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VIOLATIONS OF STOCHASTIC TRANSITIVITY ON CONCURRENT CHAINS: IMPLICATIONS FOR THEORIES OF CHOICE

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The concurrent-chains procedure has been used to measure how choice depends on various aspects of reinforcement, such as its delay and its magnitude. Navarick and Fantino (1972, 1974, 1975) have found that choice in this procedure can violate the condition of stochastic transitivity that is required if a unidimensional scale for reinforcements is to be possible. It is shown in this paper that two simple unidimensional models of choice on concurrent chains can produce violations of stochastic transitivity. It is argued that such violations may result from the complex contingencies of the concurrent-chains procedure.

Key words: transitivity, concurrent chains, reinforcement delay, reinforcement magnitude, preference, delay reduction, matching

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Navarick and Fantino (1972, 1974, 1975) showed that relative allocation on concurrent chains can violate stochastic transitivity. In this paper I consider various models of choice when terminal links differ in terms of magnitude and delay. It is shown that two relatively simple models can produce such violations. Navarick and Fantino have argued that their results indicate that there is no unidimensional scale that can predict choice probabilities. Because my models are based on a unidimensional scale, I argue that although choice behavior on concurrent chains does sometimes satisfy the formal definition of a violation of stochastic transitivity, this may be a consequence of the complex contingencies of the concurrent-chains procedure. It may be possible to find a unidimensional scale that characterizes reinforcements in terms of their magnitude and delay.

DEFINITIONS

The Concurrent-Chains Procedure

In the concurrent-chains procedure an animal can make responses on one of two simultaneously available initial links. Responses on these links are reinforced, not by direct access to food, but by access to a terminal link that provides a certain amount of food after a certain delay. Once a response on one initial link provides access to a terminal link, the initial link of the other alternative becomes inoperative until the terminal links is completed. Further details and examples can be found in Fantino (1969, 1977) and Navarick and Fantino (1972, 1974, 1975, 1976).

In this paper I designate the two alternatives or sides by the subscripts 1 and 2. The initial links are independent variable-interval (VI) schedules with rates λ_1 and λ_2 (i.e., $1/\lambda_1$ and $1/\lambda_2$ are the programmed schedule intervals). Thus, a VI 60 s means that $\lambda = 1/60$ per second. The terminal links are characterized by their delay to reinforcement, D, and by their magnitude of reinforcement, M. In general, these delays and magnitudes can be random variables, but for simplicity I will consider them to have no variability. The model proposed by Killeen (e.g., Killeen, 1982) explicitly incorporates variability in the terminal links, and can be used as the basis for further explorations of violations of stochastic transitivity.

To summarize the notation, the initial link on Side 1 provides access to a terminal link with delay D_1 and reinforcement magnitude M_1, and the initial link on Side 2 leads to a terminal link with delay D_2 and reinforcement magnitude M_2. The proportion of time spent on the initial link on Side 1 is denoted by p. A chain is an initial link and a terminal link, and hence is specified by the three parameters

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λ_i, M_i and D_i. When $\lambda_1 = \lambda_2$, the common value is denoted by λ.

The proportion of time spent (or responses made) on an initial link is known as the animal's relative allocation. It is also sometimes called "preference" (e.g., Green & Snyderman, 1980; Killeen, 1968; Navarick & Fantino, 1972), which suggests that it is a measure of the animal's preference for the terminal links. Fantino (1969) showed that, for given terminal links, relative allocation depends on the initial links. This means that relative allocation is not measuring only the reinforcement value of the terminal links.

Stochastic Transitivity and Related Concepts

Let $a, b, c, \ldots$ be possible terminal links, and let $p(a, b)$ be the relative allocation to the initial link leading to a when the terminal links are a and b. Behavior satisfies strong stochastic transitivity (SST) if

$$p(a, b) \geq 0.5 \text{ and } p(b, c) \geq 0.5 \text{ imply } p(a, c) \geq \max\,[p(a, b), p(b, c)]. \quad (1)$$

Behavior satisfies weak stochastic transitivity (WST) if

$$p(a, b) \geq 0.5 \text{ and } p(b, c) \geq 0.5 \text{ imply } p(a, c) \geq 0.5. \quad (2)$$

Behavior satisfies the substitutability condition if

$$p(a, c) > p(b, c) \text{ implies } p(a, b) > 0.5 \quad (3)$$

and

$$p(a, c) = p(b, c) \text{ implies } p(a, b) = 0.5. \quad (4)$$

Tversky and Russo (1969) showed that SST is equivalent to substitutability. It follows that, if behavior violates the substitutability condition, SST cannot hold. Tversky and Russo also showed that SST is equivalent to simple scalability. Behavior satisfies simple scalability if there exist real-valued functions F and u such that, for all a, b in the set of possible terminal links,

$$p(a, b) = F[u(a), u(b)], \quad (5)$$

where F is strictly increasing in its first argument and strictly decreasing in its second argument.

Simple scalability is closely related to the idea of a unidimensional model of choice. If alternatives can vary in more than one attribute, a unidimensional theory involves combining these attributes into a single dimension that determines choice. In contrast, a multidimensional theory does not combine the attributes but keeps them separate. For example, let the alternatives differ in terms of reward magnitude and associated delay. A unidimensional theory would combine these attributes, perhaps into some sort of ratio of magnitude divided by delay. A possible multidimensional theory might say that choice is made on the basis of magnitude if the relevant difference exceeds a critical value. If it does not, then the choice is made on the basis of delay. This "lexicographic semiorder" is able to produce violations of transitivity (see Tversky, 1969, for further discussion).

The aim of this paper is to investigate the implications of violations of stochastic transitivity on concurrent chains for theories of choice. The findings of Navarick and Fantino can be summarized as violations of SST but not of WST. In response to such results, Navarick and Fantino suggest that the direction of choice may be predictable by unidimensional theories, but exact choice probabilities may require multidimensional theories. I argue that there is another possibility. Unidimensional theories may be able to predict exact choice probabilities (and also produce violations of SST).

