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ORDER AND CHAOS IN FIXED-INTERVAL SCHEDULES OF REINFORCEMENT

MARK S. HOYERT

INDIANA UNIVERSITY NORTHWEST

Fixed-interval schedule performance is characterized by high levels of variability. Responding is absent at the onset of the interval and gradually increases in frequency until reinforcer delivery. Measures of behavior also vary drastically and unpredictably between successive intervals. Recent advances in the study of nonlinear dynamics have allowed researchers to study irregular and unpredictable behavior in a number of fields. This paper reviews several concepts and techniques from nonlinear dynamics and examines their utility in predicting the behavior of pigeons responding to a fixed-interval schedule of reinforcement. The analysis provided fairly accurate a priori accounts of response rates, accounting for 92.8% of the variance when predicting response rate 1 second in the future and 64% of the variance when predicting response rates for each second over the entire next interreinforcer interval. The nonlinear dynamics account suggests that even the "noisiest" behavior might be the product of purely deterministic mechanisms.

Key words: nonlinear dynamics, fixed-interval schedule, response variability, predictability, behavioral dynamics, key peck, pigeons

Nonlinear dynamical systems theory (commonly known as chaos theory) has recently gained much publicity and has been used to further understanding of phenomena in many different fields (Gleick, 1987). Nonlinear dynamics has distinguished itself by demonstrating that turbulent, chaotic behavior can be produced by very simple, completely deterministic mechanisms. Nonlinear dynamics also offers procedures that can sometimes identify the underlying order in very "noisy" phenomena. This paper introduces several concepts and techniques from nonlinear dynamics and examines their utility in understanding a classic problem in behavior analysis, that of characterizing performance on a fixed-interval (FI) schedule of reinforcement.

THE BASIC PHENOMENON

The FI schedule of reinforcement arranges a reinforcer for the first response following the passage of a specified period of time. Although only one response is required in each interval, the schedule typically generates many responses that occur in a highly stereotyped pattern. Skinner (1938) suggested that the pattern involves the organization of the following four types of variability.

1. Between-session variability: Response rates and other measures of performance, computed over the entire session, oscillate between sessions.
2. Between-interval variability: Response rates and other measures of behavior, computed over each interval, fluctuate from interval to interval.
3. Within-interval variability: Responding is typically absent at the onset of the interval and gradually increases in frequency until reinforcer delivery.
4. Response clustering: Individual responses tend to occur in groups of two or three responses.

The first and fourth dynamic effects have traditionally been viewed as characteristic of all schedules (Zeiler, 1977). The second- and third-order deviations, however, have been viewed as truly characteristic of FI schedules. Within-interval variability is the more commonly known effect. The conventional description holds that there is a pause at the beginning of the interval that is followed by an acceleration in responding to a high terminal rate. The terminal value is sometimes described as constant (e.g., Dews, 1970, 1978). This general pattern of responding is maintained across consecutive intervals. However, the duration of the postreinforcement pause, the speed of response-rate acceleration, the terminal response rate, and the total number of responses

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emitted during the interval can all change drastically between intervals. The most common description of this between-interval variability is derived from Skinner (1938) and Ferster and Skinner (1957). These researchers noticed that there was a general tendency for intervals with many responses to be followed by intervals with few responses and for intervals with few responses to be followed by intervals with many responses. The same type of cyclic pattern is also said to occur with duration of the postreinforcement pause (Shull, 1971). However, interval-to-interval dynamics are not that simple. Both response number and postreinforcement-pause duration increase and decrease irregularly and assume many high, low, and intermediate values (Dews, 1970; Gentry & Marr, 1982; Randolph & Sewell, 1968; Shull, 1971; Wearden, 1979; Wearden & Lowe, 1983). No consistent relationship between measures of responding in successive intervals has yet been found.

THE TRADITIONAL APPROACH TO DESCRIBING VARIABILITY

Behavior analysts have tended to regard irregular or highly variable behavior as a problem of experimental control (Sidman, 1960). It has been assumed that a finite number of variables produces all observed variability. It has also been assumed that the important independent variables produce dynamics that are inherently stable (i.e., in the absence of disturbance, the dependent measures tend toward a stable equilibrium). The behavior analyst must identify those independent variables and control their effects through experimental manipulation. Typically, variables are studied in isolation, and their effects are explored over a range of values. If the dependent measures continue to fluctuate, this can only mean that external forces from some unknown source of control or some poorly controlled extraneous variable is still operating. In this event, new candidates are selected and experimentally controlled. The procedure is repeated until variability is minimized. The ultimate goal of this approach involves constructing a list of the important variables affecting performance and then describing how the variables act singly and in concert.

