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STIMULI, REINFORCERS, AND BEHAVIOR: AN INTEGRATION

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We propose that a fundamental unit of behavior is the concurrent discriminated operant, and we discuss in detail a quantitative model of the concurrent three-term contingency that is based on the notion that an animal's behavior is controlled to differing extents by both stimulus–behavior and behavior–reinforcer relations. We show how this model can describe performance in a variety of experimental procedures: conditional discrimination and matching to sample, both with and without reinforcement for responses that are traditionally identified as errors; conditional discrimination with more than two stimuli and choice alternatives; delayed matching to sample and delayed reinforcement in matching to sample; second-order and complex conditional discrimination; and multiple and concurrent schedules. Although the model is incomplete in its coverage, and may be incorrect, we believe that this conceptual approach will bear fruit in the development of behavior theory.

Key words: discriminated operant, conditional discrimination, stimulus control, reinforcement, detection models, matching, discriminability

> An adequate formulation of the interaction between an organism and its environment must always specify three things: (1) the occasion upon which a response occurs, (2) the response itself, and (3) the reinforcing consequences. The interrelations among them are the contingencies of reinforcement. (Skinner, 1969, p. 7)

With the specification of the three-term contingency quoted above, Skinner defined the *discriminated operant,*[1] which we take to be a fundamental analytic unit for the science of behavior. Some experimenters have concentrated on ''the occasion upon which a response occurs''—the antecedent stimulus. In general, they have arranged maximally different consequences in the presence of two stimuli, such as reinforcement versus extinction, and then varied some aspects of one or both stimuli. Conversely, those experimenters who have concentrated on ''the reinforcing consequences'' have usually explored the effects of various schedule contingencies within a single undifferentiated session, with no explicit antecedent stimuli. When two or more schedules have been studied within a single session, they have generally been correlated with highly distinctive stimuli. Although there are many exceptions to these overly simple generalizations, there have been few systematic efforts to study the joint effects of variations in reinforcing consequences and in accompanying stimuli. Neither is there much systematic information on the effects of variations in ''the response itself,'' and how different response definitions may interact with stimulus and reinforcer control. Here, we show that variations in each of the three terms of the discriminated operant may have functionally similar effects, and we present a

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[1] The definition of this and other technical terms can be found in Appendix A.

	Response 1	**Response 2**
Stimulus 1	R_1	
Stimulus 2		R_2

Fig. 1. The contingencies of the concurrent discriminated operant. Each of two different responses is reinforced in the presence of a different discriminative stimulus.

simple algebraic model that provides an economical summary of these effects in many standard experimental situations.

The model has two major components. The first component characterizes an organism's history of reinforcement for different responses in the presence of different stimuli during prolonged exposure to an experimental condition. Two parameters are identified with the confusability of the relations among the stimuli, responses, and reinforcers defining two or more discriminated operants; these parameters are used to derive an algebraic expression of the effective allocation of reinforcers accruing to those operants. The second component characterizes the way in which the effective allocation of reinforcers determines steady-state behavior.

The model will be developed initially for experimental paradigms that explicitly define two discriminated operants, where Response 1 is reinforced in the presence of, or following, Stimulus 1, and Response 2 is reinforced in the presence of, or following, Stimulus 2, as shown in the matrix of Figure 1. The responses are available concurrently, and the stimuli are presented successively. In some standard paradigms, Response 2 may be unspecified, as in single-response "go/no-go" successive discriminations. However, Herrnstein (1970) rightly pointed out that when an experimenter arranges for a response (B_1) to be reinforced (R_1), there exists by definition a complementary (or extraneous) class of responses (not-B_1, designated B_e by Herrnstein) that is reinforced by a complementary (or extraneous) class of reinforcers (not-R_1, designated R_e). The term *extraneous* is to be understood only in relation to the experimenter's arrangement of contingencies. Thus, all behavior occasioned by an antecedent stimulus occurs in a context of concurrent alternatives, whether measured or not. We therefore suggest that the fundamental unit of behavior is the *concurrent discriminated operant* and not (as suggested by Skinner, 1969) the single discriminated operant.

We begin by illustrating some qualitative similarities in the effects of the three terms defining concurrent discriminated operants within some standard experimental paradigms: (a) successive discriminations and multiple schedules, which correspond to one column of the matrix of Figure 1; (b) simultaneous discriminations and concurrent schedules, which correspond to collapsing the two rows of the matrix into a single row; and (c) conditional discriminations, which correspond to the full matrix.

The model that we develop here is based on previous modeling efforts by ourselves and our colleagues over a number of years, which are reviewed briefly. We then present the model for the basic conditional discrimination paradigm illustrated in Figure 1, and develop it for progressively more complex cases. The model's predictions are compared with conditional discrimination data from a variety of paradigms, including signal recognition and matching to sample with two or more defined stimuli and responses. We also treat the effects of reinforcing responses that are conventionally construed as errors in these paradigms. We then return to cases in which extraneous responses and reinforcers must be considered, and discuss ways to incorporate the effects of reinforcer magnitude or quality in future models. Although all of the data that we compare with model predictions are from nonhuman subjects, primarily pigeons, we conclude by considering the relevance of our model to research and application with humans.

SOME EQUIVALENCES AMONG THE TERMS OF THE DISCRIMINATED OPERANT

A number of experiments have demonstrated that varying one of the three terms of the discriminated operant may be functionally equivalent to varying another term, thus suggesting the possibility of a unified descriptive account.

Free-Operant Successive Discrimination and Multiple Schedules

Perhaps the simplest arrangement for the study of discriminated operants is the successive go/no-go or S^D/S^Δ free-operant paradigm. For example, a pigeon is trained to peck a key for food reinforcers arranged by a variable-interval (VI) schedule of reinforcement in the presence of one stimulus (designated $S+$, S^D, or more generally S_1), where S_1 alternates with a second stimulus (designated $S-$, S^Δ, or more generally S_2) in the presence of which responses never produce food. This procedure is known as a multiple VI extinction schedule of reinforcement, with its components defined by the stimuli and the schedules they accompany. As noted above, the procedure corresponds to the left column of the matrix in Figure 1. In effect, the procedure defines two successive discriminated operants: S_1:(peck → VI food) and S_2:(peck → no food). Our notation is intended to signify that the stimuli set the occasion for a specified outcome if a specified response occurs. We will subsequently refer to these relations by saying that a given stimulus signals a specified contingency of reinforcement.

The usual result of this procedure is a high rate of responding during S_1 and a near-zero rate during S_2. However, the response rate during S_1 depends on the rate of reinforcement arranged by the VI schedule: The higher the rate of reinforcement, the higher the rate of responding. At the same time, the rate of responding during S_2 depends on the physical difference between S_1 and S_2: The smaller the S_1-S_2 disparity, the higher the rate of S_2 responding. In the limit, with zero S_1-S_2 difference, the response rates become identical, at least if the occurrence of reinforcement or the passage of time cannot serve as cues for the component in effect. This would require that the two components alternate irregularly, that S_1 components end after each reinforcer, and that S_2 components end at random times (Alsop & Davison, 1991).

Cumming (1955) systematically explored both the rate of reinforcement during S_1 and the S_1-S_2 disparity with pigeons as subjects and with S_1 and S_2 defined by two luminance levels of a white keylight. He found that the ratio of response rates during S_1 versus S_2 increased with the S_1-S_2 disparity, and that for any given S_1-S_2 disparity, both response rates varied directly with the rate of reinforcement during S_1 so that their ratio was approximately constant. An alternative approach to the study of discriminated operant performance in multiple schedules is to hold the stimuli constant and vary the rates of reinforcement during both S_1 and S_2. For example, Reynolds (1963) trained pigeons to peck at red (S_1) and green (S_2) keys, where S_1 and S_2 alternated every 3 min and independent VI schedules were arranged in their presence. Over successive conditions, the VI schedules were varied systematically. In general, response rates were positively related to the rate of reinforcement arranged by the VI schedule in each component, and the ratio of response rates was an orderly increasing function of the ratio of reinforcer rates. From this line of research, it is clear that response rates are equal under two sorts of conditions: first, when the schedules are different and the accompanying stimuli are the same; and second, when the accompanying stimuli are different but the schedules are the same. Colloquially, in the first of these the subject "cannot" discriminate because the stimuli are indiscriminable, whereas in the second the subject "will" not discriminate in the sense that, if the consequences are indiscriminable, equal responding is obligatory.

Clearly, there are two continua to be explored here: the difference between the stimuli, and the difference between the reinforcement schedules. However, research on multiple schedules exemplifies the point that analyses of the effects of the stimuli and of the reinforcement schedules have been largely independent. One body of literature has followed the tradition of Guttman and Kalish (1956), using multiple VI extinction schedules while varying one or more stimulus dimensions during a maintained generalization test (for review, see Heinemann & Chase, 1975). A separate literature has followed the early work of Reynolds (1961), exploring the effects of various reinforcement schedules and component durations with constant, highly distinctive stimuli (for review, see Davison & McCarthy, 1988). Only a few studies (e.g., White, Pipe, & McLean, 1984) have systematically examined the joint effects of both determiners of discriminated operant performance in multiple schedules.

Choice: Simultaneous Discrimination and Concurrent Schedules

Traditional studies of discrimination learning have often presented two stimuli simultaneously in discrete trials, with their spatial locations (typically left-right) varied irregularly, and with reinforcement available for a single response directed toward one stimulus (S_1) but not to the other (S_2). Performance is usually measured as percentage choices of S_1, conventionally identified as "correct" responses. In most research, the stimuli differ substantially, and interest typically centers on the acquisition, transfer, or reversal of the discrimination as affected by other variables such as prior learning history or physiological intervention.

Clearly, acquisition and maintained accuracy will depend on the difference between the stimuli, and for this reason the simultaneous discrimination procedure has been used for psychophysical assessment of sensory sensitivity in well-trained animal subjects (e.g., Mentzer, 1966). In the limit, when the stimuli are identical, accuracy should fall to chance levels. Performance on a difficult luminance discrimination, under which accuracy was maintained at about 75% to 80%, was studied by Nevin (1967). He arranged discrete trials that ended after 2 s if no response occurred and found that the probability or schedule of reinforcement for correct choices affected the overall probability of response but not the ratio of S_1 to S_2 responses (and thus the percentage of correct responses). This result parallels Cumming's (1955) findings with free-operant multiple schedules described above. Although most simultaneous discrimination research has allowed only one response per trial, as in Nevin's (1967) study, free-operant simultaneous discrimination procedures with extended stimulus presentations, and with VI reinforcement of responses on one alternative and extinction of responses on the other, have also been studied (e.g., Honig, 1962).

Whether the procedure involves discrete trials or extended stimulus presentations, it may be construed as involving either two or four operants. The two-operant interpretation neglects response locations and emphasizes the stimuli: Respond to S_1 → food, and respond to S_2 → no food. In effect, this approach defines the responses by the stimuli toward which they are directed (where "to" is intended as shorthand for "directed toward"). The paradigm corresponds to collapsing the upper and lower rows of the matrix in Figure 1 into a single row, in which the contingencies are signaled by simultaneous presentation of S_1 and S_2. The four-operant interpretation refers to both stimulus and response locations: S_1 left:(respond left → food); S_1 left:(respond right → no food); S_1 right:(respond right → food); and S_1 right: (respond left → no food), thereby corresponding to the full matrix of Figure 1. There has been considerable theoretical debate over the correct interpretation (see Mackintosh, 1974, for review), with no generally accepted conclusion. In view of our interest in the response term, we opt for the four-operant approach.

A separate line of research has studied the effects of simultaneously available reinforcement schedules, known as *concurrent* schedules, which may be continuously available for free-operant responding or arranged in discrete trials (e.g. Herrnstein, 1961; Nevin, 1969a; see Davison & McCarthy, 1988, and Williams, 1988, for reviews). Because this work informs much of our thinking, it will be described in some detail.

In an early study of concurrent-schedule performance, Herrnstein (1961) trained pigeons to peck at either of two simultaneously available keys, with food reinforcers arranged by independent VI schedules. In one condition, for example, the average interval between reinforcers was 2.25 min for pecks on Key 1 and 4.5 min for pecks on Key 2. As a result, the birds could obtain about 27 reinforcers per hour on Key 1 and about 13 per hour on Key 2. In the course of a 60-reinforcer session, the birds made about 6,000 key pecks, with about 4,000 on Key 1 and 2,000 on Key 2. Thus, responding was distributed in about the same ratio as the obtained reinforcer rates. This result held for several other schedule combinations. The general result is expressed algebraically as

$$\frac{B_1}{B_2} = \frac{R_1}{R_2}, \tag{1}$$

where B_1 and B_2 are the numbers of responses emitted on the two keys, and R_1 and R_2 are the numbers of reinforcers obtained by pecks to those keys.

It is important to note that these response ratios were not constrained by the procedure: For example, all reinforcers could have been obtained by simply alternating from one key to the other, in which case B_1/B_2 would be 1.0 regardless of the two schedule values. To reduce the likelihood of this pattern of responding, most conditions of the experiment involved a penalty for changes from one key to the other known as a *changeover delay* (COD). Specifically, Herrnstein arranged that pecks could not be reinforced until at least 1.5 s had elapsed since a changeover from one key to the other. This COD prevented immediate reinforcement of simple alternation, and may be interpreted as establishing the independence of the two operants: Peck Key 1 → food, and peck Key 2 → food. Herrnstein found that switching was much less frequent and response allocation more nearly approximated exact matching when the COD was in effect, suggesting that matching may be the normative result if the two operants are indeed independent.

Another method for arranging concurrent schedules had been described earlier by Findley (1958). His method involved correlating the two VI schedules with different stimuli on a main key, as in multiple schedules, but allowing the subject to change over from one to the other by pecking a second *switching* or *changeover* key. In this arrangement, the concurrent operants are defined by the explicit stimuli on the main key rather than topographically by key location, as in Herrnstein's (1961) study. Despite these differences, the results were similar to Herrnstein's, in that the ratio of responses, and of times spent in the presence of each stimulus, roughly equaled or matched the ratio of the reinforcer rates. However, there were some systematic deviations from matching (see Nevin, 1984, for reanalysis of Findley's data), and Equation 1 must be modified to describe them. A simple modification that captures these and many other results very well is the generalized matching law (Baum, 1974, 1979):

$$\frac{B_1}{B_2} = c\left(\frac{R_1}{R_2}\right)^a, \qquad (2)$$

where c represents a constant bias toward one or the other operant, evident in unequal responding when the reinforcer rates are equal, and a represents sensitivity to reinforcement. When c is 1.0 and a is 1.0, Equation 2 reduces to Equation 1 and describes strict matching, as found by Herrnstein (1961). When a is 0, response ratios are constant regardless of the reinforcer ratios, a result that would arise if the subject collected all reinforcers simply by pecking the two keys (or stimuli) in strict alternation, or in any other pattern that was independent of the two schedule values.

This discussion suggests that concurrent operants may be defined either by the response (e.g., by key location) or by the stimulus signaling the schedule in effect, with, as far as we know, roughly equivalent results. Moreover, some data suggest that sensitivity to reinforcement depends on the extent to which the two operants are differentiated with respect to reinforcement by the COD (see Davison & McCarthy, 1988, for a discussion of these findings). Another method for varying the difference between concurrent operants was described by Miller, Saunders, and Bourland (1980). They employed the Findley switching-key procedure and varied the relative rates of reinforcement. Also, across groups of birds they varied the similarity of the stimuli defining the two operants (lines of various orientations projected on the main key). For one group, the stimuli were lines of the same orientation; for the second group, the lines differed by 15°; and for the third group, the lines differed by 45°. The values of a were about 0.17, 0.33, and 0.99 for these three sets of stimulus disparities. (With 0° disparity, the value of a should have been 0; however, as Alsop & Davison, 1991, suggested, the reinforcer rates may themselves have provided cues to the different schedules.) The general conclusion is that sensitivity to reinforcement depends on the extent to which concurrent operants are differentiated by the variables that define them.

Conditional Discriminations

Conditional discriminations combine the successive stimulus presentations of multiple schedules and the simultaneous availability of two choices with their associated schedules, as in concurrent schedules. As shown in the matrix of Figure 1, reinforcement is conditional

Table 1

Summary examples of the four discriminated operants in various paradigms described in the text.

	Stimulus	Response	Consequence
Signal detection	Signal + noise	"Yes"	Payoff
	Signal + noise	"No"	Penalty
	Noise	"Yes"	Penalty
	Noise	"No"	Payoff
Matching to sample	Red sample	Peck red	Food
	Red sample	Peck green	No food
	Green sample	Peck red	No food
	Green sample	Peck green	Food
Free operant	Vertical line	Peck Key 1	VI food
	Vertical line	Peck Key 2	No food
	Horizontal line	Peck Key 1	No food
	Horizontal line	Peck Key 2	VI food

upon the current or prior stimulus. The standard yes-no signal-detection experiment provides one example: If a signal is presented, "yes" is followed by a payoff and "no" is followed by a penalty; if the signal is not presented, "no" is followed by a payoff and "yes" is followed by a penalty. The well-known matching-to-sample procedure provides another example: If the sample color on the center key of a three-key chamber is lighted red, and the side keys are then lighted with red and green comparison colors, food is given for pecks to red; but if the sample is green, food is given for pecks to green. Although both of these examples employ discrete-trial presentations, conditional discriminations may also be arranged for free-operant behavior during extended stimulus presentations. For example, White (1986) trained pigeons in a two-key chamber with VI reinforcement of pecks on Key 1 and extinction of pecks on Key 2 when both keys had vertical lines projected on them. Conversely, he arranged VI reinforcement for pecks on Key 2 and extinction of pecks on Key 1 when both keys had horizontal lines projected on them. The four discriminated operants in these examples are summarized in Table 1. In each example, two discriminative relations are successive, determined by the stimulus presentation, as in multiple schedules, and two are simultaneous, as in concurrent schedules. Thus, a full account of performance in the conditional discrimination paradigm should encompass multiple- and concurrent-schedule performances as well.

The results of several studies suggest that conditional discrimination performance depends on stimulus, response, and reinforcement terms in closely interrelated ways. The accuracy of performance obviously depends on the physical difference between the conditional stimuli. For example, Swets (1959) varied the signal-to-noise ratio in auditory signal detection with human subjects. With pigeons, McCarthy and Davison (1980a) varied signal duration, and Wright (1972) varied wavelength differences of lighted keys. All found that accuracy (i.e., the degree to which responses conformed to the experimenter's definition of reinforceable responses) increased with stimulus disparity.

Not surprisingly, conditional discrimination performance also depends on the differentiation between the responses. For example, Eckerman (1970) trained pigeons to peck different locations along a lighted 25-cm strip in the presence of different wavelengths. Three groups differed according to the response definition on the strip key. For Group 1, both responses were defined near the center of the strip; for Group 2, responses about 4 cm to the right of center were reinforced on 506-nm trials, and responses 4 cm to the left of center were reinforced on 583-nm trials; and for Group 3, responses about 8 cm to the right of center were reinforced on 506-nm trials, and responses 8 cm to the left of center were reinforced on 583-nm trials. Group 1 made as many "errors" as correct responses; Group 2 made relatively few errors; and Group 3 made virtually none. This result complements the findings of Miller et al. (1980) with responses defined by their location rather than by the stimulus.