MODELS THAT CANNOT RESULT IN VIOLATIONS OF SST

If

$$p(a, b) = \frac{u(a)}{u(a) + u(b)} \quad (6)$$

("strict utility"; Luce & Suppes, 1965), then simple scalability holds, and hence behavior cannot violate SST. This conclusion still holds if $u(a)$ depends not only on a but also on the initial link that leads to it, and $u(b)$ depends not only on b but also on the initial link that leads to it. The implication is that any unidimensional theory that assigns a single number to a terminal link or even to a chain and then uses a rule of the form shown in Equation 6 cannot produce violations of SST.

The Models of Killeen and Vaughan

In this section the models proposed by Killeen (1982) and Vaughan (1985) are used to illustrate the above definitions. The models also give us an indication of why SST might not hold. Killeen's Equation 7 is

$$B_1/(B_1 + B_2) = S_1/(S_1 + S_2), \quad (7)$$

where the left-hand side is the relative rate of responding and the right-hand side is the relative strength of the chains. If we assume that $B_1/(B_1 + B_2) = p$ and that S_i depends only on the parameters of side i, then it is clear that Equation 7 is of the same form as Equation 6; therefore, SST cannot be violated. The crucial issue is what determines S_i. Killeen (1982) says that the strength, S, of a schedule is given by

$$S = K_p(\exp(-K_q t/M) + 1/t)/(IL + T) \quad (8)$$

where K_p and K_q are constants, t is the time from the start of the terminal link until reinforcement, T is the duration of the terminal link and IL is the duration of the initial link. (Killeen's p, q, and I have been changed to K_p, K_q, and IL to avoid clashes with the notation used in this paper.) If more than one reinforcer is delivered in the terminal link, or if the end of the terminal link occurs after the end of reinforcement, then $t \neq T$ (Killeen, 1982, p. 219). For the schedules considered here, $t = T$. Adopting the notation of this paper, we have

$$S_i = K_p\,(\exp(-K_q D_i/M_i) + 1/D_i)/(IL_i + D_i). \quad (9)$$

The only way in which this strength might depend on the parameters of the other chain is through IL. Killeen (1982, p. 219) says that when there are large differences in the number of entries into terminal links, IL for a given side must be calculated by dividing the total obtained time in the initial links by the number of entries into the terminal link on that side. This means that IL_i can depend on behavior and because behavior may depend on all the parameters, IL_i may depend on the terminal link on the other side. To summarize, if IL_i depends on just the parameters on side i, then behavior cannot violate SST, but if IL_i depends on the parameters of the other side, then it is possible for SST to be violated.

In the modified version of incentive theory presented by Killeen and Fantino (1990), K_q is replaced by a term that is proportional to $1/T$, where T is the average time between reinforcements. As will be seen from Equation 30, T depends on D_1 and D_2, and hence S_1 involves D_2 and S_2 involves D_1, so that Equation 7 is not of the same formation as Equation 6.

Vaughan (1985) presents a model of choice based on the equalization of the value of the two keys in the concurrent-chains procedure. The model is an extension of the principle of melioration (Herrnstein & Vaughan, 1980; Vaughan, 1982) that results in a form of matching at equilibrium. Vaughan works with the concept of the value of a key, which is taken to be its strength as a conditioned reinforcer. In a concurrent-chains procedure, Vaughan considers value at three points on a given side. V_3 is the value of reinforcement at the moment that it is presented, V_2 is the value of entering the terminal link, and V_1 is the value of the initial link. Any given value depends on the next value in the sequence and on the rate of transitions to the next stage. Vaughan (1985) develops an equation for allocation, based on linear VIs on the initial links. (On a linear VI, the local rate of reinforcement is equal to the programmed rate divided by the proportion of time spent on the VI.) Vaughan's equation cannot predict violations of SST. Vaughan (1985, Appendix 3) gives the following equations for the value V_{1L} and V_{1R} of the left and right initial links:

$$V_{1L} = [(R_L/t_L)V_{2L}]/[(R_L/t_L) + \alpha] \quad (10)$$

$$V_{1R} = \{[R_R/(1 - t_L)]V_{2R}\}/\{[R_R/(1 - t_L)] + \alpha\} \quad (11)$$

where t_L and $1 - t_L$ are the relative times on the left and right initial link, respectively, R_L = number of transitions per second programmed on the left concurrent schedule, R_R = number of transitions per second programmed on the right concurrent schedule, V_{2L} = value of left terminal link on entering it, V_{2R} = value of right terminal link on entering it, and α is a positive constant, whose value depends on whether the terminal links are FIs or VIs (see Vaughan, 1985, p. 390). (I have used α where Vaughan uses a to avoid confusion with the use of a for a terminal link in this paper.)

The equilibrium matching condition is

$$V_{1L} = V_{1R}. \quad (12)$$

In the notation of this paper, t_L corresponds to p and R_L and R_R correspond to λ_1 and λ_2, so with the assumption that $\lambda_1 = \lambda_2 = \lambda$, Equations 10 to 12 give

$$\frac{V_{2L}}{\lambda + p\alpha} = \frac{V_{2R}}{\lambda + q\alpha}. \quad (13)$$

where $q = 1 - p$. To simplify the notation, I will write V_L and V_R for V_{2L} and V_{2R}, respectively. Now put a standard terminal link on the left side and determine a terminal link on the right side that results in a certain allocation p to the left side. From Equation 13

$$V_L = V_R(\lambda + p\alpha)/(\lambda + q\alpha). \quad (14)$$

From Vaughan's definitions of value, the magnitude and delay associated with a given terminal link enter the model only by way of the relevant V. Thus, from Equation 14 all terminal links that produce a given value of p against a standard terminal link with value V_L have the same value. It follows that when any two such terminal links are used in a test of substitutability, the relative allocation will be .5. (Appendix 1 outlines a different experimental context in which Vaughan's model could result in violations of SST.)

