circ W^*(m_2)\},$$
$$r \boxdot W^*(m_1) = \{W^*(m) \,|\, m \in r \diamond W^*(m_1)\}. \quad \square$$

Remark 3. Let M be an LA-hypermodule and N be an LA-subhypermodule of M, then the left identity of $[M : N^*]$ is $\{N\}$.

Proposition 3. *Let M be an LA-hypermodule such that for all $x \in M$, $x \circ x^{-1} = e$ and N be an LA-subhypermodule of M. Then for every $h_1, h_2 \in M$, following statements are equivalent:*
(1) $h_1 \in N \circ h_2$,
(2) $h_1 \circ h_2^{-1} \subseteq N$,
(3) $h_1 \circ h_2^{-1} \cap N \neq \emptyset$.

Proof. (1) $\Longrightarrow$ (2): Let $h_1 \in N \circ h_2$

$$\Longrightarrow h_1 \circ h_2^{-1} \subseteq (N \circ h_2) \circ h_2^{-1} \text{ (as } h_1 \in N \circ h_2)$$
$$= (h_2^{-1} \circ h_2) \circ N \text{ (by left invertive law)}$$
$$= e \circ N$$
$$\Longrightarrow h_1 \circ h_2^{-1} \subseteq N.$$

(2) $\Longrightarrow$ (3): Let $h_1 \circ h_2^{-1} \subseteq N$

$$\Longrightarrow h_1 \circ h_2^{-1} \cap N \neq \emptyset.$$

(3) $\Longrightarrow$ (1): Let $h_1 \circ h_2^{-1} \cap N \neq \emptyset$, this implies that, there exists an element $x \in h_1 \circ h_2^{-1} \cap N \Longrightarrow x \in h_1 \circ h_2^{-1}$ and $x \in N$. As, $x \in h_1 \circ h_2^{-1}$

$$\Longrightarrow h_1 \in x \circ (h_2^{-1})^{-1} = x \circ h_2$$
$$\subseteq N \circ h_2$$
$$\Longrightarrow h_1 \subseteq N \circ h_2. \quad \square$$

Definition 12. Let U be an LA-subhypermodule of an LA-hypermodule M. We define the set $\Omega(U)$ as follows:

$$\Omega(U) = \{t \in M \,|\, t \circ t^{-1} \subseteq U\}.$$

Example 10. Let $S = \{j, k, l\}$ be a set with the hyperoperations $+$ and $\times$ defined as follows:

$+$	j	k	l
j	j	S	S
k	S	$\{k,l\}$	$\{k,l\}$
l	S	S	S

$\times$	j	k	l
j	j	j	j
k	j	S	l
l	j	S	S

Then $(S, +, \times)$ is an LA-hyperring. Let $M = \{0, 1, 2, 3\}$ be a set with the hyperoperation $\circ$ defined as follows:

$\circ$	0	1	2	3
0	0	1	2	3
1	2	$\{1,2\}$	$\{0,1\}$	3
2	1	$\{0,2\}$	$\{1,2\}$	3
3	3	3	3	$\{0,1,2\}$

Then $(M, \circ)$ is an LA-polygroup. Now, we define the external product $\diamond : S \times M \to P^*(M)$ as follows:

$\diamond$	0	1	2	3
j	0	0	0	0
k	0	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$
l	0	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Then M is an LA-hypermodule over the LA-hyperring S, and $N = \{0, 1, 2\}$ is an LA-subhypermodule of M. So, $\Omega(N) = \{0, 1, 2, 3\} = M$.

Proposition 4. *Let M be an LA-hypermodule and $p_1, p_2 \in \Omega(\{e\})$, then, $p_1 \circ p_2$ is a singleton set.*

Proof. Let $p_1, p_2 \in \Omega(\{e\})$, such that $p_1 \circ p_1^{-1} \subseteq \{e\}$, and $p_2 \circ p_2^{-1} \subseteq \{e\}$. Let $x, y \in p_1 \circ p_2$, then,

$$\begin{aligned} x \circ y^{-1} &\subseteq \left(p_1 \circ p_2\right) \circ \left(p_1 \circ p_2\right)^{-1} \\ &= \left(p_2 \circ p_1\right) \circ \left(p_2^{-1} \circ p_1^{-1}\right) \\ &= \left(p_2 \circ p_2^{-1}\right) \circ (p_1 \circ p_1^{-1}) \\ &= e \circ e \\ &= e. \end{aligned}$$

Thus $x \circ y^{-1} \subseteq \{e\}$, this means that $x = y$. This implies that $p_1 \circ p_2$ is a singleton set. □

Proposition 5. *Let M be an LA-hypermodule. Then $\Omega(\{e\})$ is an abelian group and for every LA-subhypermodule N, $\Omega(\{e\}) \subseteq N$.*