The reinforcer is, of course, the third term of the discriminated operant, and conditional discrimination accuracy also depends on whether the consequences of the two correct responses (i.e., those specified for reinforcement) in a standard two-stimulus two-response conditional discrimination are the same or different. For example, Peterson, Wheeler, and Trapold (1980) trained pigeons in a conditional discrimination problem in which a green center key signaled that a peck to the side key with vertical lines was correct, and a red center key signaled that a peck to the side key with horizontal lines was correct. One group of pigeons received food accompanied by a tone for both kinds of correct responses; another group received food plus tone for one kind of correct response and tone alone (i.e., no food) for the other. The latter group was substantially more accurate than the former, especially when delays were introduced between the center-key color and the side-key choice. This exemplifies the *differential outcome effect* first reported by Trapold (1970), and is here interpreted as resulting from the larger difference between discriminated operants for the latter group. These results are entirely consistent with the dependency of multiple- and concurrent-schedule performances on the degree of differentiation between the two discriminated operants, as discussed above.

In the model that we develop below, we employ a theoretical parameter, d_{sb}, that measures the distinctiveness of the relation between the conditional stimuli and the responses they occasion for one discriminated operant relative to another. The value of this parameter should be affected, for example, by the difference between the conditional stimuli and by the delay between the conditional stimuli and responses. We employ a second parameter, d_{br}, to represent the distinctiveness of the relation between behavior and reinforcement for one discriminated operant relative to another. The value of d_{br} reflects the joint effects of variables that influence response–reinforcer contingencies such as the qualities or delays of the outcomes and the topographical differentiation of responses. It is important to observe that response differentiation will be reflected in both parameters: For example, in Eckerman's (1970) study, increasing the separation between correct responses would increase both d_{sb} and d_{br}.

A MODEL OF DISCRIMINATED OPERANT BEHAVIOR

In view of the discussion above, an adequate model must include terms for the degree of differentiation between two operants based on the stimulus–response relation (what response goes with what stimulus) and, separately, the response–reinforcer contingencies that define those two operants (what reinforcer goes with what response). We begin with a brief review of earlier modeling efforts, partly to set the stage for the present model and partly because they will be referred to below.

	B_1	B_2
S_1	11	12
S_2	21	22

Fig. 2. The conditional discrimination matrix. The stimuli are designated *S* and the responses *B*, and the cells of the matrix are designated as the stimulus–response combinations.

Background

Nevin, Jenkins, Whittaker, and Yarensky (1977,[2] 1982) proposed a model of signal-detection performance based on the direct and generalized strengthening effects of reinforcers obtained in the cells of the matrix of Figure 2, which simply expands Figure 1 with notation for all four cells. The basic notion was that reinforcers for B_1 on S_1 trials (R_{11}) would also strengthen B_1 on S_2 trials, to the extent that S_1 and S_2 are confusable. Thus, although R_{21} is actually zero, it may effectively be greater than zero. Likewise, reinforcers for B_2 on S_2 trials (R_{22}) would also strengthen B_2 on S_1 trials. The subject was assumed to match the ratio of B_1 and B_2 to the ratio of direct and generalized reinforcers, separately on S_1 and S_2 trials (for full rationale and equations, see Nevin, 1981; Nevin et al., 1982).

Davison and Tustin (1978) arrived at a similar formulation by a different route. Taking the generalized matching law (Equation 2) as their starting point, they proposed that S_1 and S_2 could be construed as biasing response allocation toward B_1 or B_2, respectively. Because we will have several occasions to refer to their formulation below, we present their equations and measures here:

$$\frac{B_{11}}{B_{12}} = cd\left(\frac{R_{11}}{R_{22}}\right)^{a}, \tag{3a}$$

and

$$\frac{B_{21}}{B_{22}} = \frac{c}{d}\left(\frac{R_{11}}{R_{22}}\right)^{a}, \tag{3b}$$

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[2] Nevin, J. A., Jenkins, P., Whittaker, S., & Yarensky, P. (1977, November). *Signal detection and matching*. Paper presented at the meetings of the Psychonomic Society, Washington, DC.

where c represents inherent bias that is constant with respect to the reinforcer ratio and a represents the sensitivity of choice allocation to the reinforcer ratio, as in Equation 2. The parameter d—stimulus bias—represents the discriminability between S_1 and S_2. If $d = 1$, signifying zero discriminability, response ratios are identical on S_1 and S_2 trials. To show that d is predicted to be independent of reinforcement, Equation 3a is divided by Equation 3b and the reinforcers cancel out. Rearranging and taking square roots,

$$d = \left(\frac{B_{11}}{B_{12}} \frac{B_{22}}{B_{21}} \right)^{0.5}. \tag{4}$$

Thus, d is measured directly by the geometric mean of the ratios of correct to incorrect responses in the presence of Stimuli 1 and 2.

To show that sensitivity to reinforcement (a) is predicted to be independent of stimulus discriminability, Equation 3a is multiplied by Equation 3b and d cancels out. Taking square roots,

$$b = \left(\frac{B_{11}}{B_{12}} \frac{B_{21}}{B_{22}} \right)^{0.5} = c \left(\frac{R_{11}}{R_{22}} \right)^{a}, \tag{5}$$

where b, the geometric mean of the ratios of responses to Alternatives 1 and 2 given Stimuli S_1 and S_2, is an overall measure of behavior allocation. Empirically, of course, d may depend on reinforcer scheduling, and a may depend on the S_1-S_2 difference. Many experiments have explored these questions (e.g., McCarthy & Davison, 1979, 1980a, 1984) with no simple conclusion emerging (for review, see Alsop & Davison, 1991).

Although the model of Davison and Tustin (1978) has been successful as a descriptive framework, it does not address the processes that determine sensitivity to reinforcement: a is simply a free parameter. With reference to the study by Miller et al. (1980), Davison and Jenkins (1985) suggested that if two concurrently available response alternatives were not well differentiated, reinforcers obtained by one response might have the effect of strengthening the other response. The idea is basically similar to the generalized strengthening effects across stimuli proposed by Nevin et al. (1977, Footnote 2). To characterize the discriminability of the response–reinforcer contingency, Davison and Jenkins introduced a second parameter, d_r. They suggested that conditional discrimination performance depended jointly on stimulus discriminability and contingency discriminability according to the following equations:

$$\frac{B_{11}}{B_{12}} = d_s \left(\frac{d_r R_1 + R_2}{d_r R_2 + R_1} \right), \tag{6a}$$

and

$$\frac{B_{21}}{B_{22}} = \frac{1}{d_s} \left(\frac{d_r R_1 + R_2}{d_r R_2 + R_1} \right), \tag{6b}$$

where d_s represents stimulus discriminability, as does d in the Davison-Tustin model, and d_r represents contingency discriminability. Davison and Jenkins showed that the value of d_r described the degree of undermatching in concurrent schedules and conditional discriminations (i.e., the extent to which $a < 1$ in Equations 2, 3a, and 3b). Moreover, d_r could be identified with parameters of the experimental contingencies in the same way that d_s could be identified with stimulus parameters.

The Davison-Jenkins (1985) model requires discrimination, as measured by d_s, to be unaffected by d_r. This may be seen by dividing Equation 6a by 6b, canceling out the reinforcer terms and showing that d_s, like Davison and Tustin's d, is given by Equation 4. This result leads to a problem. If $d_r = 1.0$, representing a total failure to discriminate which reinforcer goes with which response, the model predicts that response ratios will be constant and independent of reinforcer ratios. Thus, performance in $S_1 = d_s$, and performance in $S_2 = 1/d_s$ for all reinforcer ratios. However, it is highly unlikely that such differential control by the stimuli could be effective in the absence of differential control by the reinforcement contingency. The situation is similar to that with identical multiple VI VI schedules signaled by red and green keylights discussed above: Although red and green may be highly discriminable by some other measure, equal response rates are forced if the contingencies of reinforcement are not discriminated.

The foregoing discussion suggests that it is essential to distinguish between stimulus *discriminability* as a theoretical parameter and stimulus control or *discrimination* as measured by Equation 4. We will have occasion to remind readers of this point as the argument proceeds.

Although stimulus discriminability and contingency discriminability were conceptualized similarly in the Davison-Jenkins (1985) model, they were not treated similarly in its equations; and it is the equations that do the work. A model that avoids the difficulty inherent in the Davison-Jenkins model, and which gives algebraic as well as conceptual equivalence to stimulus and contingency discriminability, was introduced jointly by Alsop (1987)[3] and by Davison (1987)[4] and first published by Alsop (1991) and Davison (1991b). It addresses steady-state behavior only, leaving for future development the consideration of transition states such as acquisition or extinction. The model will be reviewed and developed here for a simple conditional discrimination performance, and then will be extended to describe performance in related cases that include complex conditional discriminations, reinforcement for conventionally defined "errors" in conditional discriminations, and multiple and concurrent schedules.

Initial Assumptions

We assume that behavioral allocation is based on strict matching of behavior ratios in the presence of (or following) conditional stimuli to the *effective* allocation of reinforcers for responses in the presence of these stimuli. In spirit, this assumption is similar to Killeen's (1994) argument that reinforcement acts on the effective response unit for the organism, which may not be the same as the unit specified by an experimental contingency. Our model is principally concerned with estimating the effective allocation of reinforcers when the stimulus–response and response–reinforcer contingencies defining the discriminated operants are confusable. Initially, a model will be developed for four discriminated operants, comprising two conditional stimulus conditions and two responses. Later, the model will be generalized to any number of discriminated operants.

As described above, the simplest conditional discrimination involves the successive and randomized presentation of one or the other of two stimuli, designated S_1 and S_2, where two response alternatives, B_1 and B_2, are simultaneously available. When S_1 is present, B_1 may be deemed correct and is reinforced according to some schedule, and when S_2 is present, B_2 may be deemed correct and is reinforced according to a separate schedule. The paradigm is summarized in the 2 × 2 matrix of Figure 2, which repeats Figure 1 with added notation. The four resulting discriminated operants are designated according to their stimulus and response identification.

For the simple case described in Figure 1, reinforcers can occur only in Cells 11 and 22, and are designated R_{11} and R_{22}. These reinforcers are assumed to strengthen the responses that produce them, designated B_{11} and B_{22}. However, to the extent that the stimuli are confusable, R_{11} will also strengthen responding in Cell 21, designated B_{21}; likewise, R_{22} will also strengthen responding in Cell 12, designated B_{12}, as suggested by Nevin et al. (1977, Footnote 2). Let us assume that the conditional stimuli, as identified with responses, are located on a dimension of psychometric space. We shall not endeavor here to locate the stimuli in an absolute sense, but just to measure their distance apart. Following Davison (1991b), we assume that the psychometric distance between two stimuli is given by d_{sbi1i2}, where $i1$ and $i2$ designate the two stimulus conditions. Such a measure ranges from one (the stimuli are completely nondiscriminable) to infinity (the stimuli are perfectly discriminable). We assume that the distances between stimuli satisfy a ratio scale, so that it is meaningful to assert that, for example, $d_{sb13} = 2{*}d_{sb12}$. This assumption implies a log interval scale, and we will sometimes report values of log d_{sb} (with values between zero and infinity).

Further, again following Davison (1991b), we assume that the generalization of reinforcer effects from Stimulus S_1 to Stimulus S_2 decays inversely with d_{sb12}. Thus, the effective reinforcer contribution of R_{11} to Response 1 in Stimulus 2 is R_{11}/d_{sb12}. The function is portrayed in Figure 3. It is similar to the expo-

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[3] Alsop, B. (1987, June). *Choice models of signal detection and detection models of choice.* Paper presented to the 10th Harvard Symposium on the Quantitative Analysis of Behavior, Boston.

[4] Davison, M. (1987, June). *Stimulus discriminability, contingency discriminability, and complex stimulus control.* Paper presented to the 10th Harvard Symposium on the Quantitative Analysis of Behavior, Boston.

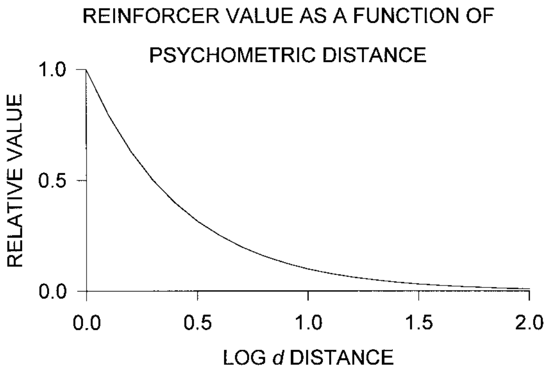

Fig. 3. The assumed decrease in reinforcer value acting on a stimulus–response pair as a function of the psychometric distance between that stimulus–response pair and the pair that gained reinforcement.

nential decay function conjectured by Shepard (1958) as the universal form of generalization gradients, but it falls off relatively more steeply at small values and less steeply at large values. Future modeling efforts may need to explore the exponential or other forms of the decay function.

As we noted above, d_{sb} is conceptually similar, in determining the effective reinforcer allocation, to the measure of stimulus discriminability (d or log d) offered by Davison and Tustin (1978). However, it is also important to note that these measures are *not* the same, because the mechanisms and equations are very different. The measure log d is calculated from discrimination performance, which (as suggested above) may be affected by variables other than the discriminability of their respective stimulus–response pairs, such as the discriminability of their respective response–reinforcer contingencies. For example, in matching to sample, log d decreases when a delay is inserted between the choice response and the reinforcer, even though the stimuli, the responses, and the relations between them (i.e., respond to the key with the same color as the sample) are unchanged (McCarthy & Davison, 1986). As we will show, this decrease in log d is compatible with invariance in d_{sb}, which would be consistent with the unchanged relation between the samples and the choice responses.

By analogy to the treatment of stimulus–response confusability, we assume that response–reinforcer contingencies are also confusable. To the extent that they are confused,

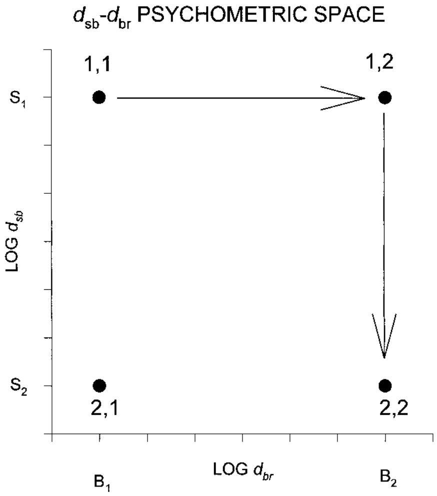

Fig. 4. How the effects of a single reinforcing event in Cell 11 are generalized via Cell 12 to Cell 22 in the conditional discrimination matrix. Equivalent processes (not shown) will also generalize this event into Cells 12 and 21. The same process is assumed to occur for reinforcers delivered in any cell of the matrix.

a reinforcer delivered for B_{11} will also strengthen B_{12}, and a reinforcer delivered for B_{22} will also strengthen B_{21}. Again following Davison (1991b), we assume that the generalization of reinforcer effects from response B_{11} to Response B_{12} decays inversely with d_{br12}. Thus, the effective reinforcer contribution of R_{11} to Response 2 in Stimulus 1 is R_{11}/d_{br12}. If the response–reinforcer contingencies are perfectly differentiable, $d_{br} = \infty$, and the reinforcer strengthens only the response that produced it. If the response–reinforcer contingencies are indistinguishable, $d_{br} = 1$, and the reinforcer strengthens both responses equally, regardless of which response produced it. As for d_{sb}, we will generally report values of log d_{br}, which ranges from zero to infinity. The parameter d_{br} is affected by such variables as the differences between the response definitions (e.g., along a strip key, as in Eckerman's, 1970, study), the differences in the outcomes of the responses (as in Peterson et al., 1980), or the delay between responses and reinforcers (as in McCarthy & Davison, 1986, where d_{br} is expected to decrease with increases in delay). Figure 4 shows how the values of log d_{sb12} and log d_{br12} affect the generalization of the effects of a single reinforcer obtained by B_1 in the presence of S_1 across stimuli and responses.

$$\begin{array}{c|c|c|}
 & B_1 & B_2 \\
\hline
S_1 & R_{11} + \dfrac{R_{22}}{d_{sb12}d_{br12}} & \dfrac{R_{11}}{d_{br12}} + \dfrac{R_{22}}{d_{sb12}} \\
\hline
S_2 & \dfrac{R_{11}}{d_{sb12}} + \dfrac{R_{22}}{d_{br12}} & R_{22} + \dfrac{R_{11}}{d_{sb12}d_{br12}} \\
\hline
\end{array}$$

Fig. 5. The effective reinforcer allocations in the four cells of the conditional discrimination matrix under conditions of reinforcement for B_1 in the presence of S_1 and for B_2 in the presence of S_2.

To get an intuitive sense of the model, consider the subject as experiencing the variations and repetitions in its behavior within a stream of environmental events—lights, tones, and the like—some of which recur from time to time. When a reinforcer disrupts the flow of the stream, its strengthening effect is felt directly on the relation between current or recent environmental events and the response being emitted when the reinforcer arrived. The strengthening effect also generalizes to other environmental events and responses to the extent that they are confusable with those that occurred in close contiguity with the reinforcer, as measured by the inverse of d_{sb} and d_{br}. We assume that this process operates following each reinforcer, as suggested in Figure 4, to increment the values of the cells in the matrix of Figure 2. The rate or probability of reinforcement correlated with a particular discriminated operant affect only the stable-state values assigned to the relevant cells. As noted earlier, we are not yet attempting to address acquisition or transition states. We assume that nonoccurrence of R_1 or R_2 does not reduce the effective reinforcement in any cell. Because it operates sequentially, reinforcer by reinforcer, the model is molecular and dynamic. However, the model is molar in the sense that, after prolonged experience under constant experimental conditions, the direct and generalized reinforcement values of the cells will settle into stable ratios, and it is these ratios that determine choice.

The effective numbers of reinforcers, direct or generalized, that have accumulated during steady-state performance are given by the expressions in the four cells in Figure 5. When the experimenter presents a particular stimulus repeatedly during prolonged exposure to the experimental conditions, the subject is assumed to emit one or the other of the measured responses in accordance with the ratio of effective reinforcers that have accrued to the two cells of the matrix that correspond to that stimulus. In its basic form, our model is concerned solely with reinforcer frequencies in the cells of the matrix and not with their values (as determined by magnitude or quality); we will discuss the extension of the model to encompass such factors in the section on concurrent schedules, below.

More generally and formally, we assume that responses allocated to the cells of the matrix in Figure 2 match the "apparent," "perceived," or "effective" long-term allocation of reinforcers, which will deviate from their veridical (i.e., experimenter-measured) allocation to the extent that d_{sb} and d_{br} are less than infinite, as shown in Figure 5. We recognize that terms like *apparent, perceived,* and *effective* reinforcers may seem loose, but here they occur as technical terms that are defined quantitatively by equations describing how, through generalization engendered by confusion between stimulus–response relations and between response–reinforcer relations, the experimenter-measured reinforcer allocations are transformed into quantities that affect behavior. The terms are always used as a convenient shorthand for the operations of our proposed equations.