Many potential controlling variables have been proposed. For instance, the duration of the interreinforcer interval (Killeen, 1975; Neuringer & Schneider, 1968), temporal discrimination (Dews, 1966, 1978; Ferster & Skinner, 1957; Staddon, 1977), and the number of responses emitted previously within the current interval (Ferster & Skinner, 1957; Rosenberg, 1986; Shull & Brownstein, 1970) have been suggested as variables that control within-interval dynamics. The number of responses emitted during the previous interval (Herrnstein & Morse, 1958; Zeiler, 1977) and the amount of time spent responding during the previous interval (Shull, 1971) have been offered as variables controlling between-interval variability. This list of variables is not exhaustive (more comprehensive reviews can be found in Dews, 1970; Zeiler, 1977, 1979). However, neither the independent effects of the variables included in this list nor the variables included in the full list can account for a large percentage of the variance (Gentry & Marr, 1982). It can be inferred that this approach has not yet led to a complete list of the independent variables. That is, not all of the variability has been accounted for. In fact, there is not even agreement that all of the suggested variables are relevant. The search for the list of variables controlling FI performance seems to have lost momentum. The most common conclusion about the factors controlling FI performance is that this is a complex phenomenon.

THE NONLINEAR SYSTEMS APPROACH

The problems experienced in behavior analysis have also been experienced in other disciplines. During the past 20 years, an alternative research approach known as nonlinear dynamical systems theory has been emerging. Like the analysis and synthesis approach described above, this strategy is deterministic. It views behavior as being caused by the combined influence of a set of controlling variables. But these variables do not work the same way as standard independent variables. Rather, each controlling variable in a nonlinear world affects all other variables. If the value of one variable is changed, the effects of all other variables subsequently change. Thus, the effects of each variable depend upon the simultaneous states of all other variables. The controlling variables in a nonlinear world are better known as state variables. Nonlinear dynamics suggests that, in principle, a complete description of any given phenomenon can be attained through a complete understanding of how each state variable acts alone and in concert. However, nonlinear dynamics suggests that this is impossible for all but the simplest phenomena. For the analysis and synthesis approach to succeed, all independent variables must be identified and measured, and their interactions understood. In most systems, as is obviously true with human and animal behavior, it has proven difficult even to identify all of the relevant variables, much less measure or understand their effects singly and in concert.

Further, nonlinear dynamics implies that studying each factor in isolation may not lead to useful knowledge. The effects of variations of a single variable may be different in the context of a system than when it is altered in isolation. The same variable whose alternation produced a smooth linear function when studied in isolation might produce wild, almost discontinuous behavior when embedded with other variables. Thus, experimenters are left with an infinitely complicated system that requires them simultaneously to understand all potential factors or else face the possibility that all knowledge of individual variables may be suspect. Nonlinear dynamics gives the researcher a tool for exploring how multiple variables might interact without identifying and measuring each of the relevant variables (Packard, Crutchfield, Farmer, & Shaw, 1980; Takens, 1981). Thus, it provides the possibility for understanding the system as a whole even if the component parts have not been identified. If the functioning of the system can be understood, further analysis can reveal how certain variables may be altered to move the whole system.

A HYPOTHETICAL SYSTEM

In order to demonstrate how perfectly understood state variables might have chaotic effects when embedded in a system of interacting independent variables, a hypothetical system will be explored. For this example, it will be assumed that the variables influencing FI performance are known and understood. If this were true, a system of equations that describe behavior could be developed. The following set of equations is representative of what such a system might look like. It is not intended to model accurately the actual mechanisms that produce observable behavior; rather, it is intended to serve as an example of how such a system could function and what kinds of behavior it might produce.