To investigate whether linear VIs on the initial links are crucial for this result, we can attempt to generalize the above argument. The terms R_L/t_L and $R_R/(1 - t_L)$ in Equations 10 and 11, respectively, are the rates of entry into terminal links on the left and right side, given that the initial VIs are linear. Let r_L (λ_L, **B**) and r_R (λ_R, **B**) be the corresponding general terms for independent VIs, where the vector **B** specifies behavior on the initial links. If behavior is uniquely specified by the relative allocation, p, then the argument carries through as before. If behavior is not uniquely specified by p, we can change behavior while holding p fixed. For example, assume that the animal spends a time ϕ_i on the initial link of side i. Then $p = \phi_1/(\phi_1 + \phi_2)$ and the stay times ϕ_1 and ϕ_2 can change while the relative allocation p remains constant. In such a case the above arguments breaks down. Equation 13 must now be replaced by

$$\frac{r_L(\lambda_L, \mathbf{B})V_L}{r_L(\lambda_L, \mathbf{B}) + \alpha} = \frac{r_R(\lambda_R, \mathbf{B})V_R}{r_R(\lambda_R, \mathbf{B}) + \alpha}. \quad (15)$$

Now imagine that we have a fixed terminal link on the left side with value V_L. We select a delay for the terminal link on the right side and seek a magnitude for the terminal link on the right side, such that a given allocation to the left side (say $p = .2$) results. The magnitude and the delay determine V_R. We then select another delay for the terminal link on the right side and seek another magnitude that results in an allocation of .2 to the left side. If the subject achieves this allocation by the same behavioral vector as in the first case, then V_R will be the same in both cases. It is possible, however, for the subject to produce an allocation of .2 by means of a different behavioral vector. As a result, V_R may be different in the two cases.

The model of Killeen (1982) has suggested that if duration of the initial link on one side is dependent on the parameters of the other side, then Equation 6 does not necessarily hold; thus, a violation of SST might be possible. The failure to find violations of SST in Vaughan's (1985) model might be a result of his general principle for allocation or his use of linear VIs on the initial links to derive an explicit equation. To clarify this issue, the first model in the next section keeps Vaughan's linear VIs but changes the form of the equalization principle that determines allocation.

UNIDIMENSIONAL MODELS THAT CAN VIOLATE SST

In this section I describe two unidimensional models that can produce violations of SST. The method that I adopt is based on the experimental procedure of Navarick and Fantino (1975). They did not investigate transitivity directly; instead they investigated the equivalent condition of substitutability. Substitutability requires that both Equation 3 and Equation 4 hold. Because I am interested in establishing that substitutability does not hold, it is only necessary to show that one of these equations does not hold. I concentrate on Equation 4. This equation can be given the following interpretation. Consider a concurrent-chains experiment involving terminal links a and c. The resulting relative allocation to the link leading to a is $p(a, c)$. Now, a new experiment is performed in which one terminal link (c) is unchanged, but the other is changed until some terminal link b is found such that $p(b, c) = p(a, c)$. In other words, preference for b against c is the same as preference for a against c. Equation 4 requires that if a and b are equally preferred to c, then a subject should be indifferent between them. This requirement is natural if relative allocation is a measure of the value of the terminal links; if a and b are equivalent in terms of choice against c, then they should have the same value and $p(a, b)$ should be .5. Navarick and Fantino followed the above procedure and found that pigeons were not necessarily indifferent between equivalent terminal links. In a similar vein, for each model considered below, I find not just two but seven terminal links that are equivalent in terms of choice against a given terminal link. (This terminal link has a reinforcement magnitude of 1 and a delay of 30.) Substitutability requires that $p = .5$ when any two of these seven terminal links are presented against each other (see Equation 4). I show that substitutability does not hold, and thus, by the result of Tversky and Russo (1969), SST and simple scalability do not hold.

Two Models Based on a Version of Matching

This section develops two models in which the allocation on the initial links is determined by a requirement that a form of local reinforcement rate on each chain is equal. To be more specific, the magnitude of reinforcement on a side, divided by the sum of the time in the initial link and the time in the terminal link, is taken to be the local reinforcement rate. It can thus be thought of as a form of matching (see Herrnstein, 1970; Vaughan, 1985). Vaughan (1985) derived his equations on the basis of linear VIs in the initial links. The first model is based on this case, and the second model uses standard VIs. It turns out that the first model is actually a special case of the second model.

Linear VIs. Vaughan (1985) defines a linear VI as a VI on which the local rate of reinforcement is equal to the programmed rate divided by the proportion of time spent on the schedule. Thus, on a concurrent-chains procedure in which the initial links are linear VIs with programmed rates λ_1 and λ_2, the rate of entry into terminal links on Side 1 is λ_1/p and the corresponding rate on Side 2 is λ_2/q, where p is the relative allocation to Side 1 and $q = 1 - p$ is the relative allocation to Side 2. Instead of adopting Vaughan's assumption about value, I will assume that what is equalized is a local rate based on reward magnitude divided by an overall time composed of time on an initial link plus delay imposed by the terminal link. For a period of time, Wp, spent on the initial link on Side 1, the expected number of terminal links obtained on Side 1 is $W\lambda_1$, so the expected reinforcement if $M_1 W\lambda_1$ and the expected delay is $D_1 W\lambda_1$. This gives a local rate on Side 1 of

$$\frac{M_1\lambda_1}{p + D_1\lambda_1}$$

which reduces to λ_1/p when $D_1 = 0$ and $M_1 = 1$.

An analogous expression holds for the local rate on Side 2. Equating these rates gives

$$\frac{M_1\lambda_1}{p + D_1\lambda_1} = \frac{M_2\lambda_2}{q + D_2\lambda_2}. \qquad (16)$$