The resulting equations that predict responding in the presence of, or following, the two conditional stimuli are, for S_1,

$$\frac{B_{11}}{B_{12}} = c\frac{R_{11} + \dfrac{R_{22}}{d_{sb12}d_{br12}}}{\dfrac{R_{11}}{d_{br12}} + \dfrac{R_{22}}{d_{sb12}}} = c\frac{d_{sb12}d_{br12}R_{11} + R_{22}}{d_{sb12}R_{11} + d_{br12}R_{22}}, \tag{7a}$$

and for S_2,

$$\frac{B_{21}}{B_{22}} = c\frac{\dfrac{R_{11}}{d_{sb12}} + \dfrac{R_{22}}{d_{br12}}}{\dfrac{R_{11}}{d_{sb12}d_{br12}} + R_{22}} = c\frac{d_{br12}R_{11} + d_{sb12}R_{22}}{R_{11} + d_{sb12}d_{br12}R_{22}}. \tag{7b}$$

The values of d_{sb12} and d_{br12} can be estimated by a nonlinear optimization program (see Appendix B) from a set of data taken across conditions that vary the ratio of R_{11} to R_{22}.

As in the generalized matching law (Baum, 1974), c represents a constant proportional preference (inherent bias; see Davison & Tustin, 1978) for one alternative response over the other. The value of c should be unaffected by changes in the conditional stimuli or in the frequency of reinforcers for the two correct responses. It need not, however, remain constant when either the response topographies or the magnitudes of reinforcers are changed.

The assumption that behavior allocation between cells strictly matches (equals) the effective reinforcer frequencies in the cells has the benefits of simplicity. It might be objected that such an assumption is too simple because it is well known (e.g., Baum, 1974; see Davison & McCarthy, 1988, for review) that such strict matching seldom occurs, and that undermatching (less change in behavior ratios than in reinforcer ratios) is the norm. However, as we now show, undermatching arises naturally from this model when the discrimination between alternatives is less than perfect.

If Equations 7a and 7b are multiplied together, we can obtain a theoretical measure of overall response bias (B):

$$B^2 = \frac{B_{11}}{B_{12}}\frac{B_{21}}{B_{22}}$$

$$= \frac{R_{11} + \dfrac{R_{22}}{d_{sb12}d_{br12}}}{\dfrac{R_{11}}{d_{br12}} + \dfrac{R_{22}}{d_{sb12}}} \cdot \frac{\dfrac{R_{11}}{d_{sb12}} + \dfrac{R_{22}}{d_{br12}}}{\dfrac{R_{11}}{d_{sb12}d_{br12}} + R_{22}}. \qquad (8)$$

In this expression, B is the geometric mean of the ratios of responses (i.e., B_1/B_2) taken across the two choice alternatives. When $d_{sb} = 1.0$, the four operants are effectively signaled by the same stimulus and the rows of the matrix in Figure 2 collapse into a single row, as in continuous free-operant two-key concurrent schedules. Equation 8 then becomes

$$B^2 = \frac{B_{11}}{B_{12}}\frac{B_{21}}{B_{22}} = c\left[\frac{R_{11} + \dfrac{R_{22}}{d_{br}}}{R_{22} + \dfrac{R_{11}}{d_{br}}}\right]^2. \qquad (9)$$

In this expression, B is equivalent to b, the combination response bias measure used by Davison and Tustin (1978) to characterize differential responding with respect to reinforcement (see Equation 5) but only when $d_{sb} = 1.0$. Note, however, that the reinforcer term is not the same as in Equation 5. We use the upper case here because B is not equivalent to b when $d_{sb} > 1.0$. (Recall that stimulus-related parameters analogous to d_{sb} canceled out when Equations 3a and 3b in the Davison-Tustin model or Equations 6a and 6b in the Davison-Jenkins model were multiplied. The same sort of cancellation does not occur when Equations 7a and 7b are multiplied unless $d_{sb} = 1.0$. The implication is that the relation between overall response bias and reinforcer allocation depends on the discriminability of stimulus–behavior relations; we return to this point below.)

Equation 9 is the same as the equation that follows Equation 11 of Davison and Jenkins (1985), who showed that it gave a good account of choice data normally construed as undermatching when fitted by the generalized matching law. When d_{br} approaches infinity, Equation 9 simplifies to

$$B = \frac{R_{11}}{R_{22}}, \qquad (10)$$

showing that overall behavior allocation to Responses 1 and 2 strictly matches the ratio of obtained (and, in this case, perceived) reinforcers.

To obtain a theoretical measure of stimulus discrimination, we begin by dividing Equation 7a by 7b, which yields

$$D^2 = \frac{B_{11}}{B_{12}}\frac{B_{22}}{B_{21}} \qquad (11)$$

$$= \frac{R_{11} + \dfrac{R_{22}}{d_{sb12}d_{br12}}}{\dfrac{R_{11}}{d_{br12}} + \dfrac{R_{22}}{d_{sb12}}} \cdot \frac{\dfrac{R_{11}}{d_{sb12}d_{br12}} + R_{22}}{\dfrac{R_{11}}{d_{sb12}} + \dfrac{R_{22}}{d_{br12}}}.$$

In this expression, D is the geometric mean of the ratios of responses normally construed as "correct" responses and "errors." Setting $d_{br} = \infty$ and taking square roots, Equation 11 simplifies to

$$D = \left(\frac{B_{11}}{B_{12}}\frac{B_{22}}{B_{21}}\right)^{0.5} = d_{sb}. \qquad (12)$$

Note that reinforcer frequencies and bias have canceled out in this expression, implying that, if $d_{br} = \infty$, D is independent of the ratio of reinforcers, real or apparent. Equation 12 is the same as Equation 4, which specified the parameter d in the Davison-Tustin model. This measure was, in their model, a pure measure of the effect of stimulus disparity because they ignored d_{br}, effectively setting it at infinity. The equivalence of D, d, and d_{sb} holds only if $d_{br} = \infty$; accordingly, we use the upper-case D here and note that when d_{br} is less than infinite, D will depend on the discriminability of response–reinforcer relations as well as stimulus–response relations.

So far, we have considered cases in which $d_{sb} = 1$ and $d_{br} = \infty$; we now explore some other cases involving extreme parameter values. When $d_{sb} = d_{br} = 1.0$, both response ratios are predicted to be equal to c, and there will be no effect of varying R_{11} or R_{22}. By contrast, when both d_{sb} and d_{br} are very large, Equation 7a will approximate $+\infty$ and Equation 7b will approximate $-\infty$, implying errorless performance that is unaffected by varying R_{11} or R_{22}.

When $d_{br} = 1.0$, the absence of any apparent differential reinforcement leads to the absence of control by changes in the reinforcer ratio for correct responses (Equation 9 = c), and no differential responding with respect to stimuli (Equation 11 = 1.0 regardless of the value of d_{sb}). The basic result, then, is that according to Equations 7a and 7b and their combinations, differential responding with respect to reinforcement can occur only if $d_{br} > 1.0$. Moreover, differential responding with respect to stimuli depends on effective (rather than arranged) differential reinforcement with respect to responding. There can be no stimulus control without effective differential reinforcement (cf. the model suggested by Davison and Jenkins, 1985, Equations 6a and 6b, in which stimulus control could occur without effective differential reinforcement).

More generally, the effects of d_{sb} and d_{br} on differential responding with respect to Stimuli S_1 and S_2 (measured by log D) and with respect to Responses B_1 and B_2 (measured by log B) as functions of R_{11}/R_{22} are summarized by the examples in Figure 6. The upper panels show the effects of the log reinforcer ratio on log B as predicted by Equation 9 for two representative values of d_{sb}: a fairly easy discrimination, $d_{sb} = 10$, in the left panel and a difficult discrimination, $d_{sb} = 2$, in the right panel. As d_{br} varies parametrically in four steps from 1 to 1,000, the bias functions become progressively steeper in both panels, and will reach an asymptote at exact matching when $d_{br} = \infty$ (not shown). This predicted steepening parallels the changes that have been observed with concurrent VI VI schedules as a function of disparity between alternatives, as reported by Miller et al. (1980). Note that the functions are roughly linear over the center of the range, and thus conform approximately to the generalized matching law when it is restated in logarithms. Note also that the functions are steeper at each intermediate value of d_{br} when d_{sb} is 2 than when it is 10, and that the curvilinearity is more pronounced when d_{sb} is smaller than when it is larger. Thus, choice allocation is predicted to be more sensitive to the reinforcer ratio when the stimuli are more confusable.

The lower panels of Figure 6 show how log D depends on the reinforcer ratio at two values of d_{br} when d_{sb} varies parametrically in four steps, as predicted by Equation 11. When $d_{br} = 10$ and $d_{sb} = 1{,}000$, the function is nearly flat and log D approaches its maximum of 1.0, the limiting value permitted by d_{br}. As d_{sb} decreases, log D decreases systematically and the functions assume an inverted-U shape until, at $d_{sb} = 1$, log D falls to 0. When $d_{br} = 2$, the maximum value of log D is 0.3 (i.e., log 2), and compared with the predictions for $d_{br} = 10$, log D is less affected by variations in log reinforcer ratios. These examples illustrate the fact that the accuracy of a discrimination depends jointly on the discriminability of the stimulus–behavior relations and the behavior–reinforcer relations.

The effects of d_{sb} and d_{br} are exactly interchangeable in their effects on log D. For example, the values of log D for $d_{sb} = 10$, $d_{br} = 2$ and for $d_{sb} = 2$, $d_{br} = 10$ are identical, as can be seen by inspection of Equation 11 and by comparison of the shallow inverted-U functions for these parameter values in the lower panels of Figure 6. By contrast, the upper panels show that the effects of d_{sb} and d_{br} on log B are not interchangeable. For example, the function for $d_{sb} = 10$, $d_{br} = 2$ is substantially shallower than the function for $d_{sb} = 2$, $d_{br} = 10$. More generally, when d_{br} approaches one, the effects of both the stimulus difference and the reinforcer ratio are progressively weakened until behavior allocation

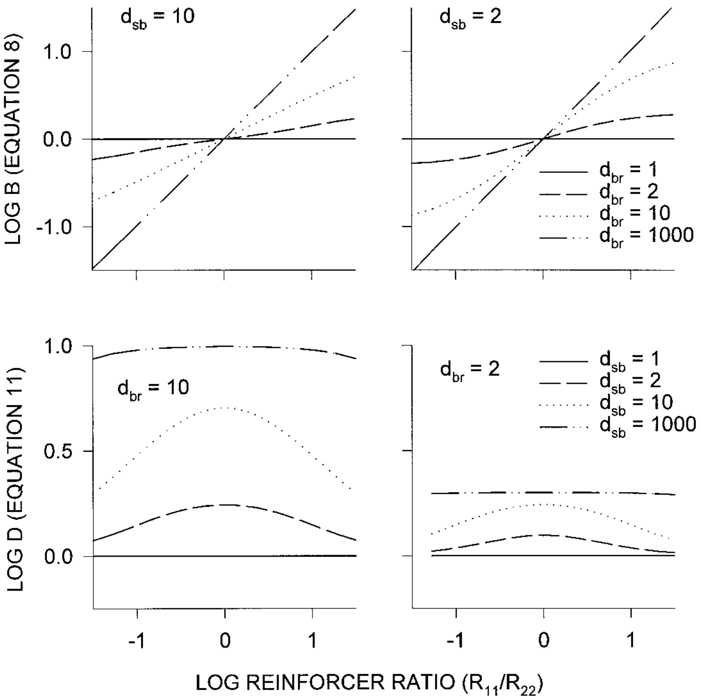

Fig. 6. Upper panels: log B (Equation 8) as a function of the log reinforcer ratio for two values of response–reinforcer discriminability (d_{br}). Lower panels: log D (Equation 11) as a function of the log reinforcer ratio for two values of stimulus–response discriminability (d_{sb}). Note that we use log B and log D to signify theoretical predictions (see Appendix A). Parameter values are for d_{br} in the upper panel and for d_{sb} in the lower panel.

is indifferent to all reinforcer ratios. However, when d_{sb} approaches one, measured discrimination decreases at all values of the reinforcer ratio, but the sensitivity of choice allocation to the reinforcer ratio increases and approaches strict matching. In this sense, then, the discriminability of stimulus–response relations and the discriminability of the response–reinforcer contingency are predicted to have different behavioral effects even though they are conceived of, and treated, in parallel.

Reinforcers for Errors with Two Stimuli and Two Responses

Although most conditional discrimination experiments employ the contingencies described above (reinforcement for B_1 only on S_1 trials, and reinforcement for B_2 only on S_2 trials), the effects of reinforcement for responses conventionally termed "errors"—B_1 given S_2, and B_2 given S_1—are of major interest. First, reinforcement of only "correct" responses is simply an extreme, and maybe unusual, point on the continuum of discriminated operant contingencies. Any serious model of contingencies of reinforcement must deal with the entire continuum, and must provide measures of d_{sb} and d_{br} that are unaffected by the distribution of reinforcers in the matrix. Second, the introduction of reinforcement for errors makes direct contact with multiple concurrent schedules, in which each of two simultaneously available responses may be reinforced on different schedules depending on the stimulus signaling them (e.g., McLean & White, 1983).

The approach taken above naturally generalizes to reinforcers delivered in any cell of the matrix, with no further assumptions. Reinforcers delivered in any cell of a matrix will have an influence on the effective reinforcer rate in other cells depending on the discriminability of both stimulus–response relations and response–reinforcer relations between those cells. The appropriate equations (for a 2 × 2 matrix, suppressing the further subscripting of d_{sb} and d_{br}) are, for S_1,

	B_1	B_2
S_1	B_{11}, R_{11}	B_{12}
S_2	B_{21}	B_{22}, R_{22}
S_3	B_{31}, R_{31}	B_{32}

	B_1	B_2
S_1	$R_{11} + \frac{R_{22}}{d_{sb12}d_{br12}} + \frac{R_{31}}{d_{sb13}}$	$\frac{R_{11}}{d_{br12}} + \frac{R_{22}}{d_{sb12}} + \frac{R_{31}}{d_{sb13}d_{br12}}$
S_2	$\frac{R_{11}}{d_{sb12}} + \frac{R_{22}}{d_{br12}} + \frac{R_{31}}{d_{sb23}}$	$R_{22} + \frac{R_{11}}{d_{sb12}d_{br12}} + \frac{R_{31}}{d_{sb23}d_{br12}}$
S_3	$R_{31} + \frac{R_{11}}{d_{sb13}} + \frac{R_{22}}{d_{sb23}d_{br12}}$	$\frac{R_{11}}{d_{sb13}d_{br12}} + \frac{R_{22}}{d_{sb23}} + \frac{R_{31}}{d_{br12}}$

Fig. 7. Upper panel: the matrix of events in a three-stimulus two-response detection matrix in which reinforcers are available for B_1 in the presence of S_1 and S_3 and for B_2 in the presence of S_2. Lower panel: the effective reinforcer matrix for the events in the upper panel.

$$\frac{B_{11}}{B_{12}} = c\frac{R_{11} + \dfrac{R_{12}}{d_{br}} + \dfrac{R_{21}}{d_{sb}} + \dfrac{R_{22}}{d_{sb}d_{br}}}{\dfrac{R_{11}}{d_{br}} + R_{12} + \dfrac{R_{21}}{d_{sb}d_{br}} + \dfrac{R_{22}}{d_{sb}}}, \qquad (13a)$$

and, for S_2,

$$\frac{B_{21}}{B_{22}} = c\frac{\dfrac{R_{11}}{d_{sb}} + \dfrac{R_{12}}{d_{sb}d_{br}} + R_{21} + \dfrac{R_{22}}{d_{br}}}{\dfrac{R_{11}}{d_{sb}d_{br}} + \dfrac{R_{12}}{d_{sb}} + \dfrac{R_{21}}{d_{br}} + R_{22}}. \qquad (13b)$$

More Than Two Stimuli and Two Responses

The conventional conditional discrimination procedure employs a pair of stimuli, S_1 and S_2. We now consider the case of more than two stimuli, some of which may share identical reinforcer contingencies. This situation requires additional stimulus–response discriminability parameters. Assuming symmetry (i.e., the discriminability of S_1 from S_2 is the same as the discriminability of S_2 from S_1), there will, for N stimuli, be $N!/(N - 1)!$ such parameters. This number quickly gets out of hand, requiring an unachievable number of experimental conditions to provide accurate estimates of the parameter values. As a formal demonstration only, here we will take three stimuli and two responses and, rather than show the equations directly, we will show a matrix of effective reinforcer values. We will assume that reinforcers are available for B_1 in the presence of both S_1 and S_3 and for B_2 in the presence of S_2. The arranged stimulus–response–reinforcer matrix is shown in the top of Figure 7 and the effective stimulus–response–reinforcer matrix is shown at the bottom.

	B_1	B_2	B_3
S_1	B_{11}, R_{11}	B_{12}	B_{13}
S_2	B_{21}	B_{22}, R_{22}	B_{23}
S_3	B_{31}	B_{32}	B_{33}, R_{33}

Fig. 8. The matrix of events in a three-stimulus three-response detection matrix in which B_1 is reinforced in the presence of S_1, B_2 in the presence of S_2, and B_3 in the presence of S_3.

Note that apparent reinforcement for S_1 and S_3 is more similar when both stimuli signal reinforcement for the same response than when they signal reinforcement for different responses. The similarity of effective reinforcement for responses to otherwise distinctive stimuli may contribute to their membership in the same stimulus class in research on categorization or stimulus equivalence.

N *Responses,* M *Stimuli*

The generalization of the present model to more than two stimuli and more than two responses is straightforward. For example, consider the conditional discrimination matrix shown in Figure 8. We require three pairwise stimulus–behavior discriminability parameters, d_{sb12}, d_{sb13}, and d_{sb23}. Likewise, we also require three behavior–reinforcement discriminability parameters, d_{br12}, d_{br13}, and d_{br23}. As in the two-stimulus two-response model, we assume that reinforcers delivered in one cell generalize to behavior in all other cells to an extent that depends on the discriminability of the stimulus–behavior relations within columns, on the discriminability of the response–reinforcer relations within rows, or on both of these. Thus, R_{11} reinforcers affect responses in Cell 12 according to R_{11}/d_{br12}, in Cell 13 according to R_{11}/d_{br13}, and so on. This operation specifies the decremental effects of what are, essentially, stimulus–behavior and behavior–reinforcer distances to elucidate for the reader the ideas underlying the model. As in the basic two-stimulus two-response case developed above, the effects of reinforcers in

$$
\begin{array}{c|c|c|c|}
 & B_1 & B_2 & B_3 \\
\hline
S_1 & R_{11} + \frac{R_{22}}{d_{sb12}d_{br12}} + \frac{R_{33}}{d_{sb13}d_{br13}} & \frac{R_{11}}{d_{br12}} + \frac{R_{22}}{d_{sb12}} + \frac{R_{33}}{d_{sb13}d_{br23}} & \frac{R_{11}}{d_{br13}} + \frac{R_{22}}{d_{sb12}d_{br23}} + \frac{R_{33}}{d_{sb13}} \\
\hline
S_2 & \frac{R_{11}}{d_{sb12}} + \frac{R_{22}}{d_{br12}} + \frac{R_{33}}{d_{sb23}d_{br13}} & R_{22} + \frac{R_{11}}{d_{sb12}d_{br12}} + \frac{R_{33}}{d_{sb23}d_{br23}} & \frac{R_{11}}{d_{sb12}d_{br13}} + \frac{R_{22}}{d_{br23}} + \frac{R_{33}}{d_{sb23}} \\
\hline
S_3 & \frac{R_{11}}{d_{sb13}} + \frac{R_{22}}{d_{sb23}d_{br12}} + \frac{R_{33}}{d_{br13}} & \frac{R_{11}}{d_{sb13}d_{br12}} + \frac{R_{22}}{d_{sb23}} + \frac{R_{33}}{d_{br23}} & R_{33} + \frac{R_{11}}{d_{sb13}d_{br13}} + \frac{R_{22}}{d_{sb23}d_{br23}} \\
\hline
\end{array}
$$

Fig. 9. The effective reinforcer matrix for the three-stimulus three-response detection matrix shown in Figure 8.

all cells, weighted by their psychological distances from that cell, add within a cell to provide an overall effective reinforcer value for that cell. Within each row of the matrix, response allocation is assumed to match the relative effective reinforcer value. The model for the three-stimulus example described here reduces directly to the two-stimulus case described above when S_3 is eliminated. The effective reinforcer matrix for the above 3 × 3 matrix (with reinforcers only as R_{11}, R_{22}, and R_{33}) is shown in Figure 9.