Proof. Straightforward. □

Proposition 6. *Let M be an LA-hypermodule and N be a proper LA-subhypermodule (i.e. $N \neq \{e\}, N \neq M$), then $\Omega(N) = M$. Moreover, $(M, \circ)$ is an abelian group iff $\Omega(\{e\}) = M$.*

Proof. Straightforward. □

Definition 13. Let M be an LA-hypermodule. We define the set $H(M)$ as follows:

$$H(M) = \left\{x | x \in t \circ t^{-1}, \text{ for all } t \in M\right\}.$$

Theorem 2. *Let M be an LA-hypermodule. Then $(M, \circ)$ is an abelian group iff $H(M) = \{e\}$.*

Proof. Suppose $(M, \circ)$ is an abelian group, then $\Omega(\{e\}) = M$. As, $H(M)$ is smallest LA-subhypermodule of M. So, $H(M) = \{e\}$. Conversely, suppose $H(M) = \{e\}$, $\Longrightarrow \Omega(\{e\}) = M$. Hence $(M, \circ)$ is an abelian group. □

Definition 14. Let M be an LA-hypermodule over an LA-hyperring S. If $(M, \circ)$ is an abelian group, then M is called multiplicative LA-hypermodule.

Corollary 1. *Let M be an LA-hypermodule and A be an LA-subhypermodule of M. Then $[M : A^*]$ is a multiplicative LA-hypermodule over the LA-hyperring S.*

Proof. Suppose that N is an LA-subhypermodule of M. Then the left identity of $[M : N^*]$ is $\{N\}$ and $([M : N^*], \boxplus)$ is an abelian group. Hence, $[M : N^*]$ is a multiplicative LA-hypermodule. □

Theorem 3. *Let M be a multiplicative LA-hypermodule over the LA-hyperring S, and there exists a left identity e' with respect to $+$ in S. If $s \diamond e = e$, $\forall s \in S$, where $e \circ t = t$, $\forall t \in M$. Then the statements given below are equivalent:*
(1) there is an element $m \in M$, such that $|e' \diamond m| = 1$,
(2) there is an element $s \in S$, such that $|s \diamond e| = 1$,
(3) $|e' \diamond e| = 1$,
(4) for all $s \in S$, $m \in M$, we have $|s \diamond m| = 1$.

Proof. (2) $\Longrightarrow$ (3), let $s \in S$, such that $|s \diamond e| = 1$. We have:

$$\begin{aligned} e' \diamond e &= \left(s + s^{-1}\right) \diamond e \\ &= (s \diamond e) \circ \left(s^{-1} \diamond e\right) \\ &= e \circ e \\ &= \{e\}. \end{aligned}$$

Hence, $|e \diamond e| = 1$.
(3) $\Longrightarrow$ (4), let $|e' \diamond e| = 1$. Let $r \neq e'$ be an element of S, we have, $e' \diamond e = \left(s + s^{-1}\right) \diamond e = (s \diamond e) \circ \left(s^{-1} \diamond e\right)$. If there exist $x \neq y$ elements of $s \diamond e$, then $e' \diamond e$ contain $x \circ y^{-1} \neq e$ and e. This makes a contradiction to the fact that $|e' \diamond e| = 1$. Now for every $s \in S$ and $m \in M$, $s \diamond \left(m \circ m^{-1}\right) = (s \diamond m) \circ \left(s \diamond m^{-1}\right)$, it follows that $s \diamond m$ contains only one element. Hence, $\forall s \in S$, $m \in M$, we have $|s \diamond m| = 1$.
(4) $\Longrightarrow$ (1), let $\forall s \in S$, $m \in M$, we have $|s \diamond m| = 1$.
Then for $s = e'$, $m \in M$, we have $|e' \diamond m| = 1$.
(1) $\Longrightarrow$ (2), let for $s = e'$, there exists $m \in M$, we have $|e' \diamond m| = 1$. Then,

$$
\begin{aligned}
s \diamond e &= s \diamond \left(m \circ m^{-1}\right) \\
&= (s \diamond m) \circ \left(s \diamond m^{-1}\right).
\end{aligned}
$$

As, $s \diamond m$ and $s \diamond m^{-1}$ contains only one element. So, $|s \diamond e| = 1$. □

Remark 4. It can be concluded from the above Theorem 3, that, if one assertion of Theorem 3 is valid, then the multiplicative LA-hypermodule M is trivial, that is, an LA-module.