Substituting $1 - p$ for q in Equation 16 we find that, when $\lambda_1 = \lambda_2 = \lambda$,

$$p = (M_1 + \lambda(M_1D_2 - M_2D_1))/(M_1 + M_2). \qquad (17)$$

Equation 16 can also be used to determine a set of terminal links that all result in a certain allocation $\hat{p}$ to the initial link on Side 1. (In developing the theoretical argument, p and $p(a, b)$ will be used for the allocation that results in an experiment with a given set of parameters, whereas $\hat{p}$ and $\hat{q} = 1 - \hat{p}$ will be used to denote a designated value of allocation that is used in determining a set of equivalent terminal links.) From Equation 16,

$$D_2 = (M_2/M_1)(\hat{p}/\lambda + D_1) - \hat{q}/\lambda. \qquad (18)$$

To determine a set of equivalent terminal links from Equation 18, it is necessary to choose a standard terminal link on Side 1 (this is the link denoted by c in Equation 4) together with a value of $\hat{p}$. Let the standard terminal link have a magnitude of 1 and a delay of 30, and let the allocation $\hat{p}$ on the initial link leading to this terminal link be .2. Then Equation 18 tells us that if the magnitude on the other terminal link is 2, then, when $\lambda = 1/60$, the delay on this side must be $2(12 + 30) - 48 = 36$. Similarly, if the magnitude on this side is 4, then the delay must be 120. A further five terminal links are listed in Table 1 in the column headed $I = 0$. Equation 18 means that D_2 increases by a constant amount for a given increase in M_2; this feature can be seen from the table. In other words, indifference curves of magnitude and delay are straight lines. All of the terminal links in the table are equivalent in terms of allocation in an experiment in which the other link has a magnitude of 1 and a delay of 30. Thus, we can test Equation 4 by letting a and b be any two terminal links from this set. If Equation 4 is to hold, putting a on one side and b on the other side must result in $p = .5$. Equation 4 can thus be checked by calling one member of the set M_1 and D_1 and another M_2 and D_2 and then using Equation 17. Alternatively, we can develop an explicit equation for the allocation when two equivalent

Table 1

Violation of SST when choice is determined by a form of matching (Equation 25). A set of equivalent pairs M_2, D_2. All members of the set result in an allocation of .2 to Side 1 when $M_1 = 1$ and $D_1 = 30$; $\lambda = 1/60$. The first D_2 column is for linear VIs on the initial links, Equation 18, which corresponds to $I = 0$ in the general model. The second D_2 column is for independent VIs with Markov switching, $I = 4$ (Equation 29).

M_2	D_2 ($I = 0$)	D_2 ($I = 4$)
2	36	41.25
4	120	131.25
6	204	221.25
8	288	311.25
10	372	401.25
12	456	491.25
16	624	671.25

Table 2

Violation of SST when choice is determined by a form of matching (Equation 25). Choice when two members of the set in Table 1 are presented as alternatives, found from Equation 22. The table shows the relative allocation on the side leading to the smaller magnitude and the shorter delay. The members of the set of equivalent terminal links shown in Table 1 are indicated here by their magnitude. For further details see text.

	4	6	8	10	12	16
2	.60	.65	.68	.70	.71	.73
4		.56	.60	.63	.65	.68
6			.54	.58	.60	.64
8				.53	.56	.60
10					.53	.57
12						.54

terminal links a and b are used. We find this allocation as follows: Let the terminal link c have magnitude M_c and delay D_c, with an analogous notation being used for terminal links a and b. Then from Equation 18 we have

$$D_a = (M_a/M_c)(\hat{p}/\lambda + D_c) - \hat{q}\lambda \quad (19)$$

and

$$D_b = (M_b/M_c)(\hat{p}/\lambda + D_c) - \hat{q}/\lambda. \quad (20)$$

When one terminal link is a and the other is b, then from Equation 17

$$p(a, b) = [M_a + \lambda\,(M_aD_b - M_bD_a)]/(M_a + M_b). \quad (21)$$

Using Equations 19 and 20 to substitute for D_a and D_b in Equation 21, it follows that

$$p(a, b) = (\hat{q}M_b + \hat{p}M_a)/(M_a + M_b). \quad (22)$$

Recall that in this equation, $\hat{p}$ is the allocation on the initial link leading to c in the determination of the terminal links a and b. It can be seen from the equation that when $\hat{p} = .5$, $p(a, b) = .5$, so in this special case Equation 4 holds. The equation also means that $p(a, b)$ decreases with a slope $(M_a - M_b)/(M_a + M_b)$ as $\hat{p}$ increases (by convention, $M_a < M_b$). The parameters of the standard terminal link c do not influence $p(a, b)$. When $M_a = 2$ and $M_b = 4$, Equation 22 indicates that $p(a, b) = .6$. The same answer can be obtained from Equation 17, if we adopt the convention that terminal link a is now on Side 1 and terminal link b is on Side 2. This case, together with further examples are given in Table 2. In every case, p is not equal to .5 when two members of the set of equivalent terminal links are chosen to form a concurrent-chains experiment. This model shows that violations of SST can occur in a model based on linear VIs. It follows that the failure to obtain violations in Vaughan's (1985) model cannot be attributed to his use of linear VIs.

Independent VIs. In this model the initial links are independent constant-probability VIs. The basic matching principle is the same as in the previous model, but it is necessary to model the relationship between time allocated to initial links and entries to terminal links. I use the Markov switching model proposed by Heyman (1979). Heyman studied pigeons on concurrent VI VI and found that stay times could be described by assuming a constant rate of switching μ_i from schedule i to the other schedule. Heyman also found that these switching rates obeyed the following constraint:

$$\mu_1 + \mu_2 = 1/I, \quad (23)$$

where I is a constant for a given animal. I has the dimensions of time. The proportion of time, p, spent on the initial link of Side 1 is given by the equation

$$p = \mu_2/(\mu_1 + \mu_2) = 1 - \mu_1 I. \quad (24)$$

Houston, Sumida, and McNamara (1987) applied Heyman's switching model to the concurrent-chains procedure. The model in this section adopts the same general approach. It is shown in Appendix 2 that the following equation for matching is obtained when $\lambda_1 = \lambda_2 = \lambda$:

$$\frac{M_1}{y_1 + D_1} = \frac{M_2}{y_2 + D_2}, \tag{25}$$

where

$$y_1 = (I + p/\lambda)/(1 + \lambda I) \tag{26}$$

and

$$y_2 = (I + q/\lambda)/(1 + \lambda I). \tag{27}$$

Equations 25 to 27 give the following equation for p:

$$p = [\lambda I(M_1 - M_2) + M_1 + \lambda(1 + \lambda I) \cdot(M_1D_2 - M_2D_1)]/(M_1 + M_2). \tag{28}$$

When $M_1 = M_2$, then for fixed D_1, D_2 and I, p tends to .5 as λ decreases (i.e., as the variable interval on the initial links increases). This means that the model is in qualitative agreement with the effect of initial-link interval reported by Fantino (1969). Equation 28 reduces to Equation 17 when $I = 0$.