The expressions in the above matrix are tedious, rather than complicated. However, using the same idea of the generalization of reinforcer effects to other stimulus–behavior pairs, equations for any $N \times M$ matrix, with reinforcers in any or all cells, can be obtained. There is also no reason why, again using the same basic theory, we should not expand the model into three dimensions (e.g., the third dimension might be the discriminability between a set of second-order conditional cues or stimulus-choice delays).

Requirements for an Effective Model

An effective model for conditional discrimination and other performances (and indeed, any model for any system) has a number of immutable requirements. These can be summarized as follows: Any parameter that purports to be a measure of an independent-variable effect must remain unaffected by the variation of other independent variables that purport to affect other parameters. For instance, d_{sb} should be affected, in an appropriate direction, by changes in conditional stimuli, but should not be affected by changes in reinforcer frequencies or, more particularly, by changes in response–reinforcer differentiation. Equally, d_{br} should be affected in an appropriate direction by changes in response–reinforcer differentiation, but not by changes in conditional stimuli. In other words, derived parameters should show what Nevin (1984) termed *parameter invariance.*

Such parameter invariance is of prime importance, and an example is useful here. When Davison and McCarthy (1980) varied the frequency of reinforcers delivered for errors (i.e., they arranged R_{12} and R_{21} reinforcers), they found that the Davison-Tustin (1978) measure of stimulus discriminability, log d, decreased with increasing error-reinforcement probability. This reinforcer manipulation should, in theory, not have affected a conditional-stimulus measure. They, and Nevin et al. (1982), went to some pains to extend the Davison-Tustin model so that the invariance of this stimulus measure could be preserved. As it turned out, neither extension was satisfactory, but the model presented here naturally, and without modification, deals with reinforcement for errors, and so has the potential to deal directly with such data and preserve parameter invariance. We shall return to an analysis of these data later.

Parameter invariance requires that the terms of our model be measured in ways that permit unambiguous assignment of numerical values, and that these values behave in accordance with the principles of measurement theory. For example, from a series of conditions of a three-stimulus three-response procedure, one can estimate log d_{sb12}, log d_{sb23}, and log d_{sb13} by a criterion of best fit. If the parameters characterize distances in psychometric space according to an interval scale, these three values of log d_{sb} must be related by the expression log d_{sb13} = log d_{sb12} + log d_{sb23}. Moreover, the appropriate values must remain unchanged when Stimuli 1 and 2, 2 and 3, or 1 and 3 are employed in two-stimulus two-response procedures. These requirements can be confirmed (or disconfirmed) if our basic parameters are quantified on log interval scales, which we assume throughout and test wherever possible, for example, with temporal, color, and luminance discrimination (Davison, 1991b; Godfrey & Davison, 1998).

APPLICATION OF THE MODEL TO CONDITIONAL DISCRIMINATION DATA

2 × 2 Conditional Discrimination

Alsop (1988) and Alsop and Davison (1991) reported an experiment that measured response–reinforcer discriminability (log d_{br}) and stimulus–response discriminability (log d_{sb}) for seven stimulus pairs ordered in terms of stimulus disparity. They used a standard signal-detection procedure with different light intensities as the conditional stimuli. These data constituted a detailed assessment of the current model: If the model is correct, the estimated value of d_{sb} should be ordinally related to the stimulus disparity, and the estimated value of d_{br} should not change systematically with changes in stimulus disparity. Figure 10 (upper panel) shows that the first of these predictions was clearly supported. The situation with regard to the second prediction (lower panel) is less clear. There does appear to be a U-shaped relation between d_{br} and stimulus disparity. However, a Friedman test for a quadratic relationship fails to find a significant quadratic trend at $p = .05$, so the U-shaped function is more apparent than real, or is not statistically evident because of large variances in some measures over subjects (i.e., in the A, B, and G sets). There is a serious problem in estimating d_{br} when d_{sb} is high (such as in Set G) because the few errors that are emitted have a very strong effect on d_{br}, and a difference of one or two responses in each error cell can radically change the value of d_{br}. For the present analysis, we used the Hautus (1995) correction on the data (see Appendix B) to try to eliminate problems of estimating the parameters when few responses are emitted in some cells of the matrix.

Reanalysis of Alsop's (1988) data reveals the predicted changes in the slope of the function relating log B to the log reinforcer ratio: As d_{sb} increases, the slope decreases (upper panels of Figure 6). However, the inverted-U form predicted for the function relating log D to the reinforcer ratio (lower panels of Figure 6) appears only at the lowest nonzero level of stimulus differences arranged in his experiment. This failure of prediction is not decisive because the reinforcer ratio was varied by less than ±1 log unit, thus capturing only the central, relatively flat part of the function. The same restriction of range applies to the data of McCarthy and Davison (1979, 1980a), who also failed to find the predicted inverted-U relation. More seriously, Whittaker (1977), using rats in a yes-no signal-detection procedure, varied the reinforcer ratio for the two choice responses over a much wider range than has been usual, and failed to find both the inverted-U relation between log D and the reinforcer ratio and the predicted change in the slope of the relation between log B and the reinforcer ratio when the disparity of the conditional stimuli was changed. The relation between log B and the log reinforcer ratio also failed to give any evidence of the predicted curvilinearity (Figure 6, top panels), and is thus incompatible with the results reported by Davison and Jones (1995). However, Whittaker used ratio schedules, so the obtained reinforcer ratio covaried with response allocation, and conducted only 8 to 15 sessions of 300 trials per session in each condition. Recent research (Davison & Jones, 1998) suggests that even when reinforcer ratios are controlled, stability takes a long time to occur at extreme ratios. Thus, Whittaker's study probably needs to be replicated.

As noted above, our model predicts an asymmetry between the effects of d_{sb} and d_{br} on the relation between B and the reinforcer ratio: B becomes more sensitive to the reinforcer ratio as d_{br} increases, but becomes less sensitive as d_{sb} increases. Exactly this result was reported by Nevin, Cate, and Alsop (1993) in an experiment in which S_1 and S_2 were bright and dim keylights, and B_1 and B_2 were defined as short or long latencies of a single response. They varied the reinforcer ratio with large or small differences between S_1 and S_2, and with large or small differences between the criteria for short or long laten-

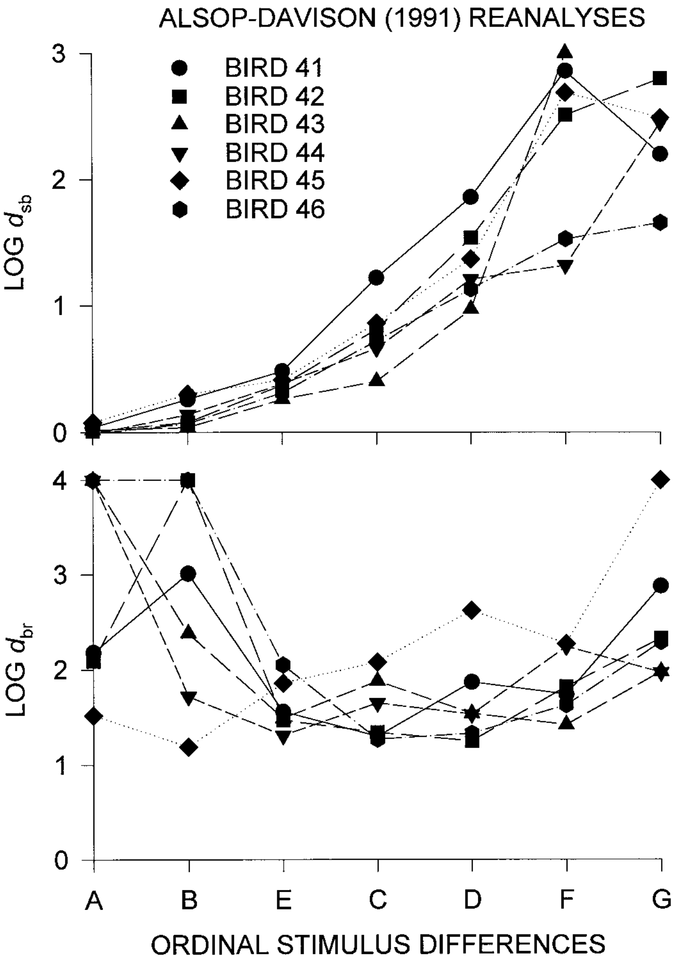

Fig. 10. Reanalyses of the signal-detection data reported by Alsop (1988) and by Alsop and Davison (1991). The upper panel shows the value of log d_{sb} as a function of the ordinally increasing disparity between the intensity of S_1 and S_2. The lower panel shows estimates of log d_{br} for each disparity level.

cies. They obtained similar values for log *d*, the Davison-Tustin measure of discrimination, in two conditions, one with a small S_1-S_2 difference and large B_1-B_2 difference and the other with a large S_1-S_2 difference and a small B_1-B_2 difference. In the first of these conditions, log *b* was a steep and orderly increasing function of the log reinforcer ratio, with *a*, the generalized-matching-law measure of sensitivity, about 0.75. In the second, log *b* was a more variable and shallower function of the log reinforcer ratio, with *a* values ranging from −0.14 to 0.40. Their results, which are shown in Figure 11, confirm a counterintuitive prediction of our model that is shown in Figure 6.

When Nevin et al. (1993) estimated the parameters of our model, d_{sb} increased with the luminance difference between S_1 and S_2, and d_{br} increased with the difference between criteria for short and long latencies. However, the estimated value of d_{br} depended on the S_1-S_2 difference, violating the requirement of parameter invariance. This violation may have resulted from the fact that some responses were not scored because their latencies did not meet the experimenters' criteria. For example, when B_1 and B_2 were defined as responses with latencies between 1.0 and 2.0 s and between 2.0 and 3.0 s, respectively, a latency of 0.9 s would not be counted even though it might belong to the functional class of "short" latencies. Moreover, obtained latencies depended on stimulus intensity as

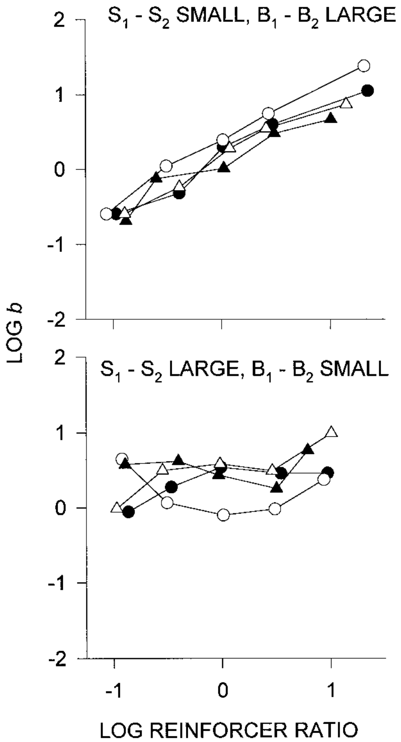

Fig. 11. Differential responding to B_1 or B_2, measured by log *b*, as a function of the obtained log reinforcer ratio (redrawn from Nevin, Cate, & Alsop, 1993).

well as the latency criteria, so the functional (as opposed to experimenter-defined) response classes were not independent of the conditional stimuli. In view of these problems, the failure of parameter invariance is perhaps not surprising.

Godfrey (1997; Godfrey & Davison, 1998) avoided these problems by defining S_1 and S_2 by the luminance of a center key, and defining B_1 and B_2 by the luminances of the choice keys, with bright and dim lights presented irregularly on the left and right keys. Her procedure identifies the choice responses by the stimuli signaling them, as in matching to sample or in a switching-key concurrent schedule, and permits the differentiation between B_1 and B_2 to be specified on the same experimental continuum as S_1 and S_2, but to be varied independently of the difference between S_1 and S_2. She varied reinforcer ratios for various different levels of conditional stimulus disparity and choice stimulus disparity and found no significant effects of the former on measures of the latter, nor vice versa. In particular, she found that discriminability measures for any particular stimulus disparity were not different according to whether they were obtained for a conditional stimulus or a choice stimulus disparity. This, then, is very strong evidence for the model.

Value Transfer

Although we have emphasized steady-state conditional discrimination performance, our model is also consistent with choice data obtained with novel pairs of stimuli. For example, Zentall and Sherburne (1994) trained pigeons on randomly alternating simultaneous discriminations with red (100% reinforcement) versus yellow (0%) and green (50%) versus blue (0%), with color assignments counterbalanced across birds. After training to criterion, they conducted probe choice tests with yellow and blue, and obtained significantly more responses to yellow. This result follows from our model if we construe red and yellow as defining B_1 and B_2 on S_1 (red-yellow) trials and green and blue as defining B_1 and B_2 on S_2 (green-blue) trials. For any value of d_{sb} greater than one (representing discrimination between the two kinds of simultaneous-discrimination trials) and d_{br} less than infinity (representing differentiation between the response–reinforcer contingencies within each simultaneous discrimination), yellow obtains greater generalized strength than blue, as shown in Figure 12. We suggest that this transfer of value from $S+$ to $S-$ within a simultaneous discrimination, which has been used to explain transitive inference in pigeons (Fersen, Wynne, Delius, & Staddon, 1991), follows naturally from our model.

Reinforcement for Errors

Three experiments with animal subjects have systematically explored the effects of reinforcing responses that are conventionally designated "errors." Davison and McCarthy (1980) trained pigeons to discriminate between S_1 (a 5-s keylight) and S_2 (a 10-s keylight) in a procedure that arranged reinforcement probabilistically for responses in each of the four cells of the matrix of Figure 2. Both B_{11} and B_{22} were reinforced with a probability of .7 throughout the experiment, and the probability of reinforcement for B_{12} and B_{21} was varied across conditions from 0 to .9.

	Red (B_1)	Yellow (B_2)
S_1 Red/Yellow trials	100	$\frac{100}{d_{br}} + \frac{50}{d_{sb}d_{br}}$
	Green (B_1)	Blue (B_2)
S_2 Green/Blue trials	50	$\frac{50}{d_{br}} + \frac{100}{d_{sb}d_{br}}$

Fig. 12. The effective reinforcer matrix for the experimental conditions arranged by Zentall and Sherburne (1994).

In this arrangement, the number of reinforcers obtained in each cell of the matrix depends directly on the frequency of responses in that cell. Nevin, Olson, Mandell, and Yarensky (1975) performed a closely comparable experiment with rats as subjects and bright or dim lights as S_1 and S_2. Although the numbers of reinforcers (the independent variables in the model) depended on the numbers and ratios of responses (the dependent variables) in both experiments, the model is applicable because it is based on obtained rather than scheduled reinforcers.

The third experiment (Nevin et al., 1982) used an alternative method for scheduling reinforcers to insure that the number of reinforcers obtained in a cell approximated the number programmed (Shimp, 1969; Stubbs & Pliskoff, 1969). The essence of the method is to arrange the availability of a reinforcer in a particular cell and withhold reinforcers in all other cells until that one has been obtained. This has come to be known as interdependent (or just dependent) scheduling, because reinforcement for one response depends on whether a reinforcer has been scheduled for and obtained by emitting another response. It has also become known, in the signal-detection literature (McCarthy & Davison, 1984), as a *controlled reinforcer-ratio procedure*, because the ratio of obtained reinforcers is specified in advance, by the experimenter, within the limits of statistical fluctuation. Nevin et al. employed pigeons as subjects with 2-s and 3-s keylights as S_1 and S_2 with interdependent scheduling to control obtained reinforcer ratios, and varied the ratio of reinforcers for B_1 and B_2 independently of the ratio of reinforcers for correct responses and errors, as traditionally defined.

Figures 13 and 14 show the results of fitting Equations 13a and 13b to Davison and McCarthy's (1980) and Nevin et al.'s (1982) data. The predictions fit the data well, with values of d_{sb} and d_{br} that are similar to those obtained for moderately confusable stimuli when only correct responses are reinforced. For Davison and McCarthy's data, mean d_{sb} values were about 10, but the d_{br} values obtained were poorly estimated because Davison and McCarthy did not explicitly vary the reinforcer ratios. Accordingly, we used a d_{br} value of 23 (the average for the Nevin et al. data) for the fits in Figure 13. For Nevin et al., the average d_{sb} and d_{br} values were 8 and 23, respectively. As a further check on the consistency of application of this model, we looked at deviations of predictions from data both as a function of the percentage of R_1 reinforcers and as a function of the percentage of reinforcers in the "correct" R_{11} and R_{22} cells. Fits to these deviations showed no significant deviations of slopes from a line of 0 slope except for Bird 60 of the Nevin et al. data.

Our approach has also been supported by a recent report by Hartl and Fantino (1996). In a conventional matching-to-sample procedure, they varied the probabilities of reinforcement for making one or the other choice response to a comparison stimulus that matched the sample, and independently, for responding to a particular comparison stimulus regardless of the sample. The latter variation effectively arranges reinforcers for errors, and their data were well explained by an earlier (but algebraically equivalent) version of our model (see their p. 23 for discussion). In conclusion, it appears that our model for the conventional two-stimulus two-response conditional discrimination extends naturally, and with the requisite invariance of d_{sb}, to situations in which reinforcers occur in all four cells of the 2 × 2 matrix.