Proposition 7. *Let M be a multiplicative LA-hypermodule and e' is the left identity in S with respect to "+". Then:*
(i) $e \in s \diamond e$, for every $s \in S$,
(ii) $e' \in e' \diamond m$, for every $m \in M$,
(iii) If A is an LA-subhypermodule of M, then $A^*(m) \in [M : A^*]$, we have:

$$
|A^*(e) \boxdot A^*(m)| = 1.
$$

Proof. Straightforward. □

5. Homomorphisms on LA-hypermodules

Definition 15. Let M and M' are two LA-hypermodules over an LA-hyperring $(S, +, \times)$. Let $f : M \to M'$ be a mapping with $f(e) = e'$. Then f is said to be:
(1) a weak homomorphism if:
(i) $f(p \circ q) \subseteq f(p) \circ' f(q)$, $\forall p, q \in M$,
(ii) $f(s \diamond p) \subseteq s \diamond' f(p)$, $\forall s \in S$ and $p \in M$.
(2) a strong homomorphism if:
(i) $f(p \circ q) = f(p) \circ' f(q)$, $\forall p, q \in M$,
(ii) $f(s \diamond p) = s \diamond' f(p)$, $\forall s \in S$ and $p \in M$.

Lemma 5. *Let g be a strong homomorphism from an LA-hypermodule M into an LA-hypermodule M'. Let N_1 and N_2 be LA-subhypermodules of M and M', respectively. Then the following results are true:*
(i) The set $g\left(N_1\right)$ is an LA-subhypermodule of M',
(ii) The set $g^{-1}\left(N_2\right)$ is an LA-subhypermodule of M.

Proof. Straightforward. □

Lemma 6. *Let g be a strong homomorphism from an LA-hypermodule M into an LA-hypermodule M', then:*
(i) $g(e) = e'$,
(ii) $g\left(x^{-1}\right) \subseteq g(x)^{-1}$.

Proof. Straightforward. □

Corollary 2. *Let M_1 and M_2 are two LA-hypermodules over an LA-hyperring $(S, +, \times)$, such that $r \diamond_1 m = e_1$ and $r \diamond_2 m = e_2$, for all $r \in S$, where e_1 is the left identity of M_1 and e_2 is the left identity of M_2. Then $f : M_1 \to M_2$ is a strong homomorphism if $Kerf = M_1$.*

Proof. Straightforward. □

Lemma 7. *Let g be a strong homomorphism from an LA-hypermodule M into an LA-hypermodule M'. Then g is a one to one mapping iff $Kerg = \{e\}$.*

Proof. Let g be a one to one mapping, then by Lemma 5, $g(e) = e'$. Now let $x \in Kerg$, then by definition of kernel $g(x) = e'$. So $g(e) = e' = g(x) \Longrightarrow x = e$, hence $Kerg = \{e\}$. Conversely, let $Kerg = \{e\}$ and consider, $g(x) = g(y)$ for $x, y \in M$. Now for $g(x) = g(y)$, we have $g(x)\circ' g(x^{-1}) = g(y)\circ' g(x^{-1})$. Then, $g(e) \in g(x\circ x^{-1}) = g(y\circ x^{-1})$, so there is an element $t \in y\circ x^{-1}$ such that $e' = g(e) = g(t)$. So, $e = t \in y\circ x^{-1}$, this implies that $x = y$. Hence, g is a one to one mapping. □

Theorem 4. *Let h be a strong homomorphism from an LA-hypermodule M into an LA-hypermodule M' with kernel K such that K is an LA-subhypermodule of M. Then $M/K \cong M'$.*

Proof. Let h be a strong homomorphism, this implies that $h(e) = e'$ and $h(m\circ n) = h(m)\circ' h(n)$, for all $m, n \in M$ and $h(s \diamond m) = s \diamond' h(m)$, for each $s \in S$, and $m \in M$. Define a mapping $\lambda : M/K \to M'$ by $\lambda(K\circ x) = h(x)$, for each $x \in M$. We first prove that the mapping λ is well defined. Let $x, y \in M$, assume that $K\circ x = K\circ y \Longrightarrow x\circ y^{-1} \subseteq K$, let $a \in x\circ y^{-1}$. Therefore, $h(a) = e'$ and $h(a) \subseteq h(x\circ y^{-1}) = h(x)\circ' h(y^{-1}) = h(x)\circ' h(y)^{-1}$. Thus $h(x) = h(y)$, this shows that λ is well defined. Now we show that λ is onto, as for every $h(x) \in M'$, there exists $K\circ x \in M/K$, such that $\lambda(K\circ x) = h(x)$. Thus λ is onto. Now we have to show that λ is one to one. For this consider $h(x) = h(y)$. Then, $e_2 \in h(x)\circ' h(y)^{-1} = h(x)\circ' h(y^{-1}) = h(x\circ y^{-1})$, so there is an element $b \in x\circ y^{-1}$ with $b \in \ker h$. So, $x\circ y^{-1} \subseteq K$, $\Longrightarrow K\circ x = K\circ y$. This shows that λ is one to one. Now we will prove that λ is a strong homomorphism. Let $K\circ x, K\circ y \in M/K$, consider,

$$\begin{aligned}\lambda((K\circ x)\circ(K\circ y)) &= \lambda(K\circ(x\circ y))\\ &= h(x\circ y)\\ &= h(x)\circ' h(y)\\ &= \lambda(K\circ x)\circ' \lambda(K\circ y).\end{aligned}$$