In this model, within- and between-interval dynamics are controlled by separate but related sources of control. Palya and Bevins (1990) suggested that within-interval variability is the joint product of competing tendencies to approach food and to engage in other activities. They suggested that the period of responding is related to the reinforcing consequences of food delivery. The probability of approaching and emitting food-related behavior gradually increases over the interval. In the current model, it is assumed that changes in response probability are proportional to the product of its current strength and the difference between it and the maximal probability of responding. Thus,

$$\Delta S_r = aS_r(S_{r\max} - S_r), \tag{1}$$

where ΔS_r is the change in response strength, a is the constant of proportionality, S_r is the current strength of responding, and $S_{r\max}$ is the maximum response strength for that interval. Integrating this equation yields

$$S_r = \frac{S_{r\max}}{1 - be^{(-cS_{r\max}t)}}, \tag{2}$$

where b is a constant related to the strength of responding at the onset of the interval, c is a constant related to the rate of growth, and t is the elapsed time. Equation 2 describes the current strength of responding at any time t. It produces an S-shaped function. That is, response strength is bounded; it has a minimum value close to zero and a maximum value close to the value of $S_{r\max}$. During the typical interval, response strength is near zero at the onset of the interval, and increases over the course of the interval. At first, response strength increases slowly, but as it increases, the rate of change also rises. Then, as response strength approaches its maximum, the rate of growth again slows.

The period of pausing is related to the reinforcing consequences of alternative behavior. The organism has a higher tendency to emit alternative behavior during the portion of the interval in which the target behavior is unlikely to produce food. The strength of the tendency to emit alternative behavior gradually declines over the interval. The rate of decline is proportional to the product of its current strength and the difference between it and the maximal strength of alternative responding. Thus,

$$\Delta S_a = dS_a(S_{amax} - S_a), \tag{3}$$

where ΔS_a is the change in response strength, d is the constant of proportionality, S_a is the current strength of responding, and S_{amax} is the maximum alternative-response strength for that interval. Integrating this equation yields

$$S_a = \frac{S_{amax}}{1 - fe^{(gS_{amax}t)}}, \tag{4}$$

where f is a constant related to the strength of alternative behavior at the onset of each interval and g is a constant related to the rate of decrease. Equation 4 describes the current strength of alternative responding at any time t. This equation also produces an S-shaped function (although inverted). However, in this equation response strength decreases. Early in the interval, response strength is close to its maximum. At first, it decreases slowly. As response strength decreases, the rate of change initially increases. As response strength approaches zero, the rate of decline slows.

Behavior during previous intervals determines the maximum response strength and maximum strength of alternative responding during subsequent intervals. The results from numerous studies suggest that the sequential dependencies involve more than performance in the immediately preceding interval (e.g., Dews, 1970; Gentry & Marr, 1982; Wearden, 1979). Dews (1970) and Zeiler (1977) suggested that the sequential dependencies may be produced by the behavior during groups of previous intervals. The current model embodies this suggestion and uses three variables that are related to prior responding. Response strength is determined by (a) the cumulative number of responses made within the session, (b) the mean response rate over the last three intervals, and (c) the number of reinforcers received during the session. In order to model the cyclic nature of between-interval variability, performance is assumed to fluctuate around a sine wave. The three variables act together to move responding over the sine function. These three variables combine to form

$$\begin{aligned} S_r &= 1 - S_a \\ &= j\{\sin(k\pi(MI + SI + T(R))\} + l, \end{aligned} \tag{5}$$

where $T(R) = f(R + m + n)$ and

$$T_{n,m}(R) = \begin{cases} 1, \ 0 + o(m+n) < R \\ \quad < m + o(m+n), \ o = 0, 1, 2, \ldots \\ 0, \ m + o(m+n) < R \\ \quad < (o+1)(m+n), \ o = 0, 1, 2, \ldots, \end{cases} \tag{6}$$

and where j, k, l, m, and n are constants, MI is the mean response strength over the last three intervals, SI is the sum of all previous response strengths in each interval, $T(R)$ is a square wave function that relates performance to the number of reinforcers delivered during the session, and R is the cumulative number of reinforcers delivered. The variable o is an integer tracking cycles of four food presentations.

Equation 5 can show nonlinear effects. Changing the values of any of the constants can alter the functioning of the system. If the three variables were separated and each had exclusive control, all three would produce very stable behavior. A sine function with either mean response rate or the cumulative number of responses as the only variable would produce stable behavior. That is, every interval would contain a postreinforcement pause with the same duration, the same number of responses, and the same terminal response rate. If the cumulative number of reinforcers were the only variable, the equations would produce an alternation among five different rates. Changing constants j, k, and l would alter the specific points, but the stability would be maintained. Combining the three variables (even with a simple additive relationship) alters the functioning of all three. In combination, they produce very wild behavior. The current state of any one affects the future output of the others.