As in the case of linear VIs, the matching condition can be used to obtain an equation for a set of equivalent terminal links. It follows from Equation 25 that

$$D_2 = (M_2/M_1)(y_1 + D_1) - y_2. \tag{29}$$

(Equation 29 has the same form as Equation 18, indeed it reduces to Equation 18 when $I = 0$). Following the procedure outlined above, Equation 29 is used to find a set of terminal links that are equivalent in terms of choice against a terminal link with reinforcement magnitude $M_1 = 1$ and delay $D_1 = 30$ when $\lambda = 1/60$ and $I = 4$. This set is shown in Table 1 in the column headed $I = 4$. To check these values, note that $p = .2$ and $I = 4$, and so, with $\lambda = 1/60$, Equations 26 and 27 can be used to find that $y_1 = 15$ and $y_2 = 48.75$. Equation 25 can now be checked by setting $M_1 = 1$ and $D_1 = 30$ and choosing any member of the set in Table 1 for the terminal link on Side 2. Both sides of the equation are approximately equal to .022. If substitutability is to hold, then the model must result in $p = .5$ if any two members of the set shown in Table 1 are used as terminal links. It is shown in Appendix 2 that the allocation $p(a, b)$, given two equivalent links a and b, obeys exactly the same equation as was found in the linear VI case (i.e., Equation 22). Thus, Table 2 applies for both models, and hence substitutability does not hold for this model either.

To summarize this section, a simple model of matching based on linear VIs produced violations of SST. A model using the same form of matching but independent VIs was then developed, and the simple model was found to be a special case corresponding to $I = 0$. Both models result in exactly the same allocation when the terminal links are drawn from the set of equivalent terminal links, and this allocation is not equal to .5.

A Modified Version of the Delay-Reduction Hypothesis

The delay-reduction hypothesis says that choice is proportional to the reduction in time to reinforcement associated with entering a terminal link. If terminal links are entered as they are set up, then the expected overall time T to reinforcement is given by

$$T = (1 + \lambda_1 D_1 + \lambda_2 D_2)/(\lambda_1 + \lambda_2). \tag{30}$$

The reduction in time associated with entering a terminal link is T minus the appropriate delay. After simplification, it is found that

$$T - D_1 = [1 + \lambda_2(D_2 - D_1)]/(\lambda_1 + \lambda_2),$$

and

$$T - D_2 = [1 + \lambda_1(D_1 - D_2)]/(\lambda_1 + \lambda_2).$$

The basic delay-reduction hypothesis states that

$$p = \frac{(T - D_1)}{(T - D_1) + (T - D_2)} \quad \text{for } D_1 \text{ and } D_2 < T; \tag{31}$$

therefore,

$$p = \frac{1 + \lambda_2(D_2 - D_1)}{2 + \lambda_2(D_2 - D_1) + \lambda_1(D_1 - D_2)} \quad \text{for } D_1 \text{ and } D_2 < T. \tag{32}$$

This equation is based on equal reinforcement magnitudes. Navarick and Fantino (1976) suggest a modification when these magnitudes are unequal. One magnitude, say M_2, is taken as a standard, and D_1 is transformed to

$$\tilde{D}_1 = M_2 D_1/M_1. \tag{33}$$

Navarick and Fantino (1976) point out that this transformation is based on an arbitrary decision about which magnitude to take as the standard; a quantitatively different result would emerge if the other magnitude was chosen as the standard. To put matters more strongly, $p(a, b) \neq 1 - p(b, a)$. In other words, changing the labels of the schedules changes the relative allocation, p. This is clearly not acceptable as a model of choice. (For completeness, I have found that substitutability does not hold in this case.)

An alternative is to modify the basic idea of delay reduction to one of delay per unit of magnitude reduction. The following points support the view that this is a natural extension of the delay-reduction hypothesis. First, it reduces to delay reduction when $M_1 = M_2$. Also, the condition $p = 1$ in Equation 32 gives a critical value of λ_1 such that Side 2 is ignored. This is the same condition for rejecting a less profitable prey type when prey have equal magnitude but unequal delays (for further discussion, see Fantino & Abarca, 1985; Houston, Sumida, & McNamara, 1987). The modified delay-reduction hypothesis gives the standard prey choice condition when $p = 1$, as is shown below.

To find the delay per unit of magnitude, we proceed as follows: The expected overall time to reinforcement is given by Equation 30. The corresponding expected reward M is given by

$$M = (\lambda_1 M_1 + \lambda_2 M_2)/(\lambda_1 + \lambda_2). \quad (34)$$

The basic delay-reduction hypothesis uses the expected overall time to reinforcement, T, and the delay remaining on entering a terminal link. The modified hypothesis uses the overall delay per unit of magnitude, T/M, and the delay per unit of magnitude on entering a terminal link. Thus, the reduction on Side 1 is

$$T/M - D_1/M_1, \quad (35)$$

and the reduction on Side 2 is

$$T/M - D_2/M_2. \quad (36)$$

Following the logic of delay reduction, we have

$$p = \frac{T/M - D_1/M_1}{T/M - D_1/M_1 + T/M - D_2/M_2}$$

$$= \frac{1 + \lambda_2(D_2 - d_1)}{2 + \lambda_2(D_2 - d_1) + \lambda_1(D_1 - d_2)} \quad (37)$$

where

$$d_1 = M_2 D_1/M_1 \quad (38)$$

and

$$d_2 = M_1 D_2/M_2. \quad (39)$$

Setting $p = 1$ in Equation 37, we obtain

$$1 + \lambda_1(D_1 - d_2) = 0,$$

so that from Equation 39 we have

$$1/\lambda_1 = D_2 M_1/M_2 - D_1. \quad (40)$$

If the magnitudes and delays are thought of as energetic content of a prey item and associated handling time, then Equation 40 is the equation for a critical value of λ_1 above which the rate of energetic gain is maximized by taking only the type of item with energetic content M_1 and handling time D_1. Thus, this modified version of the delay-reduction hypothesis generalizes the relation between delay reduction and optimal prey choice.