Matching to Sample and Its Variants

As we have noted above, the conventional three-key matching-to-sample paradigm is a two-stimulus two-response conditional discrimination in which the sample, presented on the center key, serves as the conditional cue and the side-key choices are defined by comparison stimuli. The procedure has been used intensively by researchers whose prima-

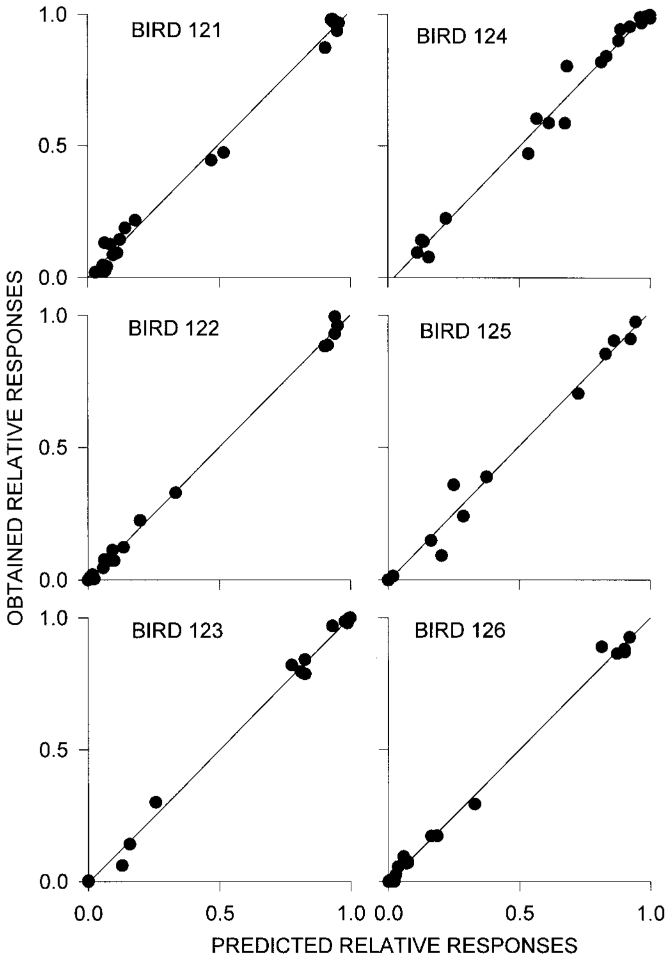

Fig. 13. Predicted and obtained relative distribution of responses for the data reported by Davison and McCarthy (1980). The straight lines were fitted by the method of least squares.

ry interests are in cognitive processes such as coding, retrieval, and limited-capacity working memory, as well as in behavior-analytic research on equivalence classes. We will not attempt to review the massive literature in this area, but to the extent that our model can account for performance in this paradigm, it will provide an alternative to explanations that invoke cognitive processes.

Identity versus symbolic matching. The best known version of the matching-to-sample paradigm, termed *identity matching,* employs comparison stimuli that are physically similar to the samples. A related version, termed *symbolic matching,* employs comparison stimuli from an independent dimension to define the choice responses. In our model, the parameter d_{sb} depends on the discriminability of the relations between the sample and choice-defining stimuli, whereas d_{br} depends on the discriminability of the relations between the choice responses and the reinforcer. According to our theory, then, in identity matching the values of d_{sb} and d_{br} should be identical, despite the fact that the conditional stimuli occur successively in time and the stimuli signaling the choice alternatives occur simultaneously. This was shown to be the case in the research reported by Godfrey (1997) and Godfrey and Davison (1998) (see the section on 2 × 2 conditional discrimination, above).

Delayed matching and delayed reinforcement. A widely studied variant of the matching-to-sample paradigm that is of special interest in re-

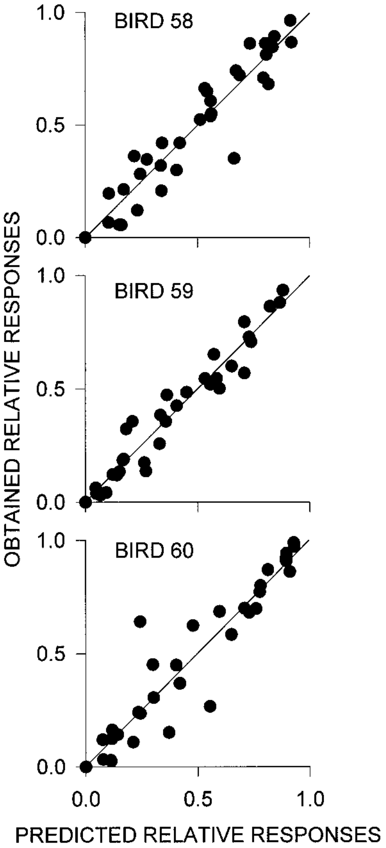

Fig. 14. Predicted and obtained relative distribution of responses for the data reported by Nevin, Jenkins, Whittaker, and Yarensky (1982). The straight lines were fitted by the method of least squares.

search on short-term memory introduces a delay (or *retention interval*) between the offset of the conditional cue and the availability of the choice responses. We will distinguish two procedures that are commonly used in research on delay of choice. The first, termed a *fixed-delay procedure,* employs a single delay value between offset of the sample and onset of the comparison stimuli throughout an experimental condition, and varies the delay between conditions in order to determine a delay gradient or forgetting curve. The second, termed a *mixed-delay procedure,* arranges a number of different delays within a single condition. A well-nigh universal effect of lengthening the delay is a progressive decrease in accuracy of discrimination as measured by percentage correct or by the Davison-Tustin measure log d (e.g., Cumming, Berryman, & Nevin, 1963; Harnett, McCarthy, & Davison, 1984; White, 1991). In the terms of our model, this decrease results from stimulus–response discriminability (d_{sb}) becoming degraded during the delay with response–reinforcer discriminability (d_{br}) remaining high. The present model, then, would predict that as d_{sb} decreases with increasing delays, log d (the Davison-Tustin measure that has been used extensively in this area) should also decrease, but sensitivity to reinforcement as measured by a, the slope of the generalized-matching-law relation between log b and the log reinforcer ratio, should increase, as shown in Figure 6.

It is also well known that introducing a delay between choice responses and the reinforcer (delay of reinforcement) also decreases log d (e.g., McCarthy & Davison, 1991). However, log d values would fall not because of decreasing stimulus–response discriminability, but because of decreasing response–reinforcer discriminability. As the delay lengthens, the delay-of-reinforcement procedure will come to function as a reinforcement-for-errors procedure, and will produce the same effects. But when response–reinforcer discriminability is compromised, sensitivity to reinforcement, as measured by a, will also decrease.

Rather than attempting to estimate d_{sb} and d_{br} from the voluminous data in this area, we will show some examples of effects that can be expected when values of log d and a (measures frequently reported in this area) are predicted from the present model using representative values of d_{sb} and d_{br}. But first we develop the model. We shall assume, for convenience, that we can specify a d_t value for each constant increment in delay time. What this means, in effect, is that in each (say) 1-s interval, a constant value of d_t—a discriminability parameter just like d_{sb} and d_{br}—operates in the usual way on the effective reinforcer matrix that exists at the start of this 1-s interval to produce a new effective reinforcer matrix. This simply has the effect of progressively moving the effective reinforcer matrix towards nondifferential reinforcement with respect to responses (i.e., equal reinforcer frequencies for the two responses in the presence of each stimulus). The selection of a 1-s step is, of course, arbitrary, and smaller steps will simply require a smaller d_t value.

Because effective reinforcer allocation

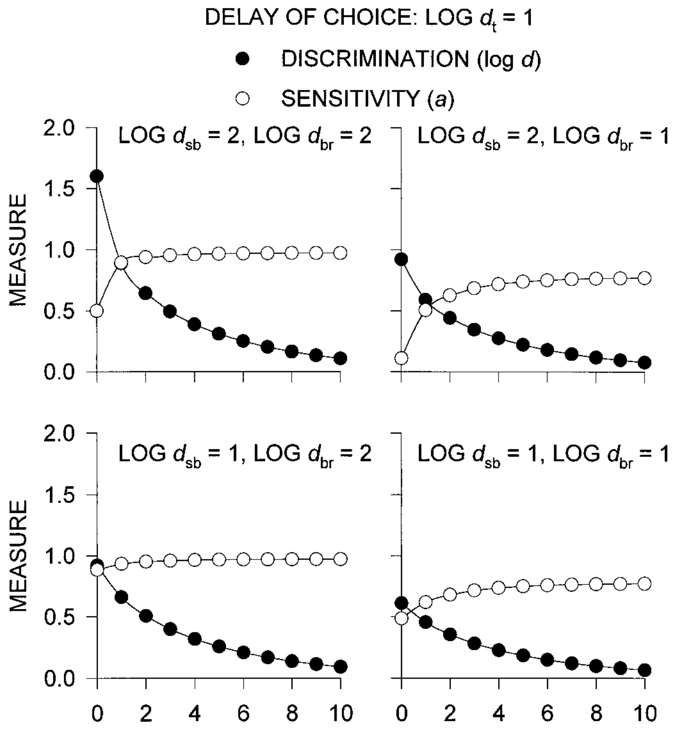

Fig. 15. Log discrimination and sensitivity to reinforcement, as measured using the Davison-Tustin (1978) model, when the delay between stimulus presentation and choice is varied for some representative values of log d_{sb} and log d_{br}. The value of d_t is constant across time.

changes over the course of the delay, the model must be applied in a successive, rather than a simultaneous, manner. That is, for the delay-of-choice situation, we need to operate progressively on stimulus–response discriminability (d_{sb}) using d_t during this delay before applying response–reinforcer confusion when the choice is emitted. In the delay-of-reinforcement situation, the matrix results of stimulus–response confusion will be progressively operated on by d_t over the delay before being operated on by response–reinforcer confusion. The particular matrix values that are current when the choice stimuli are presented, or when the reinforcer is delivered, will be used to predict response ratios and calculate log d.

Figure 15 shows how predicted measures of discrimination and sensitivity to reinforcement change with increasing delay of choice with some representative values of d_{sb} and d_{br}. Discrimination, measured by log d, falls under all conditions, and sensitivity to reinforcement, measured by a, either increases or may appear to remain constant when d_{br} is relatively high and d_{sb} is relatively low.

Figure 16 shows delay-of-reinforcement predictions. Both log d and a fall with increasing delay with sensitivity being generally higher when d_{br} is greater, though sensitivity is similar when d_{sb} and d_{br} are both high and when they are both low.

In an extensive experiment, using mixed rather than fixed delays and quite high conditional stimulus discriminability values (log d values generally between 1.3 and 1.9), Jones and White (1992) reported a statistically significant increase in sensitivity to reinforcement with increasing stimulus–choice delay, confirming the predictions in the upper panels of Figure 15. In a related experiment with fixed rather than mixed delays, Harnett et al. (1984) found the usual decrement in log d with increasing delay, and but no statistically significant change in sensitivity to reinforcement. Because their log d values were some-

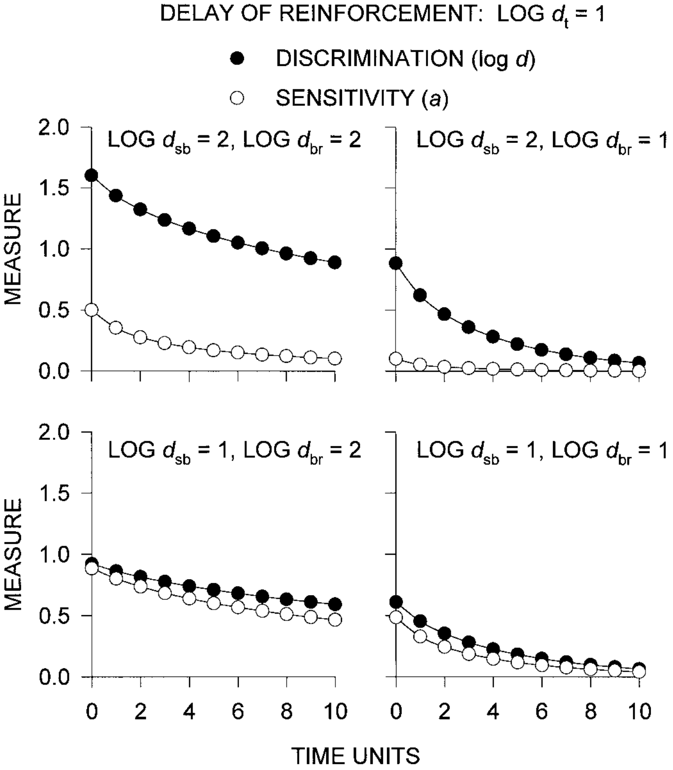

Fig. 16. Log discrimination and sensitivity to reinforcement, as measured using the Davison-Tustin (1978) model, when the delay between choice and reinforcement is varied for some representative values of log d_{sb} and log d_{br}. The value of d_t is constant across time.

what lower than those of Jones and White, this lack of a statistically significant change accords reasonably with the shallow sensitivity function predicted in the lower left panel of Figure 15.

McCarthy and Davison (1986) reported that both delay of choice and delay of reinforcement decreased log *d*, but that the effect for delay of choice was greater than that for delay of reinforcement. This difference is predicted for cases in which d_{br} is high, as can be seen by comparing the left panels of Figures 15 and 16 (but note that the predicted decreases in log *d* are identical when d_{br} is moderate, as in the right panels). McCarthy and Davison (1991) replicated this effect, and also measured sensitivity to reinforcement (*a*) for each stimulus–choice and choice–reinforcer delay. Sensitivity to reinforcement decreased (though not greatly) when both delays were varied. Although our model predicts the decrease in *a* with increasing choice–reinforcer delay, the decrease in *a* with increasing stimulus–choice delay, given that log *d* values were around 1.0, is inconsistent with the predictions in Figure 15. McCarthy and Davison made no correction for zero cells, and such a correction would have had the effect of marginally decreasing the sensitivity (and log *d*) values at short delays. However, it remains puzzling that no increase in sensitivity was found. This last result, then, appears partially but not strongly to argue against the present model. In passing, it is important to note that McCarthy and Davison *did* fit Equations 7a and 7b to their data, and found some changes in d_{sb} and d_{br} with increasing stimulus–choice and choice–reinforcer delays. This lack of invariance is not an argument against the full model, however, because they did not use the model with increased confusion with elapsing delays as presented above.

Finally, McCarthy and Voss (1995), using a fixed-delay procedure, provided clear evidence that sensitivity to reinforcement fell with increasing stimulus–choice delay, both for small and large reinforcer durations, contrary to our prediction. Given the wide range of results reported on the relation between sensitivity to reinforcement and delay in delayed matching to sample, it is not surprising that we have some difficulty in modeling in this area. The majority of results seem to suggest that sensitivity falls with delay, which is incompatible with our model. However, we would argue that the final assessment of our model for delay of choice should await the empirical resolution of just what variables are critical to produce increases or decreases in the value of *a* with delay; until, in other words, we have a clear empirical result to model.[5]

Second-order discrimination of mixed delays. Many researchers, but notably White and his associates (e.g., White, 1985), have varied stimulus–choice delays within sessions rather than across conditions. Recently, White and Cooney (1996) varied the reinforcer ratio separately following two delays that occurred in irregular order. For example, in one set of conditions, reinforcer probabilities for correct responses to red and green choice keys varied across conditions from .1 to .9 when the stimulus–choice delay was 0.1 s but were constant at .5 when the stimulus–choice delay was 4 s. In effect, the length of the delay may be construed as a second-order conditional cue signaling differential reinforcer probabilities. Overall, White and Cooney found that the reinforcer ratio arranged at one delay did not affect performance at the other delay with a different reinforcer ratio, and concluded that ''performance at one retention interval is independent of factors that influence performance at another'' (p. 55).

The two delays that White and Cooney (1996) used were highly discriminable (McCarthy & Davison, 1980a), but if their delays had been 3.8 and 4 s, they probably would have found interdependence of reinforcer-ratio effects. To address the generalization of the effects of reinforcers across different delays, as well as across stimuli and responses, we introduce the notion of the discriminability between stimulus–choice delays, d_d. The model for mixed stimulus–choice delays is effected by degrading d_{sb} initially by d_t, as in the single-delay model above, and then degrading each of the resulting reinforcer values by the appropriate d_d values. The development is done here for just two mixed delays, but may logically be applied to any number. With the loss of only a little generality (the effects of d_t), we shall present the development for a second-order conditional discrimination in which two stimuli such as continuous versus flashing house lights, on a separate dimension from the first-order stimuli, signal different sets of first-order stimulus–behavior–reinforcer contingencies. These stimuli are designated S_A and S_B, and represent the two delays in White and Cooney's procedure.

The reinforcer matrix we shall use is shown in Figure 17. Note that the B_1 and B_2 contingencies are reversed between the second-order stimuli, S_A and S_B. The discriminability of the second-order stimuli is d_2, and we assume that the response–reinforcer discriminability (d_{br}) and the first-order stimulus–response discriminability (d_{sb}) are the same under both second-order conditional stimuli.

Figure 18 shows the predicted effects of varying the reinforcer ratio (R_{a11}/R_{a22}) in the presence of S_A (analogous to the 0.1-s delay in the example from White and Cooney, 1996) on responding with respect to S_1 and S_2 (log *D*) and with respect to B_1 and B_2 (log *B*) in the presence of S_B (analogous to the 4-s delay) which, in our example, offers equal R_{b12} and R_{b21} reinforcer rates. The values of d_{sb} and d_{br} are both 10, and the value of d_2 is 2 (upper panel) and 10 (lower panel). As would be expected, when there is little discrimination between S_A and S_B, varying the reinforcer ratio in S_A has a strong effect on responding in the presence of S_B. The value of log *D* in S_B is negative because of the reversed contingencies of reinforcement. These values are affected by the discriminability between S_A and S_B, and are smaller in an absolute sense when d_2 is smaller and are nonlinear with respect to the S_A reinforcer ratio.

This analysis shows that the mixed-delay procedure will affect measurements of stimulus discriminability more than the fixed-delay procedure will, and will provide lower estimates of discrimination (log *d*) than the single-delay procedure even when reinforcer ratios are the same at all delays. The exact

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[5] White and Wixted (1999) recently described an inverse relation between discrimination and sensitivity to relative reinforcement in delayed matching to sample.

Second-order stimulus	First-order stimulus	B_1	B_2
a	S_1	R_{a11}	-
a	S_2	-	R_{a22}
b	S_1	-	R_{b12}
b	S_2	R_{b21}	-

Second-order stimulus	First-order stimulus	B_1	B_2
a	S_1	$R_{a11}+\frac{R_{a22}}{d_{sb}d_{br}}+\frac{\frac{R_{b12}}{d_{br}}+\frac{R_{b21}}{d_{sb}}}{d_2}$	$\frac{R_{a22}}{d_{sb}}+\frac{R_{a11}}{d_{br}}+\frac{R_{b12}+\frac{R_{b21}}{d_{sb}d_{br}}}{d_2}$
a	S_2	$\frac{R_{a22}}{d_{br}}+\frac{R_{a11}}{d_{sb}}+\frac{R_{b21}+\frac{R_{b12}}{d_{sb}d_{br}}}{d_2}$	$R_{a22}+\frac{R_{a11}}{d_{sb}d_{br}}+\frac{\frac{R_{b12}}{d_{sb}}+\frac{R_{b21}}{d_{br}}}{d_2}$
b	S_1	$\frac{R_{b12}}{d_{br}}+\frac{R_{b21}}{d_{sb}}+\frac{R_{a11}+\frac{R_{a22}}{d_{sdr}}}{d_2}$	$R_{b12}+\frac{R_{b21}}{d_{sb}d_{br}}+\frac{\frac{R_{a11}}{d_{br}}+\frac{R_{a22}}{d_{sb}}}{d_2}$
b	S_2	$R_{b21}+\frac{R_{b12}}{d_{sb}d_{br}}+\frac{\frac{R_{a22}}{d_{br}}+\frac{R_{a11}}{d_{sb}}}{d_2}$	$\frac{R_{b12}}{d_{sb}}+\frac{R_{b21}}{d_{br}}+\frac{R_{a22}+\frac{R_{a11}}{d_{sb}d_{br}}}{d_2}$

Fig. 17. Upper panel: the matrix of events in a second-order conditional discrimination in which B_1 is reinforced in the presence of S_1 and B_2 in the presence of S_2, both when Stimulus A is presented. The contingencies of reinforcement are reversed when Stimulus B is presented. Lower panel: The effective reinforcer matrix for the events shown in the upper panel.

pattern of results obtained will depend critically on the distribution and spacing of stimulus–choice delays. More generally, it shows that the discriminability of second-order cues correlated with different outcome matrices in conditional discriminations, as in Hobson (1978), will affect the estimation of model parameters.