Observed behavior within each interval is a joint function of the strength of food-related behavior and the strength of alternative responding:

$$B = (S_r - S_a)p, \tag{7}$$

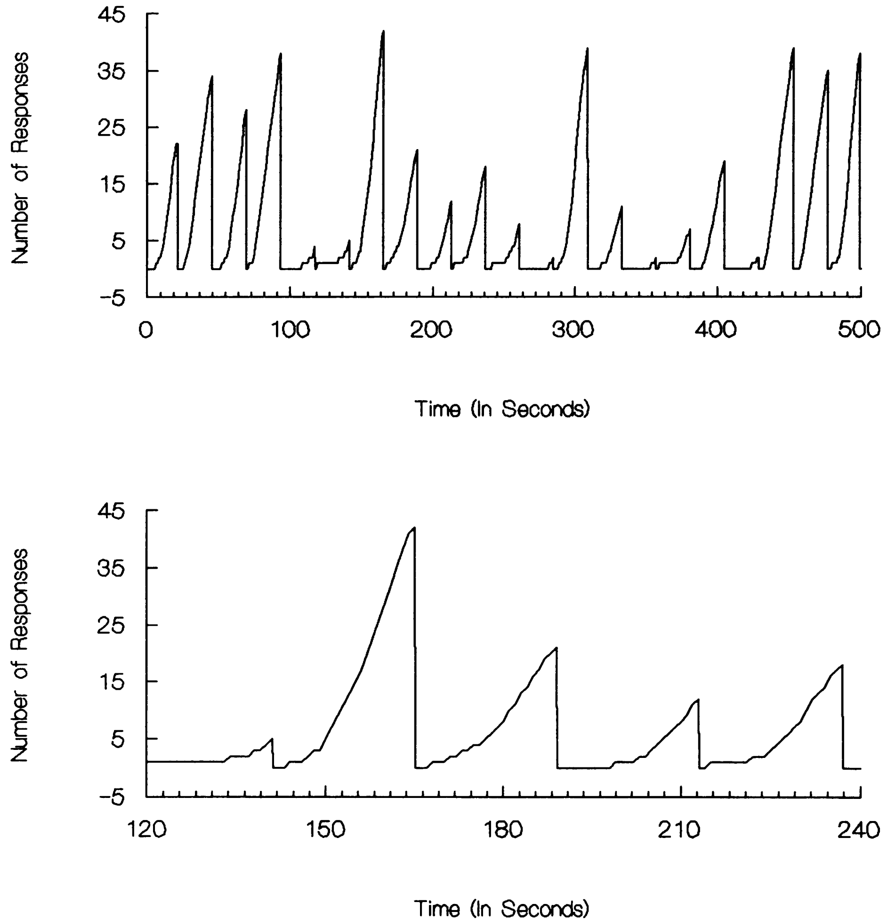


Fig. 1. Sample cumulative record of responding to an FI 20-s schedule generated by the hypothetical system. The upper panel displays 21 complete interreinforcer intervals and reveals clear between-interval dynamics. The lower panel shows five of the intervals in greater detail.

where B is a measurable response rate and p is a constant that transforms response strength into a measurable quantity. Equation 7 combines the opposing tendencies to emit the target response and to engage in other behavior. At the onset of the typical interval, S_a is relatively high and S_r is relatively low. Therefore, B is below zero. Over the course of the interval, S_a decreases and S_r increases. B increases gradually. The part of the interval with positive values represents observable food-related behavior.

Equation 7 produces performance that is similar to that emitted by pigeons receiving reinforcement periodically. Response rates for each second were computed for 300 consecutive FI 20-s intervals. The values of all constants used to produce these response rates are included in Appendix 1. Each interval is composed of a period of pausing and a period of responding. Response rates gradually accelerate over the course of the interval. A sample cumulative record (Figure 1) reveals the characteristic scallop shape and large interval-to-interval dynamics.

Between- and within-interval variability can also be assessed with several quantitative procedures. Traditionally, within-interval dynamics has been measured by computing an index of curvature (Fry, Kelleher, & Cook, 1960). This measure generates a single value that designates the degree to which responding is evenly distributed throughout the interval. The measure is positive when accelerated responding is present. It is zero when response rate remains constant over the duration of the interval. It is negative when response rate declines. The index of curvature computed for the output of Equation 5 was 0.374, which indicates accelerated responding and is similar to those found in previous reports (e.g., Palya & Bevins, 1990; Gentry, Weiss, & Laties, 1983).