When $\lambda_1 = \lambda_2 = \lambda$ we can use Equation 37 to find a set of terminal links that results in a certain allocation $\hat{p}$ to the initial link on Side 1. It follows from Equation 37 that

$$D_2 = \frac{\hat{q}(M_2/M_1)D_1 + \hat{p}D_1 - (1 - 2\hat{p})/\lambda}{\hat{q} + \hat{p}M_1/M_2}, \quad (41)$$

where $\hat{q} = 1 - \hat{p}$.

An example of a resulting set of equivalent terminal links is given in Table 3. These values can be checked from Equations 37 through 39. Thus when both initial links are VI 60 s and one terminal link is $M = 1$, $D = 30$ and the other terminal link is $M = 2$, $D = 20.0$, the allocation to the side leading to the $M = 1$, $D = 30$ terminal link is .2. The same allocation results to this side when the terminal link on the other side is changed to $M = 4$, $D = 77.73$ or to $M = 6$, $D = 136.89$ and so forth. If substitutability is to hold, Equation 4 requires that if one terminal link is one member of this set of equivalent links and the other terminal link is another member of the set, then p must be .5. Table 4 gives p (from Equation 37) when members of this set are given as terminal links. The convention is that the table gives the relative allocation on the initial link leading to the smaller magnitude and shorter delay. The table shows that when one terminal link has a magnitude of 2 and a delay of 20.0 and the other has a magnitude of 4 and a delay of 77.73, then the allocation on the initial link leading to the former terminal link is .70, so substitutability does not hold. As the difference in reward magnitude increases (i.e., as we move along a row) the deviation from .5 increases.

Table 3

Violation of SST when choice is determined by the modified delay-reduction hypothesis, as given by Equation 37. A set of equivalent pairs M_2, D_2 from Equation 41. All members of the set result in an allocation of .2 to Side 1 when $M_1 = 1$ and $D_1 = 30$. $\lambda = 1/60$.

M_2	D_2
2	20.0
4	77.7
6	136.8
8	196.4
10	256.1
12	315.9
16	435.7

Table 4

Violation of SST when choice is determined by the modified delay-reduction hypothesis, as given by Equation 37. Choice when two members of the set in Table 3 are presented as alternatives, found from Equation 37. Conventions as in Table 2.

	4	6	8	10	12	16
2	.70	.80	.85	.88	.90	.93
4		.63	.72	.78	.82	.87
6			.60	.67	.72	.80
8				.58	.64	.73
10					.56	.66
12						.60

By following the procedure used to obtain Equation 22, we can derive an equation for the allocation when two equivalent terminal links a and b are used in a concurrent-chains experiment. (Details are given in Appendix 3.) The resulting equation is

$$p(a, b) = (1 + z_1)/(2 + z_1 + z_2), \quad (42)$$

where

$$z_1 = q(1 - 2p)[(M_b/M_a) - 1]/(q + pM_c/M_b) \cdot (q + pM_c/M_a) \quad (43)$$

and

$$z_2 = q(1 - 2p)[(M_a/M_b) - 1]/(q + pM_c/M_b) \cdot (q + pM_c/M_a). \quad (44)$$

Like Equation 22, Equation 42 is independent of λ.

DISCUSSION

The models presented here are not intended to be complete accounts of allocation in concurrent-chains experiments. The main purpose of the models is to show that a fundamentally unidimensional framework can produce violations of SST. To highlight this point, these models have been kept simple. In particular, they do not include a representation of the variability in the terminal-link delay of reinforcement. As a result, the models cannot account for the preference for variable as opposed to fixed delays (e.g., Herrnstein, 1964; Killeen, 1968) nor for intransitivities associated with fixed and variable delays (e.g., Navarick & Fantino, 1972). (Mazur & Coe, 1987, p. 296, sketch a model that can account for the effect of variable delays.)

The results presented here, together with those of Houston, Sumida, and McNamara (1987), show that three different models of choice on the concurrent-chains procedure can produce results that satisfy the formal requirements of a violation of SST. I use this wording because, although the models are used in a way that can be interpreted in terms of the procedure for testing SST, they actually do not involve the sort of choice between alternatives that is usually considered in this context. In saying that SST is violated, we interpret the models as making a choice between terminal links in which each link is characterized by a value. The models are not based on such a procedure. Although it is difficult to give an exact definition of a unidimensional model, all three models combine the parameters of reinforcement magnitude and delay, rather than treating them separately. Thus, we have a series of models in which consistent choices on the basis of a single scale or dimension can produce results that violate SST. It is instructive to consider the reasons that underlie these violations. Houston, Sumida, and McNamara (1987) showed that the maximization of overall rate on concurrent chains could produce what they called "apparent" violations of SST. The reason for the violation in this case is that choice is not determined by a comparison of the values of the terminal links, but by a comparison of the overall consequences of all possible allocations to the initial links. In the simple version of matching based on linear VIs (Equation 16), the equation for relative allocation (Equation 17) does not have the form of Equation 6 but involves terms that are the product of the magnitude on one terminal link and the delay on the other terminal link. This feature is also present in the more general model of matching (Equation 28). In my modification of the delay-reduction hypothesis, the relative reduction equations (Equations 35 and 36) involve a comparison of each link with an overall delay per unit of magnitude. As in the model based on matching, the equation for allocation involves terms that are the product of the magnitude on one terminal link and the delay on the other terminal link. Thus, in all of these models, allocation does not give us some sort of relative value of the terminal links. Gollub (1977) said that it was naive to regard the concurrent-chains procedure as giving such a scaling, but the continuing use of "preference" for relative allocation on the initial links may encourage the belief that relative allocation corresponds to relative value. There are empirical and theoretical objections to this belief. For example, Fantino (1969) found that relative allocation depends on the schedule used for the initial links, and the analysis of Houston, McNamara, and Sumida (1987) shows that relative allocation can change as a result of changes in the initial links, even if the subject's behavior at the molecular level does not change.

Navarick and Fantino (1975, p. 181) say, "If weak stochastic transitivity were supported, but functional equivalence were not, one might posit a single dimension to account for the direction of preferences (unidimensional model) while positing several dimensions to account for exact choice probabilities (multidimensional model)." Such a state of affairs is possible, but is it desirable? What would be the status of the unidimensional model? Should it not be discarded?