Now consider,

$$\lambda(K\circ e) \;=\; h(e) \;=\; e'$$

and,

$$\begin{aligned}\lambda(r \circledast (K\circ x) &= \lambda(K\circ(r \diamond x))\\ &= h(r \diamond x)\\ &= r \diamond' h(x).\end{aligned}$$

This shows that λ is a strong homomorphism. As $\lambda : M/K \to M'$ is a bijective strong homomorphism. Hence $M/K \cong M'$. □

Theorem 5. *If N_1 and N_2 are LA-subhypermodules of an LA-hypermodule M, then $N_2/\left(N_1 \cap N_2\right) \cong \left(N_2\circ N_1\right)/N_1$.*

Theorem 6. *If N_1 and N_2 are LA-subhypermodules of an LA-hypermodule M, such that $N_1 \subseteq N_2$, then $\left(M/N_1\right)/\left(N_2/N_1\right) \cong M/N_2$.*

6. Regular relations

Definition 16. Let U be an LA-subhypermodule of an LA-hypermodule M and U^* be an equivalence relation on M. We extend U^* to non-empty subset of M by β^* and γ^* as follows:
Let $A, B \in P^*(M)$, where $P^*(M)$ is the family of all those subsets of M that contain at least one element. Now define:
$A\beta^* B \Longleftrightarrow$ for every $a \in A$, there exists an element $b \in B$ such that aU^*b, and for every $b \in B$, there exists an element $a \in A$ such that, bU^*a.
$A\gamma^* B \Longleftrightarrow$ for each $a \in A$, and for each $b \in B$, one has aU^*b.
where aU^*b, we mean $(a, b) \in U^*$.
An equivalence relation U^* on M is called regular (respectively strongly regular), if for all $p, q, x \in M$, and $s \in S$,
(i) $pU^*q \Longrightarrow (p\circ x)\,\beta^*\,(q\circ x)$ and $(x\circ p)\,\beta^*\,(x\circ q)$
(respectively $pU^*q \Longrightarrow (p\circ x)\,\gamma^*\,(q\circ x)$ and $(x\circ p)\,\gamma^*\,(x\circ q)$),
(ii) $pU^*q \Longrightarrow (s \diamond p)\,\beta^*\,(s \diamond q)$ (respectively $pU^*q \Longrightarrow (s \diamond p)\,\gamma^*\,(s \diamond q)$).

Theorem 7. *Let U be an LA-subhypermodule of an LA-hypermodule M. Let U^* be a regular relation on M, then $M/U^* = \{U^*(t) | t \in M\}$ is an LA-hypermodule over the LA-hyperring S with the following hyperoperations:*

$$\begin{aligned}U^*(t_1) \boxplus U^*(t_2) &= \left\{U^*(t)\,|\,t \in U^*(t_1)\circ U^*(t_2)\right\}\\ s \boxdot U^*(t_1) &= \left\{U^*(t)\,|\,t \in s \diamond U^*(t_1)\right\}.\end{aligned}$$

Proof. Straightforward. □

Corollary 3. *Let U be an LA-subhypermodule of an LA-hypermodule M. Then the equivalence relation defined as $(xU^{*}y$ iff $x \circ U = y \circ U)$ is strongly regular relation. Hence $([M : N^{*}], \boxplus)$ is an abelian group.*

Proof. Straightforward. □

7. Conclusion

We have introduced a new concept within hypermodules called the left almost hypermodule, briefly referred to as LA-hypermodule. We have presented a detailed analysis of prominent features and prospective consequences of the left almost hypermodule, instigating further exploration into its attributes. There are multiple characteristics of hypermodules which are true in nature for LA-hypermodules as well. The difference between hypermodules and LA-hypermodules is due to medial law, which holds for LA-hypermodules with respect to hyperoperation "$\circ$" as defined in M. Therefore, all theorems and subsequent outcomes in relation to the concept of normality are different for LA-hypermodules. We have used the idea of left almost polygroups to investigate the vicissitude from left almost polygroup to left almost hypermodule. The three isomorphism theorems are also valid for LA-hypermodules. Furthermore, we have defined the strongly and weakly regular relations on LA-hypermodules to study the outcomes in relation to homomorphism and regular relations.

Ethical approval

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CRediT authorship contribution statement

Nabilah Abughazalah: Visualization, Supervision, Methodology. **Shehzadi Salma Kanwal:** Writing – original draft, Investigation. **Mudsir Mehdi:** Writing – original draft, Formal analysis. **Naveed Yaqoob:** Writing – review & editing, Supervision, Project administration, Methodology, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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