Between-interval dynamics has been measured with two statistics: the runs test (Siegel, 1956) and Lag 1 autocorrelations (Weiss, Laties, Siegel, & Goldstein, 1966). The runs test can measure whether behavior changes relative to performance in the immediately preceding interval. Thus, intervals are scored as to whether the measures of performance were higher or lower than in the preceding interval. The statistic can detect two types of deviations from chance: (a) A positive score indicates more alternation between higher and lower scores that would be expected from random processes, and (b) a negative score indicates less alternation than expected by chance. A score near zero indicates about the same amount of alternation as expected from chance. A runs test was computed for both postreinforcement pause ($Z = 3.4$, $p < .05$) and response number ($Z = 3.3$, $p < .05$). Both values were statistically significant and were similar to those found in previous research (e.g., Dews, 1970; Shull, 1971; Wearden, 1979).

The second procedure, Lag 1 autocorrelations (Weiss et al., 1966), provides a measure of absolute effects. This statistic computes a correlation between response rate (or any other measure of behavior) in one interval and that same measure during the next interval. The statistic measures both strength and direction of periodicity. The Lag 1 autocorrelation computed from the model were also similar to previous results (e.g., Gentry & Marr, 1982; Shull, 1971; Wearden, 1979; Wearden & Lowe, 1983). A correlation coefficient of .01 was obtained with postreinforcement pause duration. A correlation coefficient of .06 was found for response number.

It appears that this system of equations can produce responding that shows both within- and between-interval dynamics similar to those exhibited by living organisms. In this system, within-interval variability was produced by the combined effect of tendencies to approach and emit food-related responses and tendencies to engage in other behavior. The duration of the postreinforcement pause, the rate of acceleration, and the terminal response rate in each interval were controlled by previous levels of responding and reinforcement. Again, it must be pointed out that the details of the proposed model are less important than the demonstration that very variable behavior can be produced by the interaction of a small number of relatively simple deterministic variables.

This hypothetical system of equations demonstrates that chaotic performance can result from the completely deterministic effects of a small number of interacting independent variables. There are no unknown sources of control or external forces, yet highly variable behavior emerges. Chaotic behavior is not a rare phenomenon. Chaotic motion can be produced if (a) the system has at least three dynamical variables and (b) the equations of motion contain a nonlinear term that couples several of the variables. Such systems are often chaotic for some choices of constants (Baker & Gollub, 1990).

THE INDETERMINACY PROBLEM

The existence of chaotic systems governed by deterministic equations poses another problem for behavior analysis. If the variability commonly dismissed as noise and carefully averaged out or statistically removed is the product of a deterministic system, then the behavior analyst must discover the determining equations. The method used to discover these in behavior analysis has been essentially to guess an approximate equation and estimate parameters from historical data. This process is well illustrated by the example of the matching relationship. This procedure can lead to problems. Specifically, the model can only be as good as the guess.

Nonlinear dynamics implies that this procedure cannot successfully describe and predict the behavior of systems even if the guess is a very good one. In a linear system, small errors in parameter estimation lead to proportionally small errors in prediction of the final behavior. However, in a nonlinear system, the current behavior of the system determines its future behavior through a feedback mechanism. A small error in estimation of the initial parameters may be amplified with each feedback loop. Small errors in estimation can become huge errors in prediction after a short period of time. To predict phenomena in a nonlinear world, one must be able to measure all variables to infinite precision. The smallest errors in estimation of the initial conditions can yield wide divergence from actual behavior, even if the true equations are known. For example, Lorenz (1963) demonstrated that a system of three simple nonlinear equations would churn out entirely different behavior if the initial conditions differed by as little as a ten thousandth of a percent. Further, the degree and direction of divergence could not be predicted before each change in initial conditions. This is referred to as the indeterminacy problem, uniqueness of trajectories, or sensitivity to initial conditions (Lorenz, 1963).

GRAPHICAL METHODS OF SEARCHING FOR ORDER

The traditional analysis and synthesis approach requires that all independent variables be discovered, their effects determined singly, and the rules for their combined influence described. Simply identifying the relevant variables has been daunting for psychology. The nonlinear systems approach offers some hope for gaining insight into the problem without identifying all of the constituent components. Nonlinear dynamics relies heavily on four mathematical constructs: phase space, the Poincaré section, attractors, and the return map (see Marr, 1992, for additional discussion).

The phase space of a dynamical system shows the position of the system at any point in time. A simple phase space can be depicted by a graph with orthogonal coordinate directions representing each of the variables needed to specify the instantaneous state of the system. For example, in the hypothetical system of equations, the response rate is specified by the momentary probability of approaching the response panel, and the maximal strength of responding.