In this paper I have developed an alternative to the approach suggested by Navarick and Fantino (1975). I have shown that some essentially unidimensional models can produce violations of SST. Both models in this paper involve choice based on a unidimensional scale but contain an assessment of the concurrent-chains procedure as a whole, rather than just the terminal links. What the violations of SST show is that we cannot find functions, u and F, that satisfy Equation 5, even if $u(a)$ depends on the initial link of chain a as well as on the terminal link. It follows that the violations of SST enable us to eliminate models of the form of Equation 6. This indicates that it is not possible to treat the links in isolation, but this could be a consequence of the chains procedure. The results of Mazur (1984, 1986) show that simple unidimensional accounts of choice can be found in discrete-trials procedures. In such a procedure, Mazur and Coe (1987) found trends in the direction predicted by Navarick and Fantino but found no significant deviations from transitivity. They state, "there was little evidence for intransitivity of choice: Averaged across subjects and replications, the obtained indifference points deviated from perfect transitivity by less than 8%, and these deviations were not statistically significant" (Mazur & Coe, 1987, p. 287). Mazur (1984) suggests that violations of SST may be a specific result of the chains procedure. One feature of my models that is unsatisfactory in this context is that allocation when two equivalent terminal links are presented is independent of the programmed rate λ on the initial links. It would be intuitively appealing to have the deviation from .5 decreased as λ increased.

Davison (1987) says, "because of the complex interactions within concurrent chains, a more gentle movement is required away from the known (e.g., concurrent VI VI schedules) into the still relatively unknown concurrent-chain procedure" (p. 234). It is possible that Navarick and Fantino (1974) were too pessimistic when they wrote, "A major revision of behavioral theories of choice may be called for. Ordinal predictions for binary choices may prove a more realistic and fruitful goal than the prediction of exact choice probabilities." I believe that experiments on operant behavior are ideally suited to the development of models that predict exact choice probabilities. It should be possible to build on models based on simple procedures, incorporating the contingencies of the chains procedure, to obtain accurate general models of choice on a wide range of schedules.

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APPENDIX 1

Violations of SST in a Version of Vaughan's Model

In the model of Vaughan (1985), α depends on whether the terminal link is an FI or a VI. It might be reasonable, however, to extend this idea such that α might take on one of a range of possible values depending on the distribution that underlies the VI. Now imagine that we have a standard terminal link c on one side of the procedure, and denote its value on being entered by V_c and its α by α_c. We can find a terminal link a with parameters V_a and α_a such that the allocation to the initial link leading to c is p. Generalizaing Equation 13, we have

$$V_c/(\lambda + p\alpha_c) = V_a/(\lambda + q\alpha_a); \quad (A1.1)$$

therefore,

$$V_a = V_c(\lambda + q\alpha_a)/(\lambda + p\alpha_c). \quad (A1.2)$$

We can now find another terminal link b that also results in the same allocation when c is the other terminal link. V_b is given by the equation

$$V_b = V_c(\lambda + q\alpha_b)/(\lambda + p\alpha_c). \quad (A1.3)$$

Terminal links a and b are equivalent in terms of allocation relative to c. Substitutability therefore requires that the relative allocation is .5 when a is one terminal link and b is the other terminal link. The following example shows that substitutability does not hold.

Let $V_c = 1$ and $\alpha_c = .2$. Then if the allocation on the initial link leading to this side is to be .2, then two possible terminal links a and b on the other side are given by $V_a = .2075$, $\alpha_a = .1$ and $V_b = .3208$, $\alpha_b = .2$. When these two equivalent terminal links are used in a test of substitutability, it is found that $p(a, b) = .538$; that is, substitutability does not hold.

APPENDIX 2

The Model of Matching

Following Houston, Sumida, and McNamara (1987), it is assumed that switching between the initial links follows the Markov model presented by Heyman (1979). A visit to Side 1 starts when the subject switches from the initial link on Side 2 to that on Side 1, and ends when the subject switches back to the initial link on Side 2. In other words, a visit to Side 1 may contain entries to the terminal link on Side 1 but may not contain any time on the initial link on Side 2. Visits to Side 2 are defined in a similar way. Visits to side i have a random duration X_i, where X_i has a negative exponential distribution with mean duration $1/\mu_i$. μ_i can be thought of as the rate of switching away from side i. The mean durations are subject to the constraint

$$\mu_1 + \mu_2 = 1/I \qquad \text{(A2.1)}$$

where I is a constant.

The proportion of time, p, spent on the initial link of Side 1 is given by the equation

$$p = 1/\mu_1/(1/\mu_1 + 1/\mu_2) = \mu_2/(\mu_1 + \mu_2). \qquad \text{(A2.2)}$$

From Equations A2.1 and A2.2 it follows that

$$p = 1 - \mu_1 I. \qquad \text{(A2.3)}$$

During a visit to side i, the expected number of entires to a terminal link is λ_i/μ_i. The probability c_i of obtaining a terminal link at the start of a visit depends on the time that has been spent away from the side. At the start of a visit to Side 1, the time that has been spent away is X_2; therefore,

$$c_1 = 1 - \mathrm{E}[\exp - \lambda_1 X_2] \qquad \text{(A2.4)}$$

where E[ ] denotes the expected value of [ ]. Given that X_2 has an exponential distribution with parameter μ_2, it follows from Equation A2.4 that

$$c_1 = \lambda_1/(\lambda_1 + \mu_2). \qquad \text{(A2.5)}$$

It is easy to modify this equation to include a changeover delay (see Houston, Sumida, & McNamara, 1987, for details). By the same argument,

$$c_2 = \lambda_2/(\lambda_2 + \mu_1). \qquad \text{(A2.6)}$$

Let $\mathrm{E}(N_i)$ be the expected number of terminal links obtained on a visit to side i. Then

$$\mathrm{E}(N_i) = c_i + \lambda_i/\mu_i, \qquad \text{(A2.7)}$$

and the local rate on side i is $\mathrm{E}(N_i)M_i/(1/\mu_i + \mathrm{E}(N_i)D_i)$.