Complex Stimulus Discrimination

An experiment reported by White et al. (1984) involved a free-operant conditional discrimination in which pecking the right key produced food according to one VI schedule when a vertical line was projected on both keys (S_1) and pecking the left key produced food according to another VI schedule when the line was tilted 15, 30, 45, 60, or 75° (S_2–S_6), presented irregularly within a single experimental condition. This paradigm is like that discussed in the section on more than two stimuli and two responses, above, but with five stimuli assigned to one of the responses. Left-key pecks given the vertical line, and right-key pecks given any other orientation, were neither reinforced nor punished. The expanded matrix defining the operants for this case is shown in Figure 19.

When the VI schedules were varied, the relation between the ratio of responses at each S_2–S_6 orientation and the associated reinforcer ratio depended on the difference in orientation of the stimuli signaling the pairs under investigation. To characterize the results in terms of the generalized matching law: *a* (sensitivity to reinforcement) decreased as stimulus disparity increased, as predicted by our model. In a related study, White (1986) varied stimulus differences between conditions and found the same result: As predicted, sensitivity to reinforcer ratios in a free-operant conditional discrimination was inversely related to stimulus disparity across successive conditions.

This finding is not limited to free-operant procedures. Davison and McCarthy (1987) trained pigeons to peck left given a fixed duration of center-key illumination (either 5 s or 20 s in different parts of their study) and to peck right given any of 12 other durations ranging from 2.5 s to 57.5 s in 5-s steps. Again, the paradigm is like that examined in

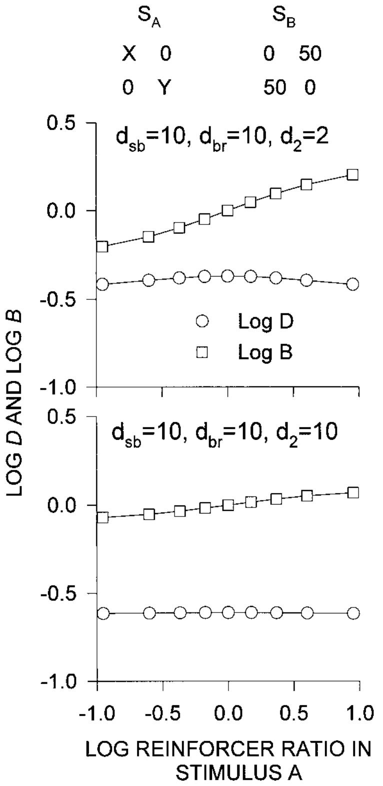


Fig. 18. The predicted effect of varying the reinforcer ratio in the presence of second-order Conditional Stimulus A on stimulus differentiation (log *D,* Equation 5) and response differentiation (log *B,* Equation 3) in the presence of second-order Conditional Stimulus B. As noted in the text, S_A is analogous to a short delay correlated with varying reinforcer ratios, and S_B is analogous to a long delay correlated with a constant reinforcer ratio. If the delays are confusable (d_2 small, upper panel), reinforcer ratios in S_A affect performance in S_B, but there is little effect if the delays are highly discriminable (d_2 large, lower panel).

the section on more than two stimuli and two responses, above, but with 12 different stimuli assigned to one of the responses. When they varied the reinforcer ratio, they found that sensitivity to reinforcement was inversely related to the discriminability of the duration examined, relative to the fixed duration.

Davison (1989) showed that the present model gave a good account of Davison and McCarthy's (1987) temporal discrimination data, providing 12 rational d_{sb} estimates and an appropriate estimate of d_{br}. More important, after best fit discriminability estimates were obtained, the model predicted almost exactly the relation between sensitivity to reinforcement and discriminability that was found empirically. In addition, Davison (1991b) reported the analysis of a set of data on color discrimination in pigeons, in which they were required to peck the left or right side keys according to which of eight color stimuli (559 to 594 nm in steps of 5 nm) had been presented. Both the reinforcer ratio for correct responses and the stimuli signaling reinforcers for pecking left or pecking right were varied. This analysis, with seven d_{sb} parameters, provided an excellent description of the data, and furthermore the d_{sb} parameters were related to wavelength in the same way as has been found for generalization, discrimination, and color-naming functions for the pigeon over this wavelength range (Shepard, 1965; Wright, 1974; Wright & Cumming, 1971). Thus, our model provides convergent measurement of the distances between stimuli in psychometric space.

	Slant	B_1	B_2
S_1	0°	B_{11}, R_{11}	B_{12}
S_2-S_6	15°	B_{21}	B_{22}, R_{22}
	30°	B_{31}	B_{32}, R_{32}
	45°	B_{41}	B_{42}, R_{42}
	60°	B_{51}	B_{52}, R_{52}
	75°	B_{61}	B_{62}, R_{62}

Fig. 19. The matrix of events in the experiment reported by White, Pipe, and McLean (1984) showing how the cells of the matrix are subscripted. Response B_1 was reinforced in the presence of a 0° slant, and B_2 was reinforced in the presence of all other orientations.

Multiple Stimuli and Multiple Correct Responses

We next consider the classical recognition task of psychophysics, in which subjects are presented with one of *N* different stimuli in random order and asked to make one of *N* different responses to indicate which stimulus was presented. Interestingly, humans have trouble identifying more than seven (±2) different tone intensities even when the tones are highly discriminable in the sense that few, if any, errors occur when fewer than seven tones are presented (e.g., Pollack, 1952). The recognition experiment has been repeated by Chase (1983) with pigeons as subjects in a chamber equipped with nine keys. The stim-

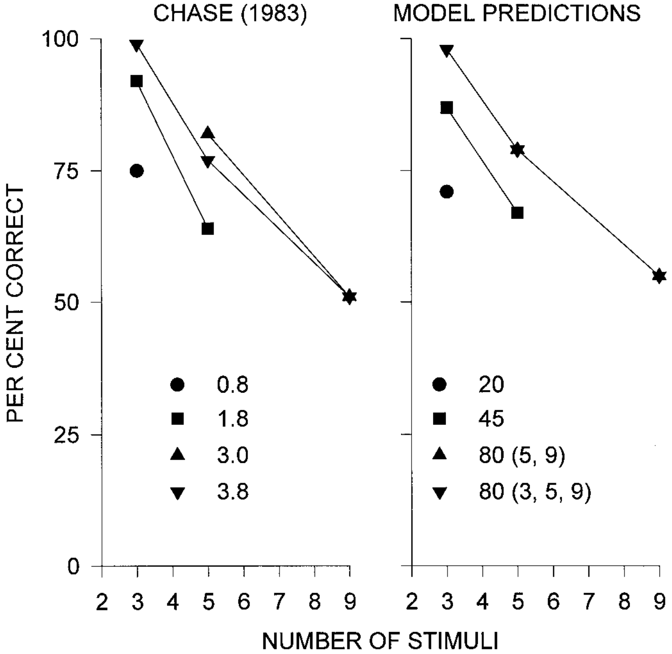


Fig. 20. Percentage correct responses for various ranges of log luminance levels (left panel) reported by Chase (1983) (represented by separate plots with distinct data points in the left panel) and model predictions assuming that a 0.4 log luminance difference corresponded to a d_{sb} value of 10 (resulting in various assumed values of d_{sb}, represented by separate plots with distinct data points in the right panel).

uli were various luminances displayed on a single rectangular key above the nine response keys, presented in discrete trials in irregular order; a single peck at the key designated as correct for each luminance produced 2-s access to food. The paradigm is like that examined in the section on *N* responses and *M* stimuli, above, with the numbers of stimuli and responses varied across conditions. In one series of conditions, Chase compared performances involving luminance ranges of 0.8 log units, 1.8 log units, 3.0 log units, and 3.8 log units, with the number of stimulus–response pairs varying from three to nine. Average percentage correct increased with the range over which the stimuli were distributed and decreased as a function of number of stimulus–response pairs defined within that range, as shown in the left panel of Figure 20. We modeled Chase's data by assuming that a 0.4 log-unit difference between stimuli corresponded to a d_{sb} value of 10 (thus, a set of nine stimuli would span a range of 80 expressed as d_{sb}). Because Chase's data were similar for the two largest ranges, 3.0 and 3.8 log units, we treated both as 3.2 log units, or 80 units on the d_{sb} scale. We also assumed a value of 10 for d_{br} for the peck–food relations between adjacent keys. The overall percentage correct predicted by our model for Chase's conditions is shown in the right panel of Figure 20. Overall, the agreement between Chase's average data and the predictions of our model is respectable. Our model also predicts that errors will be most frequent in the middle of the stimulus–response matrix, as found by Chase with pigeons and by Pollack (1952) with humans.

Recognition tasks usually arrange ordinal stimulus–response mapping: The least intense stimulus is identified with Response 1, the next with Response 2, and so on. With pigeons, Chase (1983) compared ordered and unordered stimulus–response identification and found that percentage correct was substantially lower with unordered identification. Our model does not predict a large decrement with unordered identification, but it does predict one aspect of Chase's results: secondary modes in response probability. Chase suggested that "Secondary peaks can be accounted for if it is assumed that discrimination among key positions is imperfect'' (1983, p. 45). This, of course, is exactly what our parameter d_{br} quantifies, and our model predicts secondary peaks at stimuli for which the correct key happened to be adjacent to the key being considered, as Chase reported. For example, when Response 9 was defined as correct for Stimulus 8, a secondary peak in the probability of pecking Key 9 appeared at Stimulus 3, which was correct for Response 8 in the unordered identification set we explored.

Godfrey (1997) reported an experiment that compared performance in 2 × 2, 3 × 3, and 4 × 4 conditional discrimination matrices. The conditional stimuli were intensities of yellow light displayed on a central key, and the choice responses consisted of pecking one of six red keys set around the central key (thus allowing the potential for 5 × 5 and 6 × 6 matrices, but these were not investigated). In one condition, two stimuli designated S_2 and S_5 were presented successively with reinforcement arranged for pecks at Keys B_2 and B_5. In a second condition, a stimulus of intermediate intensity (S_4) was also presented, with reinforcement arranged for pecks at Key B_4. In a third condition, another stimulus of intermediate intensity (S_3) was included, with reinforcement arranged for pecks at Key B_3. Thus, it was possible to compare the discriminability of a given pair of stimulus–behavior relations (e.g., d_{sb25}) across conditions with different numbers of discriminated operants. In addition, it was possible to determine whether log d_{sb} values combined additively, as we have assumed. The average parameter estimates and the additivity predictions are presented in Figure 21. Across conditions, there were no systematic changes in any d_{sb} parameter estimates when further stimulus–response–reinforcement contingencies were added or subtracted. For example, log d_{sb} for S_2 versus S_5 when studied alone (upper panel) is not substantially affected by adding one (middle panel) or two (lower panel) intervening stimuli. The model thus works well with complex stimulus–response–reinforcer relations. Further, as Figure 21 also shows, the additivity requirement (that, e.g., $\log d_{sb23} + \log d_{sb34} = \log d_{sb24}$) seems to be approximately true. However, in all four cases that could be investigated with these data, the predicted value of log d_{sb} was somewhat less than the obtained value. If such findings are replicated, they could indicate that psychometric space is nonlinear in log terms (and may be linear in other terms). Clearly, more research is needed in this area.

One surprising aspect of the results was that the actual behavior ratios between pairs of three-term contingencies were unaffected by the addition of further contingencies, a result that appears to be incompatible with the present model. Consider a 2 × 2 matrix of the form $S_2{:}(B_2 \rightarrow R_2)$ and $S_5{:}(B_5 \rightarrow R_5)$. If a further contingency $S_4{:}(B_4 \rightarrow R_4)$ is added, in which S_4 is closer physically to S_5 than to S_2, we would expect that S_5 performance would gain more reinforcer value than S_2 performance, leading to a decrement in the S_2/S_5 behavior ratios. However, given reasonable discriminability between S_4 and S_5, the effect would be rather small, and may not be discernible in the general error variance. This effect also requires further investigation. Despite such uncertainties, it is clear that our model provides an effective account of stimulus control and choice in a wide variety of conditional discrimination paradigms.

MULTIPLE AND CONCURRENT SCHEDULES

We began this article by discussing some qualitative similarities that result when the differences among stimuli, responses, and reinforcers are varied. Our argument began with conventional multiple and concurrent schedules, but the remainder of our development dealt with experimental paradigms in which the reinforcement contingencies for two or more concurrently available choice responses depend on the values of two or more successively presented stimuli. In effect, all such paradigms can be characterized as multiple concurrent schedules. We now apply the general model that successfully accounted for a large array of findings in these paradigms to the presumably simpler cases: multiple and concurrent schedules.

Multiple Schedules

As we noted at the outset, all measured behavior occurs in a setting that includes unmeasured, extraneous behavior and reinforcers. A 2 × 2 matrix that incorporates extraneous behavior (B_e) and reinforcers (R_e)

for a standard single-key multiple schedule with distinctive stimuli S_1 and S_2 to signal the components would look like Figure 22, and the resulting equations, including discriminability parameters, would be, for S_1,

$$\frac{B_{11}}{B_{1e}} = \frac{R_{11} + \dfrac{R_{21}}{d_{sb}} + \dfrac{R_{1e}}{d_{br}} + \dfrac{R_{2e}}{d_{sb}d_{br}}}{R_{1e} + \dfrac{R_{2e}}{d_{sb}} + \dfrac{R_{11}}{d_{br}} + \dfrac{R_{21}}{d_{sb}d_{br}}} \tag{14a}$$

and, for S_2,

$$\frac{B_{21}}{B_{2e}} = \frac{R_{21} + \dfrac{R_{11}}{d_{sb}} + \dfrac{R_{2e}}{d_{br}} + \dfrac{R_{1e}}{d_{sb}d_{br}}}{R_{2e} + \dfrac{R_{1e}}{d_{sb}} + \dfrac{R_{21}}{d_{br}} + \dfrac{R_{11}}{d_{sb}d_{br}}}. \tag{14b}$$

Dividing these equations gives us an expression for the relation between the response ratio, B_1/B_2, and the reinforcer rate in each component, in Equation 15, below. Some simplifying assumptions are needed to reduce the number of free parameters in Equation 15. It may seem reasonable to assume that the discriminability of response–reinforcer contingencies such as (B_1:key peck → R_1: food) and (B_e:scratch → R_e:relief of itch) would be essentially infinite. However, key pecking for magazine grain and pecking at the chamber floor for spilled grain might be moderately confusable, but the frequencies of these events are unknown, and we do not wish to adopt the probably incorrect assumption that R_e is the same in both components. If variable-time reinforcement is explicitly used to simulate R_e, the extraneous reinforcers are countable, but d_{br} may range from one to near infinity, depending on the subject's response rate, interresponse-time distribution, and reinforcement history. Related difficulties arise with B_e: Unless we adopt Herrnstein's (1970) restrictive (and probably incorrect) assumptions that $B + B_e = k$, and that k is the same for both schedule components, the number of free parameters in Equations 1 and 2 is excessive.

It is possible to model multiple VI extinction schedules, which are common in the study of stimulus control, by assuming that d_{br} is infinite. Then, Equation 15 simplifies to

$$\frac{B_1}{B_2} = d_{sb} \cdot \frac{B_{1e}}{B_{2e}} \cdot \frac{(R_{2e} + R_{1e}/d_{sb})}{(R_{1e} + R_{2e}/d_{sb})}. \tag{16}$$

This expression implies that if the ratios of extraneous responses and their reinforcers in S_1 and S_2 remain constant, the ratio of measured responses depends on d_{sb} but is independent of reinforcer rate in S_1, as found by Cumming (1955).

Finally, if R_e is simulated by reinforcement for a specified alternative response, the matrix for the resulting multiple concurrent schedule is just like that for reinforcement for "errors," for which performance is readily modeled (Equations 13a and 13b). Research on the simulation of extraneous reinforcers in multiple schedules has been reported by Davison (1993), Lobb and Davison (1977), McLean and White (1983), and McLean (1992, 1995). The findings in general suggest that simulated extraneous reinforcers are reallocated between components, especially when they are arranged on ratio schedules. The present model cannot directly, without further assumptions, predict the degree of reallocation of R_e between multiple-schedule components; it simply allows for it (but see Davison, 1993). One result is clearly predicted from Equations 13a and 13b: "Successive independence" (the independence of log response ratios in one component from the conditions of reinforcement in the other component; McLean & White, 1983) will occur when d_{sb12} is asymptotically large (as with, e.g., red vs. green signaled components; see Charman & Davison, 1983). However, successive independence will not hold when d_{sb12} is small because the effects of reinforcers in one component will generalize and contribute to

$$\frac{B_{11}}{B_{21}} = \frac{B_{1e}\left(R_{11} + \dfrac{R_{21}}{d_{sb}} + \dfrac{R_{1e}}{d_{br}} + \dfrac{R_{2e}}{d_{sb}d_{br}}\right)\left(R_{2e} + \dfrac{R_{1e}}{d_{sb}} + \dfrac{R_{21}}{d_{br}} + \dfrac{R_{11}}{d_{sb}d_{br}}\right)}{B_{2e}\left(R_{1e} + \dfrac{R_{2e}}{d_{sb}} + \dfrac{R_{11}}{d_{br}} + \dfrac{R_{21}}{d_{sb}d_{br}}\right)\left(R_{21} + \dfrac{R_{11}}{d_{sb}} + \dfrac{R_{2e}}{d_{br}} + \dfrac{R_{1e}}{d_{sb}d_{br}}\right)}. \tag{15}$$

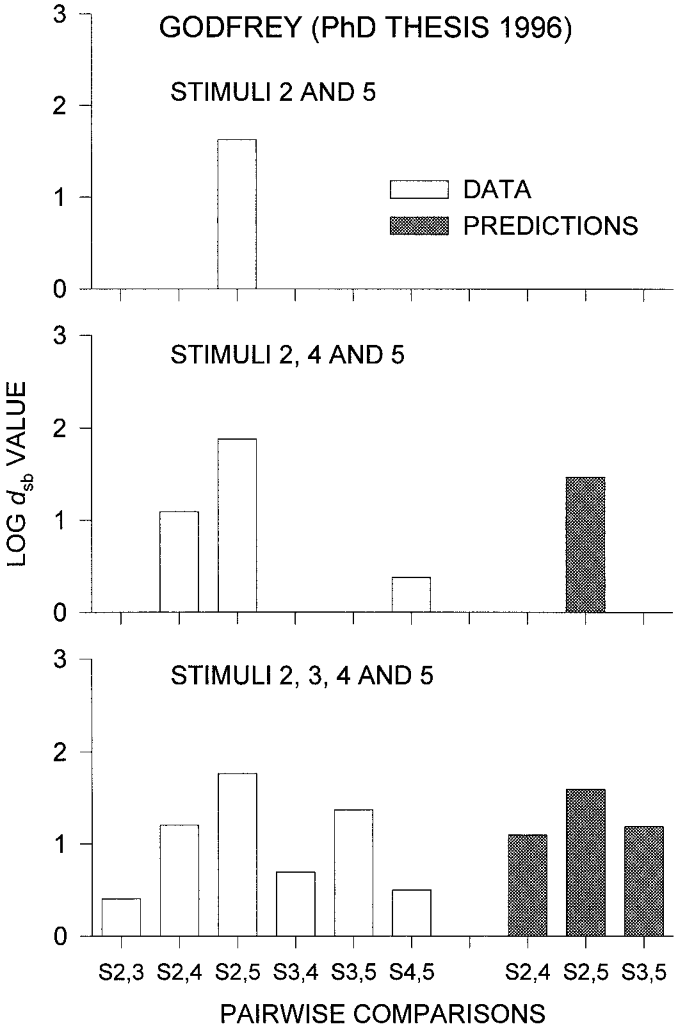
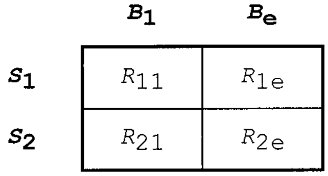


Fig. 21. Left: data from Godfrey (1997) comparing obtained d_{sb} in 2 × 2, 3 × 3, and 4 × 4 conditional discrimination tasks using the same stimuli. Right: predicted stimulus–behavior discriminability values for some pairwise combinations obtained by adding log d_{sb} for component pairs (e.g., using log d_{sb23} + log d_{sb34} to predict log d_{sb24}). The top panel shows data from Stimuli 2 and 5 in a 2 × 2 task, the center panel shows data from Stimuli 2, 4, and 5 in a 3 × 3 task, and the bottom panel shows data from Stimuli 2, 3, 4, and 5 in a 4 × 4 task.

effective reinforcement in the other component and vice versa, as explained above for 2 × 2 conditional discriminations.

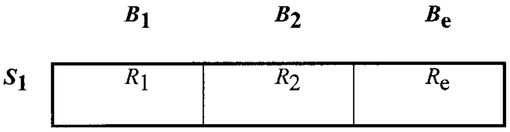
Concurrent Schedules

Conventional two-key concurrent schedules are arranged in the presence of a single stimulus condition, so S_1 and S_2 are equivalent and its matrix (Figure 23) would collapse into a single row (which simplifies subscripting). The resulting equation for the B_1/B_2 ratio is



	B_1	B_e
S_1	R_{11}	R_{1e}
S_2	R_{21}	R_{2e}

Fig. 22. A detection matrix for multiple schedules incorporating extraneous behavior and reinforcers.



	B_1	B_2	B_e
S_1	R_1	R_2	R_e

Fig. 23. A detection matrix for concurrent schedules incorporating extraneous behavior and reinforcers.

$$\frac{B_1}{B_2} = \frac{R_1 + \dfrac{R_2}{d_{br12}} + \dfrac{R_e}{d_{br1e}}}{R_2 + \dfrac{R_1}{d_{br12}} + \dfrac{R_e}{d_{br2e}}}. \tag{17}$$

If we assume that $B_e \rightarrow R_e$ is infinitely discriminable from $B_1 \rightarrow R_1$ and $B_2 \rightarrow R_2$, the equation simplifies to

$$\frac{B_1}{B_2} = \frac{R_1 + \dfrac{R_2}{d_{br12}}}{R_2 + \dfrac{R_1}{d_{br12}}}, \tag{18}$$

which, as noted above, provides a good account of concurrent VI VI schedule performance.

The predictions of this approach are that the ratio of responses B_1 and B_2 will be unaffected by the value of R_e if, and only if, d_{br1e} and d_{br2e} are infinite (no confusion). Such a situation is shown in the right graphs in Figure 24. Interpreting this situation as a three-alternative concurrent schedule, the constant-ratio rule (i.e., constancy of choice ratios between a pair of alternatives when a third is added or removed; Luce, 1959) will be correct under these conditions—that is, only when both differences between defined and extraneous response–reinforcer contingencies are highly discriminable. If they are less than highly discriminable, then because the effects of R_e are additive, increasing extraneous reinforcer rates (or decreasing arranged R_1 and R_2 reinforcer rates) will decrease response differentiation between B_1 and B_2 (Figure 24, left and center panels). In generalized-matching terms, sensitivity to reinforcement is predicted to fall as arranged R_1 and R_2 reinforcer rates decrease, assuming a constant third-alternative reinforcer rate. If extraneous reinforcers are assumed to exist

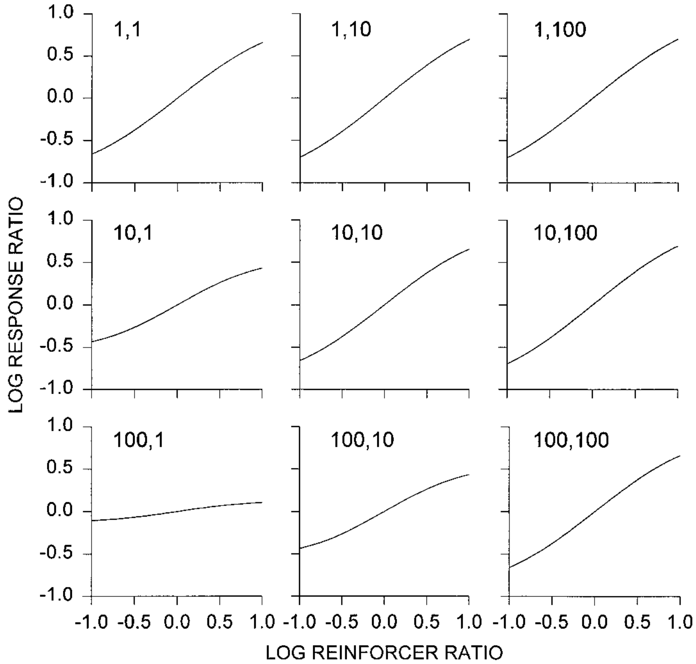


Fig. 24. The predicted relation between log response ratios and log reinforcer ratios in concurrent VI VI schedules. The parameters of each graph are, respectively, the value of the extraneous reinforcer rate (or the third-alternative reinforcer rate) and the value of the response–reinforcer discriminability between the first two alternatives and the third alternative. The data were simulated by distributing a total of 40 reinforcers between the first two alternatives.

in their experiments, this result was reported by Alsop and Elliffe (1988) and by Elliffe and Alsop (1996). The present model predicts that overall reinforcer rate does not affect response ratios under two specific conditions: (a) if the extraneous reinforcer rate has a value of zero or (b) if there is perfect discrimination between the pair of response–reinforcer contingencies under consideration and any substantive extraneous reinforcer rate. The same conclusions follow for explicitly defined and reinforced third-alternative responses.

We have assumed in this discussion that if the two alternatives being examined are Alternatives 1 and 2, the response–reinforcer discriminability between Alternatives 1 and 3 and between Alternatives 2 and 3 are equal. However, the values of these discriminabilities may not be equal if B_1 and B_2 are topographically different.

As Davison and Jenkins (1985) pointed out, and as we illustrate in Figure 24, Equation 18 implies that the relation between the log response ratio and the log reinforcer ratio is nonlinear whenever d_{br} is less than infinity. The nonlinearity should become more apparent as reinforcer ratios become more extreme. Using a switching-key procedure with only moderately discriminable keylight luminances to signal the alternatives, Davison and Jones (1995) arranged reinforcer ratios that were as extreme as 160:1. They reported statistically significant nonlinearity between log response and reinforcer ratios in the direction predicted by our model.

A second implication of Equation 18 is that, if d_{br12} is not infinite, responding will be maintained in the extinction component of concurrent VI extinction schedules, as has been reported by several researchers (e.g., Catania & Cutts, 1963; Davison & Hunter, 1976). Moreover, when R_2 is zero, Equation 18 simplifies to $B_1/B_2 = 1/d_{br12}$; that is, the response ratio is constant and independent of R_1. Davison and Jones (1998) reported a switching-key experiment supporting this prediction. Subsequent research in Davison's laboratory, using extended training under each experimental condition, has provided further support, and it would be interesting to explore the effects of systematically varying the difference between the stimuli that define the alternatives. The generalized matching law, of course, cannot predict anything other than the absence of responding on the extinction alternative.

Choice-Controlling Variables

It is well known that many other variables in addition to reinforcer rate, such as reinforcer magnitude, delay, and quality, determine choice allocation in concurrent schedules (see Davison & McCarthy, 1988, for review). Such variables can be incorporated into our model, but not without raising some theoretical problems. When some variable such as reinforcer magnitude is arranged differentially for two or more choices, both the values of the reinforcers and the discriminability of the response–reinforcer relations are altered. This issue has not arisen previously because we have modeled only cases in which the same reinforcer was arranged for all discriminated operants, and only reinforcer frequency varied. We now explore several ways of addressing additional reinforcer variables, with reinforcer magnitude (M) as the relevant variable, and neglecting (for this purpose) extraneous behavior and reinforcers.

Research has suggested the following generalizations that should be accommodated by the model without adding outrageous numbers of parameters:

Result 1. When both reinforcer magnitudes are varied, log response ratios are a monotonic increasing, probably linear, function of log magnitude ratio. Linearity cannot be strongly asserted because researchers have not varied reinforcer magnitudes over a wide range of values. The most extensive data come from Schneider (1973), who used four magnitude combinations and found no obvious deviations from linearity. It is possible that Result 1a, below, is more general.

Result 1a. If one magnitude is held constant while the other varies, log response ratios are a monotonic increasing but nonlinear function of log magnitude ratios (Davison & Hogsden, 1984).

Result 2. When reinforcer magnitudes are different but constant for two alternatives and the ratio of reinforcer rates is varied, log response ratios are a biased linear function of log reinforcer-rate ratios. Again, the data supporting this rather widely accepted generalization are sparse, as pointed out by Davison and McCarthy (1988).

Result 3. When reinforcer magnitudes are different but constant for two alternatives and reinforcer rates are kept equal while their absolute value is varied, response ratios are a decreasing function of absolute reinforcer rate (Davison, 1988).

Result 3a. However, when reinforcer magnitudes are the same for two alternatives, sensitivity to variations in reinforcer rate is an increasing function of absolute reinforcer rate (Alsop & Elliffe, 1988; Elliffe & Alsop, 1996; Logue & Chavarro, 1987).

Result 4. When reinforcer magnitudes are kept in a constant ratio while their absolute value increases, response ratios decrease (Logue & Chavarro, 1987).

We consider several ways of dealing with this complex pattern of results. First, consider simply multiplying each reinforcer term in Equation 18 by its corresponding magnitude:

$$\frac{B_1}{B_2} = \frac{R_1 M_1 + \dfrac{R_2 M_2}{d_{br}}}{R_2 M_2 + \dfrac{R_1 M_1}{d_{br}}}. \tag{19}$$

Unfortunately, the apparent simplicity of this approach is undermined by the fact that the value of d_{br} will depend on the difference between M_1 and M_2, as in the differential outcome effect. Any positive ordinal relation predicts an inverse S-shaped curve for Result 1 and fails to predict the constancy of sensitivity in Result 2. Equation 19 can account for Result 3a by allowing for extraneous reinforcers, as noted above, but this approach predicts the reverse of Result 3. It is possible to account for Results 1a and 4 by assuming an appropriate function for d_{br} in relation to the absolute and relative values of M_1 and M_2, but this would not be satisfying unless the function was based on independent evaluation of the discriminability of different reinforcer magnitudes.

Second, consider writing separate, multiplicative terms for reinforcer frequency and amount:

$$\frac{B_{11}}{B_{12}} = c\frac{R_{11} + \dfrac{R_{12}}{d_{br12}}}{R_{12} + \dfrac{R_{11}}{d_{br12}}} \cdot \frac{M_{11} + \dfrac{M_{12}}{d_{bm12}}}{M_{12} + \dfrac{M_{11}}{d_{bm12}}}, \tag{20}$$

where d_{bm} is the discriminability of the relation between responses and the values of the reinforcer magnitudes M_1 and M_2, and d_{br} is interpreted as the discriminability of the relation between responses and the frequency of reinforcers. In this expression, d_{bm} will be some function of the difference or the ratio of the reinforcer magnitudes. Equation 20 will behave much like the concatenated generalized matching law for reinforcer frequency and magnitude, in that a given pair of unequal magnitudes M_1 and M_2 will establish a constant bias when reinforcer frequency is varied (Result 2). However, in all other respects, Equation 20 behaves much like Equation 19.

A second problem with both of the foregoing equations is that the value of a reinforcer (i.e., its direct and generalized strengthening effects) may not be linearly related to its physical magnitude (as specified, say, by duration of food access; e.g., Epstein, 1981). This leaves us in the uncomfortable position of having to conjecture both the value of each reinforcer as a function of M and the discriminability of the response–reinforcer relations as a function of M_1 and M_2. Exactly the same problems arise when other reinforcer parameters, such as delay or quality, are varied. If our model is to be extended to choice-controlling variables other than frequency of reinforcement, a substantial research effort will be required to identify independent functions for the discriminability and value of the consequences of choice.

Finally, in all fairness, it should be pointed out that the problems we have in determining an effective model for the interaction of reinforcer rates and magnitudes are just as much problems for the generalized matching law, unless, of course, one is willing to allow sensitivities to reinforcer rates and magnitudes to vary as a dependent variable in that description.

The Question of Overmatching

It is clear by inspection of Equations 17 and 18 that strict matching is the upper limit of sensitivity to reinforcement. However, even with this clear prediction, we must expect that the variance in estimating sensitivity to reinforcement in experimental situations will provide a distribution of values that may well extend on occasion to sensitivities of greater than one, as reported by Baum (1979).

Overmatching, however, can also be produced experimentally in two, probably related, cases: First, overmatching occurs when concurrent VI VI performance is punished by contingent electric shock (de Villiers, 1980; Farley, 1980; Farley & Fantino, 1978). These researchers suggested that punishers obtained on a response alternative subtracted from the reinforcers obtained at that alternative. The value of an alternative, under their matching approach, was thus $R_i - \alpha P_i$, where P is the punishment frequency and α is a scaling parameter that equates one reinforcer to one punisher. Second, overmatching has been reported when subjects are required to work for a period of time to change over from one alternative to another (Baum, 1982; Davison, 1991a). A similar approach to that used for the quantification of punishment—a subtractive model—has proved effective here (Davison), and recently Temple, Scown, and Foster (1995) have shown that Davison's model predicts the effect of varying changeover delays in concurrent VI VI schedules. Naturally, then, in the present approach, overmatching will require a similar subtractive model.

It is relatively easy to include subtractive terms of the form $-w$ (for a constant punisher caused by a changeover delay or a travel time) to the numerator and denominator of our concurrent-schedule equation (Equation 18) as suggested by Davison and Jenkins (1985). However, a further question arises: Should this factor be subtracted from the arranged reinforcer frequencies, or from the frequencies after generalization has occurred? We know of no data to guide us in this, but the logic of Davison's (1991a) model suggests that the subtraction should be after generalization. Finally, in line with the logic of our approach and our treatment of reinforcer magnitude, an additional parameter might be necessary that captures the discriminability of the response–punishment contingency.

In summary, the concatenation of choice-affecting variables (e.g., reinforcer magnitude, reinforcer delay, and punishment) with reinforcer frequency to predict choice remains a challenge to this, and other, treatments of choice. The amount of research that has been reported in this area is considerable. However, some experimental results appear to be inconsistent with each other, and none of the experiments have been explicitly designed as tests of the present approach. Further detailed research on, for example, the interaction of reinforcer frequencies and magnitudes should be able to provide an unambiguous guide to the structure of the equation for concurrent-schedule behavior allocation.

DISCUSSION

We have presented a particular theoretical model of discriminated operant behavior that exemplifies a general perspective that we find valuable: unified treatment of the ways in which antecedent stimuli and reinforcers affect behavior, where stimulus and reinforcer effects are treated equivalently and interchangeably. This approach follows from qualitative similarities among the effects of experimental variations in stimulus–behavior relations and behavior–reinforcer relations, as specified by the experimenter's definitions of these terms and the experimentally arranged contingencies linking them.

The process of translating our general approach into a particular model was guided by the more specific and quantitative approach of signal-detection theory. As initially put forward by Tanner, Swets, Green, and their associates in the 1950s and 1960s (Green & Swets, 1966), signal-detection theory specifies two independent model parameters—d' and β—that correspond to two aspects of behavior—discrimination between successive stimulus presentations and bias toward one or the other choice response. Signal-detection research demonstrated that d' was in fact independent of bias, with discrimination depending on the physical properties of the stimuli and bias depending on such nonstimulus variables as payoffs, costs, and instructions. That research did not, however, examine whether β was independent of discrimination. The demonstration of invariance of d' across variations in these nonstimulus variables, and between procedures differing in how responses were defined and recorded (such as yes-no, forced choice, or ratings), permitted identification of d' with stimulus discriminability. This was a major contribution to psychophysics and a challenge to behavior theory. Could we do the same sort of thing across a yet wider range of procedures?

We have followed the detection-theory approach by defining two independent model parameters that reflect the discriminability of stimulus–behavior and behavior–reinforcer relations as specified for a particular discriminated operant relative to others arranged within the same experimental session. The model advanced here has the following general properties: First, stimulus–behavior and behavior–reinforcer relations are conceptualized similarly and their discriminabilities are quantified identically. Second, reinforcers are presumed to strengthen the particular experimentally defined response that produced it in the presence of the particular environmental stimulus that the experimenter has currently or recently presented. Third, the strengthening effect of the reinforcer generalizes to another discriminated operant to the extent that the stimulus–behavior relation and the behavior–reinforcer relation characterizing the second operant, taken separately, are similar to those of the reinforced operant. Fourth, similarity is expressed as the inverse of distance in a psychometric space with orthogonal axes corresponding to stimulus–behavior and behavior–reinforcer relations, within which all experimentally defined discriminated operants are located. The distance between any pair of operants is given by the city-block sum of distances on the stimulus–behavior and behavior–reinforcer axes.

Models and Model Domains

In this paper, we have applied a consistent conceptual model to a number of research areas. This model hinges on (a) the suggestion that the effects of stimuli and reinforcers on behavior are symmetrical, (b) that they act in a particular way in flat and orthogonal psychometric space, and (c) that generalization between points in psychometric space occurs according to a specific quantitative process. In many ways, only the first of these is critical to our thinking, and the model developments done here may have required a degree of specificity about the second and third points that is premature. Unfortunately, there are indefinitely many alternative psychometric spaces and quantitative generalization processes, and quantitative predictions derived from many of these are indiscriminable from those derived from many others, given the usual noise in the data. Over the course of our 20 years of informal collaboration, we have explored many of these. We are not firmly wedded to the specific model we have offered here, but naturally we have been guided towards this model from both theoretical considerations and from fits to data.