Thus, the matching equation that determines the proportional allocation is

$$\mathrm{E}(N_1)M_1/[1/\mu_1 + \mathrm{E}(N_1)D_1] = \mathrm{E}(N_2)M_2/[1/\mu_2 + \mathrm{E}(N_2)D_2]. \qquad \text{(A2.8)}$$

From these equations it follows that

$$\frac{M_1[\lambda_1/(\lambda_1 + \mu_2) + \lambda_1/\mu_1]}{1/\mu_1 + D_1[\lambda_1/(\lambda_1 + \mu_2) + \lambda_1/\mu_1]} = \frac{M_2[\lambda_2/(\lambda_2 + \mu_1) + \lambda_2/\mu_2]}{1/\mu_2 + D_2[\lambda_2/(\lambda_2 + \mu_1) + \lambda_2/\mu_2]}. \qquad \text{(A2.9)}$$

When $\lambda_1 = \lambda_2 = \lambda$ this equation can be rearranged to

$$\frac{M_1}{(\lambda + \mu_2)/(\mu_1 + \mu_2 + \lambda) + D_1\lambda} = \frac{M_2}{(\lambda + \mu_1)/(\mu_1 + \mu_2 + \lambda) + D_2\lambda}. \qquad \text{(A2.10)}$$

But $\mu_1 + \mu_2 = 1/I$ (Equation A2.1), and from Equation A2.3 $\mu_1 = q/I$, and thus $\mu_2 = p/I$. Therefore,

$$\frac{M_1}{y_1 + D_1} = \frac{M_2}{y_2 + D_2}, \qquad \text{(A2.11)}$$

where

$$y_1 = (I + p/\lambda)/(1 + \lambda I) \qquad \text{(A2.12)}$$

and

$$y_2 = (I + q/\lambda)/(1 + \lambda I). \qquad \text{(A2.13)}$$

From Equations A2.11 to A2.13 it follows that

$$M_1[(I + q/\lambda)/(1 + \lambda I) + D_2] = M_2[(I + p/\lambda)/(1 + \lambda I) + D_1]. \qquad \text{(A2.14)}$$

Substituting $1 - p$ for q and rearranging, we find that

$$p = [\lambda I(M_1 - M_2) + M_1 + \lambda(1 + \lambda I) \cdot (M_1D_2 - M_2D_1)]/(M_1 + M_2). \quad \text{(A2.15)}$$

Now consider a standard terminal link c with parameters M_c and D_c and use Equation A2.11 to find two terminal links a and b that are equivalent in terms of allocation when c is the other terminal link. It follows that

$$D_a = (M_a/M_c)(y_1 + D_c) - y_2 \quad \text{(A2.16)}$$

and

$$D_b = (M_b/M_c)(y_1 + D_c) - y_2. \quad \text{(A2.17)}$$

Applying Equation A2.15 to allocation when the terminal links are a and b, we have

$$p(a, b) = [\lambda I(M_a - M_b) + M_a + \lambda(1 + \lambda I) \cdot (M_aD_b - M_bD_a)]/(M_a + M_b). \quad \text{(A2.18)}$$

Using Equations A2.16 and A2.17 to eliminate D_a and D_b, it follows that

$$p(a, b) = (\hat{q}M_b + \hat{p}M_a)/(M_a + M_b). \quad \text{(A2.19)}$$

Equation A2.19 is the same as Equation 22, which gives $p(a, b)$ for the model based on linear VIs. As is mentioned in the section on these models, the linear VI model is actually a special case of the model described here, corresponding to $I = 0$. This can be seen from the fact that the rate of entry into the terminal link on side i is $E(N_i)/(1/\mu_i)$, so that, in the case of Side 1, this rate is

$$\begin{aligned}&(\mu_1\lambda_1)/(\lambda_1 + \mu_2) + \lambda_1\\ &= (q\lambda_1/I)/(\lambda_1 + p/I) + \lambda_1\\ &= (q\lambda_1/(\lambda_1 I + p) + \lambda_1.\end{aligned}$$

It follows that when $I = 0$, the rate is $q\lambda_1/p + \lambda_1 = \lambda_1/p$, which is the equation for a linear VI.

APPENDIX 3

Allocation Between Two Equivalent Terminal Links Under the Modified Delay-Reduction Hypothesis

Let the standard link c have parameters M_c and D_c and let the required allocation on the initial link leading to c be $\hat{p}$. Then, from Equation 41,

$$D_a = [(M_a/M_c)D_c\hat{q} + \hat{p}D_c - (1 - 2\hat{p})/\lambda]/(\hat{q} + \hat{p}M_c/M_a) \quad \text{(A3.1)}$$

and

$$D_b = [(M_b/M_c)D_c\hat{q} + \hat{p}D_c - (1 - 2\hat{p})/\lambda]/(\hat{q} + \hat{p}M_c/M_b). \quad \text{(A3.2)}$$

When one terminal link is a and the other is b and $\lambda_1 = \lambda_2 = \lambda$, then, from Equation 37, the allocation on the initial link leading to a, $p(a, b)$ is given by the following equation:

$$p(a, b) = [1 + \lambda(D_b - d_a)]/[2 + \lambda(D_b - d_a) + \lambda(D_a - d_b)] \quad \text{(A3.3)}$$

where

$$d_a = M_bD_a/M_a \quad \text{(A3.4)}$$

and

$$d_b = M_aD_b/M_b. \quad \text{(A3.5)}$$

Using Equation A3.1 to eliminate D_a from Equation A3.4 and Equation A3.2 to eliminate D_b from Equation A3.5, it is found that

$$p(a, b) = (1 + z_1)/(2 + z_1 + z_2), \quad \text{(A3.6)}$$

where

$$z_1 = \hat{q}(1 - 2\hat{p})[(M_b/M_a) - 1]/(\hat{q} + \hat{p}M_c/M_b)(\hat{q} + \hat{p}M_c/M_a) \quad \text{(A3.7)}$$

and

$$z_2 = \hat{q}(1 - 2\hat{p})[(M_a/M_b) - 1]/(\hat{q} + \hat{p}M_c/M_b)(\hat{q} + \hat{p}M_c/M_a). \quad \text{(A3.8)}$$