What alternatives are there, in general, to the approach taken here? The first set of alternatives is in the nature of psychometric space. It seems to us exceedingly unlikely that the log d_{sb} − log d_{br} psychometric space is flat in the way we have assumed, as suggested in Figure 4. If it is not flat, then the distances between reinforcing events and the point at which they affect another response will be inaccurately measured in our approach. There are a number of different techniques of determining such spaces, but lacking sufficient data, we have made the simplest assumption. Equally, as Davison (1991b) discussed, there are indefinitely many ways of specifying the effective distances in psychometric space. We have assumed city-block measurement (see Figure 4), in which effects are determined by the distance along the perimeter of the triangle in psychometric space. Straightforward alternatives to this distance measurement are the Euclidean (hypotenuse distance) measurement, and the supremum measurement (the effective distance is the largest of the d_{sb} or d_{br} distances). Davison suggested that the supremum could be better than the city-block measurement in predicting detection performance, but we have not followed his suggestion here because the gains were slight, and the equations were more difficult. Other measures remain to be investigated, and, of course, the appropriate measurement and the flatness or otherwise of the psychometric space interact considerably.

The specific quantitative process of generalization (we assumed a reciprocal process for d_{sb} and d_{br} in Equations 7a and 7b) could be modeled by almost any monotonically decreasing nonlinear or even linear function. Given the noise in the available data, it is not at all easy to choose among these, and our decision to use the reciprocal function is based on simplicity of equation form, together with the expectation of greater generalization decrements close to the reinforcing event than far from it.

Thus, taking all the above considerations into account, we can estimate the probability of our having lit upon the correct model as probably something rather less than 1 in 10^6. However, at a more general and conceptual level, we have, we hope, defined an *approach* to the problem of predicting how three-term contingencies in the context of other three-term contingencies will affect behavior.

What the Model Does Well

In procedures that arrange two stimuli and two responses, our model parameters exhibit independence and invariance when either stimulus–behavior or behavior–reinforcer relations are varied experimentally. In addition, our model correctly predicts the opposite effects of d_{sb} and d_{br} on sensitivity to variation in reinforcer ratios, and the effects of reinforcers for ''errors.'' The model extends naturally to procedures involving more than two stimuli or responses, and correctly predicts the effects of varying the numbers of discriminated operants and the differences among them. Moreover, the model parameters are well behaved in that they change monotonically with experimentally defined variables and roughly satisfy additivity within psychometric space. The model can account for many of the effects of delay of choice and delay of reinforcement on accuracy of discrimination (measured as log d) in delayed matching to sample and related procedures. And finally, it gives a rational account of undermatching on concurrent VI VI schedules. Thus, its effectiveness is not limited to a single paradigm.

What the Model Does Not Do Well

Despite its effectiveness across several different paradigms, the model encounters difficulties in some of them. In particular, the model predicts an inverse relation between the discriminability of stimulus–behavior relations and sensitivity to reinforcer ratios that is confirmed when stimulus–behavior discriminability is varied by changing the stimuli themselves. However, it is not always confirmed when stimulus–behavior discriminability is degraded by imposing delays between the conditional stimuli and the choice responses in delayed matching to sample. Although the available data are equivocal, our model's treatment of stimulus–choice delay may need to be reconsidered.

In addition, in conditional discrimination performance, the inverse-U relation between log D and the reinforcer ratio predicted by our model and shown in Figure 6 is not generally obtained, and the data of Whittaker (1977) do not support the predicted relations between log B and the reinforcer ratio.

A different sort of problem arises in the treatment of performance on concurrent VI VI schedules when variables other than reinforcer rate, such as reinforcer magnitude, are considered. First, there is some uncertainty about the best way to incorporate terms reflecting such variables in the model's basic equations; and second, the data now available are so complex as to defy effective modeling without ad hoc assumptions concerning application to each data set.

Finally, the application of our model to multiple VI VI schedules is complicated by the nature of the dependent variable. In the paradigms in which our model is generally successful, behavior is readily measured as a ratio of concurrently available responses. In standard multiple schedules, by contrast, the dependent variable is the rate of a single response. To accommodate multiple-schedule (and single-schedule) performance, some rational and empirically meaningful way to reexpress response rate will be required.

Things Not Modeled Here

We have simply ignored many variables that affect behavior in the paradigms we have considered. For example, in discrete-trial conditional discriminations, we ignore the interval between trials and its well-known effects. Stated more generally, we do not address the role of the context within which a given discriminated operant is defined, except for the generalized effects of other defined discriminated operants.

Another variable that we neglect for the present is whether the reinforcers for the two sorts of correct responses in a conditional discrimination differ in amount, delay, or quality. In the section on conditional discriminations, we cited the research of Peterson et al. (1980) showing that qualitatively different outcomes enhanced accuracy in symbolic matching to sample, especially when delays were imposed between stimulus offset and choice, as an example of the effect of increased discriminability of the response–reinforcer relation. One way to model this differential outcome effect is to incorporate a parameter that characterizes reinforcer discriminability into our basic equations and allow it to increase with differences in reinforcer amount, delay, or quality. However, as described in the section on choice-controlling variables and summarized above, it is not clear how best to incorporate such changes in our basic model, and we leave the issue as a challenge for the future.

Another issue that we ignore for the purposes of this model is the role of the subject's history of reinforcement in conditions prior to the current condition, or stated otherwise, the length of the effective time window within which the direct and generalized effects of reinforcers accumulate. It seems likely that the length of the time window depends on the frequency with which experimental conditions are altered, and we simply assume sufficient exposure to insure control by the conditions of reinforcement that are currently arranged, and no influence of prior conditions.

Relatedly, we do not attempt to model behavior during acquisition or transitions between experimental conditions despite the fact that our approach is readily translated into dynamic equations. This is clearly an important direction to pursue, but there are many different ways to treat dynamic processes, and we need to be sure that our model is as effective as possible for steady-state behavior before extending its approach to trial-by-trial or reinforcer-by-reinforcer changes.

Also relatedly, we do not treat the effects of reinforcement on resistance to change of discriminated operant behavior (e.g., Nevin, 1992). There are two reasons for postponing attempts to treat resistance to change within our general model. First, Nevin's analyses have been concerned almost entirely with changes in the rates of responding in multiple schedules, and response rates (as noted above) are not comfortably handled by our model without further assumptions. Second, Nevin has argued that resistance to change depends on the stimulus–reinforcer relation and not on the behavior–reinforcer relation. In our model, the discriminability of the stimulus–reinforcer relation is implied by the joint values of the stimulus–behavior and behavior–reinforcer discriminabilities, complicating direct application to Nevin's work.

Research and Applications with Humans

Finally, we consider some ways in which our model applies to human performance. First, of course, it models the general properties of human signal-detection performance; that is what inspired our approach in the first place (Nevin, 1969b). We should, however, note some specific aspects of detection performance that are not fully captured here. In the human psychophysics literature, the well-known ROC or isosensitivity curve is often reported to be linear in double-normal coordinates, as predicted by classical signal-detection theory (Green & Swets, 1966). Our model predicts an isosensitivity curve that is concave in such coordinates, with the degree of concavity being minimal at large values of d_{br} but increasing as d_{br} decreases. This is not a serious problem because d_{br} should be large when ''yes'' and ''no'' are well differentiated and are followed immediately by differential payoffs, penalties, or other explicit feedback. Thus, the isosensitivity curve should be nearly straight in double-normal coordinates, and the usual noise in the data precludes detection of slight curvilinearity. Second, many human isosensitivity curves are asymmetrical, with slopes less than 1.0, whereas our model predicts symmetry with an average slope of 1.0. Signal-detection theory accommodates asymmetry and nonunit slopes by adding a parameter that reflects the ratio of variances in the postulated distributions of sensory effects that account for detection performance (see Egan, 1975, for treatment of this and related approaches). We could do likewise by allowing d_{sb} to take different values for signal (S_1) and noise (S_2) trials, but for the present we refrain from this added degree of complexity (and parametric freedom).[6]

More generally, our model may help to un-

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[6] Many of these issues are discussed by Alsop (1998), whose approach is closely related to ours. In particular, he states that ''Ultimately, signal-detection performance is the product of a variety of discriminations involving the sample stimuli, the response alternatives, and the feedback or outcomes for these choices'' (p. 249). Our model attempts to formalize and quantify these determiners of detection performance.

derstand some sources of variation in human performance on conditional discriminations and concurrent schedules. For example, Baron and Surdy (1990) varied the magnitudes of payoffs and penalties in a continuous recognition task with young (age 18 to 26 years) and older (age 62 to 75 years) adults. The paradigm was a 2 × 2 conditional discrimination analogous to signal detection, in which the ''signal'' was prior exposure to an item. Recognition performance was generally less accurate for the older adults, although the difference decreased with extended practice. Interestingly, the older adults were also less sensitive to variations in payoff and penalty magnitudes. In terms of our model, the implication is that d_{br} was lower for the older subjects. Because measured discrimination depends on d_{br}, as shown in Figure 6, the age difference in recognition may have been overestimated. The general message here is that d_{br} must be equated across groups of subjects before differences in discrimination performances with the same stimuli can be compared.

It may be that the age difference in sensitivity to the magnitude of the payoffs and penalties reported by Baron and Surdy (1990) could be reduced by enhancing the distinctiveness of the response–reinforcer relations. Some suggestive data on this issue have been obtained by Stine-Morrow, Soederberg Miller, and Nevin (in press), who studied young and older adults in a lexical decision task. The paradigm was a 2 × 2 conditional discrimination employing spoken words and confusable nonwords as stimuli. Over several variations in the context of stimulus presentations, the older subjects were generally less accurate in distinguishing between words and nonwords when feedback and accumulated payoffs were given only at the end of a block of trials. However, when immediate feedback signaling money earned was provided for correct identifications of words and nonwords on each trial, discrimination increased markedly for the older adults and the age difference was eliminated. In terms of our model, the provision of immediate feedback may be construed as enhancing d_{br}.

Our model may also help to interpret performance deficits in patients with Korsakoff's syndrome. Oscar-Berman, Heyman, Bonner, and Ryder (1980) compared the performances of Korsakoff patients and normal subjects on concurrent VI VI schedules, and found that choice allocation was less sensitive to the reinforcer ratio for the Korsakoff patients. In fact, the Korsakoff group median estimated value of a in Equation 2 above was 0.03, suggesting a very low value of d_{br}. Korsakoff patients also exhibit deficits in various discrimination learning and delayed discrimination tasks. These stimulus-discrimination deficits may result, at least in part, from the patients' difficulty in distinguishing the response–reinforcer contingencies in these stimulus-control tasks. Interventions designed to enhance the discriminability of response–reinforcer contingencies might ameliorate some of the apparent deficits in stimulus discrimination. At the least, therapists must take the discriminability of both stimulus–behavior and behavior–reinforcer relations into account in the functional analysis and modification of behavior in applied settings. In this way, a quantitative approach to behavior therapy (Davison, 1992; McDowell, 1982) may be developed.

CONCLUSION

The basic approach behind the model we present here is based on ideas that have informed the experimental analysis of behavior for many years. For example, reinforcer rates have been treated as functionally equivalent to environmental stimuli in the generalization-decrement account of extinction: The richer the schedule during training, the greater the discriminability of nonreinforcement. We have tried to make the equivalence of the effects of stimuli and reinforcers explicit and to quantify the discriminability of their relations with behavior in concurrent discriminated operants. We have proposed some equations suggesting how their effects may combine and have applied them as broadly as we are able without ad hoc modification. The outcome, we believe, is favorable enough to encourage further efforts along these lines. We will continue to explore alternatives in the domain of integrative models suggested by our basic approach and apply them yet more broadly. It is our special hope that readers will do likewise.

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APPENDIX A

GLOSSARY

$S_{i1,2,...n}$ — a set of experimentally specified stimuli.

$B_{j1,2,...n}$ — a set of experimentally specified responses.

R_{ij} — an outcome contingent on B_j given S_i.

Discriminated operant — a fundamental behavioral unit defined as S_i:$(B_j \rightarrow R_{ij})$.

c — inherent bias in choice between B_1 and

B_2 that is independent of S_1, S_2, and R_{11}/R_{22}; estimated as the intercept of the least squares fit to data relating log b (see below) to log R_{11}/R_{22}.

a — sensitivity of the ratio B_1/B_2 to the ratio of reinforcers R_1/R_2 obtained by B_1 and B_2, estimated as the slope of the least squares fit to data relating log b (see below) to log R_{11}/R_{22}. Note that when S_1 and S_2 are indistinguishable or undefined, as in two-alternative concurrent schedules, c and a are estimated from the logarithmic form of the generalized matching law: $\log(B_1/B_2) = a \log(R_1/R_2) + \log c$.

d_{sbij} — a theoretical parameter characterizing the discriminability of the stimulus–behavior relations S_{i1}:B_{j1}, S_{i2}:B_{j2}.

d_{brij} — a theoretical parameter characterizing the discriminability of the behavior–reinforcer relations $B_{j1} \rightarrow R_{11}$, $B_{j2} \rightarrow R_{22}$.

d — stimulus discriminability in the model of Davison and Tustin (1978); calculated as the geometric mean of B_{11}/B_{12}, B_{22}/B_{21}. Frequently reported as log d, estimated as the difference between intercepts of least squares fits to data relating log B_{11}/B_{12} and log B_{21}/B_{22} to log R_{11}/R_{22} over several conditions.

b — overall bias in the model of Davison and Tustin (1978); calculated as the geometric mean of B_{11}/B_{12} and B_{21}/B_{22}. The model predicts that $\log b = a \log (R_{11}/R_{22}) + \log c$.

d_s — stimulus discriminability in the model of Davison and Jenkins (1985); conceptually and algebraically equivalent to d, and calculated as for d above.

d_r — the discriminability of response–reinforcer relations in the model of Davison and Jenkins (1985); conceptually equivalent to d_{br} above, and estimated by nonlinear optimization.

B — overall bias in the present model, calculated as for b above. However, it is not predicted by the same equation and depends on both d_{sb} and d_{br} (see text, Equation 8). The upper case is intended to distinguish its theoretical origin, and signifies the value predicted by our model rather than calculated or estimated from data.

D — measured discrimination in the present model, calculated as for d above. However, in the present model it depends on both d_{sb} and d_{br} (see text, Equation 11). The upper case is intended to distinguish its theoretical origin, and signifies the value predicted by our model rather than calculated or estimated from data.

d_t — a theoretical parameter characterizing the degradation of stimulus–behavior discriminability, d_{sb}, over time separating S_i and B_j, or equivalently, the degradation of behavior–reinforcer discriminability over time separating B_j and R_{ij}.

d_d — a theoretical parameter characterizing the discriminability of different delays intervening between S_i and B_j.

d_2 — a theoretical parameter characterizing the discriminability of different second-order stimuli signaling different first-order relations among S_i, B_j, and R_{ij}.

d_{bm} — a theoretical parameter characterizing the discriminability of reinforcers R_1 and R_2 that differ in magnitude or quality.

P_{ij} — punishers contingent on S_i:R_j.

α — a scale factor equating the weakening effects of a given punisher with the strengthening effects of a given reinforcer.

d' — a parameter in the theory of signal detection corresponding to the discriminability of a signal in a background of noise, interpreted as the difference between the means of two hypothetical distributions of sensory effects divided by their standard deviation.

β — a parameter of the theory of signal detection corresponding to response bias, interpreted as the location of a response criterion on the sensory-effect continuum.

APPENDIX B

Fitting the Model

A note is in order here about fitting the model we have presented. To estimate the various parameters, various data could be used in various different ways. For example, as data, we might fit a proportional model $[B_i/(B_i + B_j)]$ to proportional predictions, using each cell as a proportion of the other cells in the presence of each stimulus. Such a proportional fit would have the effect of differentially weighting data that occur around a response proportion of .5 in comparison with those more extreme. This would produce parameter estimates that would be more accurate for wide generalization (small discriminability) values. The opposite weighting can be achieved if simple ratios are used, in which case small degrees of generalization (large discriminabilities) would be more accurately estimated. The middle way, which may have the benefit of equalizing variances across all values of response measures, is to use log ratio fits. Tustin and Davison (1978) showed that log ratio measures were homoscedastic in concurrent VI VI performance, and perhaps this would also apply to the signal-detection situation. We recommend this procedure, at least as an interim measure.

A problem arises, however. If we use Equations 1 and 2 directly to obtain values of d_{sb} and d_{br}, optimization programs will frequently keep increasing the values of both d_{sb} and d_{br} without bound. This results from the nature of the equation, whereby perfect discrimination has an infinite value. Stable and sensible fits can, however, be achieved if the equations are algebraically converted to optimize for values of p_{sb} and p_{br} [defined as $d_{sb}/(1 + d_{sb})$ and $d_{br}/(1 + d_{br})$]. Because these parameters have a range of 0.5 to 1.0, they can easily be constrained to fall within this range, or, more usually, to be less than or equal to 1. We must, however, remember that if such fitted parameter values are found to be either consistently above 1.0, or less than 0.5, these values contain important information on the adequacy (indeed, inadequacy) of the model under investigation.

It is still necessary to decide which data should be used in log ratios. Again, we recommend pairwise ratios of responses within each stimulus (e.g., in a three-response choice, B_1/B_2, B_2/B_3, and B_3/B_1. This provides a set of data that are well distributed between positive and negative values and also provides some data signal (systematic variance) for the relative performance (and hence, the parameter estimate) between B_2 and B_3 (d_{br23}). Notice, though, that the present models do allow ratios to be taken vertically in a matrix (e.g., B_{11}/B_{21}) when the frequency of presentation of all stimuli is the same (signal-presentation probability = .5 in a 2 × 2 matrix), or if the response and reinforcer numbers are normalized between stimuli.

A common problem in fitting data from conditional discrimination situations is low response numbers in some cells. If both d_{sb} and d_{br} are close to infinity, error responses may only occasionally be emitted. As a result, parameter estimates may be unattainable (if response counts are zero), or may be poor in accuracy. This can be overcome in a number of ways. First, and this is definitely *not* recommended, data can be collected until there is at least one response in each error cell. This procedure will bias the estimate of d_{sb} and d_{br}, usually towards being too large. Second, the fits can be done as relative numbers rather than log ratios. Although this allows a fit to be carried out, parameter estimates of small confusions will be in error. Third, Hautus (1995) has published a theoretical analysis of the ways in which these problems can be overcome showing that the procedure of simply adding 0.5 to response counts in all cells will usually provide a better estimate of parameters like d_{sb} and d_{br}. We recommend that procedure.

Finally, and more technically, what is the best way of actually carrying out the fit? The best we have found is to use a spreadsheet that incorporates an optimizer. Quattro-Pro® is particularly good in this regard, because it contains built-in statistical functions that avoid many columns of calculation. Excel® is satisfactory, and SigmaPlot® is fast. Happily, they all seem to give very similar answers! The usual caveats, of course, apply: You should have seriously more data than the number of parameters you need to fit. Also, when fitting large numbers of parameters, optimization can take some time, even on a fast computer.