Figure 2 displays the phase portrait or state space of the hypothetical system. To display a phase space, a graph in which the axes are the independent variables is constructed. Each point on the graph has coordinates $X(t)$, $Y(t)$, and $Z(t)$. By connecting the points in temporal sequence, the trajectory of the system over time is graphed. As can be seen in Figure 2, this is clearly not a random jumble of points. Instead,

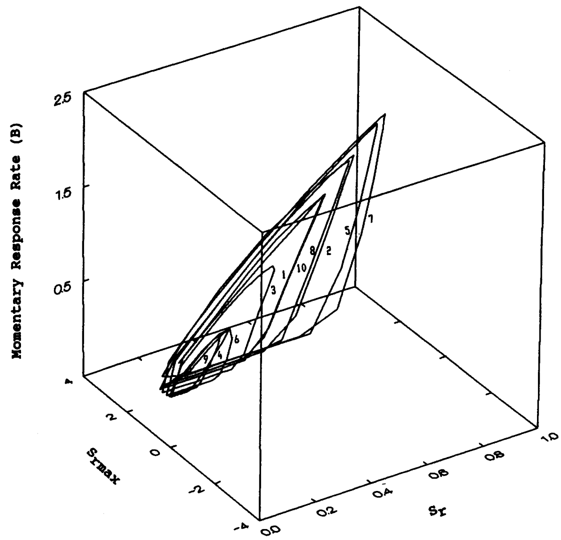


Fig. 2. Phase portrait of the hypothetical system. The axes represent the three state variables that interact to determine the position of the system: strength of responding, maximal response strength, and observable response rate. The 10 orbits represent responding in each of the first 10 intervals displayed in Figure 1. Each orbit can provide the same type of information available in a cumulative record. Within each interval, response rate is initially close to zero, increases gradually, and finally decreases rapidly.

the trajectory follows an apparently quasi-periodic and bounded orbit (i.e., it is confined to a finite region of the phase space).

In Figure 2, the axes represent the momentary response rate (B), the probability of approaching the manipulandum (S_r), and the maximal strength of responding (S_{rmax}). Ten trajectories are displayed. These correspond to the first 10 intervals shown in the upper panel in Figure 1. Each trajectory traces a similar path through phase space. Each orbit starts in the lower left hand corner of the graph. At this point, both response rate and response strength are near zero. Initially, response strength increases while response rate remains near zero. Then response rate and response strength increase concurrently. Finally, both decrease to the near-zero levels. The size of each orbit is determined by the maximal rate of responding. Smaller orbits (such as Orbit 9) have lower values of S_{rmax}. Larger orbits (such as Orbit 7) have larger values of S_{rmax}. Examining the response rate for any one orbit can provide the same kind of information found in a cumulative record. Follow Orbit 7 through phase space. Response rate is close to zero at the onset of the orbit. As time passes (distances on the trajectories represent time), response rate at first remains near zero; then it increases until it reaches a high rate. This terminal rate is maintained for a short duration, and then response rate quickly decreases (following reinforcer delivery). This corresponds to the postreinforcement pause, the period of response acceleration, the terminal rate, and decrease following reinforcement. Orbits that travel a shorter excursion from the origin (such as Trajectory 9) indicate intervals that have a lower terminal response rate. These intervals also have fewer responses. Trajectories that traveled far from the origin (such as Orbit 7) contain much more responding.

The phase portrait shows the position of the system in phase space. An attractor is the force that determines the shape, size, and other properties of the system (Farmer, Ott, & Yorke, 1983). The attractor is, in turn, the product of the interacting state variables. There are numerous types of attractors. Each has a characteristic organizing effect on the behavior of the system. These include point attractors, limit cycles, toroidal flow, strange attractors, and turbulence.

Point attractors draw the behavior of the system towards one single value. In the absence of perturbations, the system approaches a stable point. An example of a possible point attractor in behavior analysis is the dynamics of a variable-interval (VI) schedule in which response rate approaches a specific level for each schedule value. Regardless of the initial response rate, the attractor will pull the behavior to a specific stable point. Consider the example of a VI 1-min schedule controlled by an attractor that pulls response rate to a level of 50 responses per minute. If a pigeon had been responding to a high-rate schedule and was then exposed to the VI 1-min schedule, responding would be pulled lower to a response rate of 50 responses per minute. If the pigeon had been responding to a low-rate schedule and was then switched to a VI 1-min schedule, responding would increase to 50 responses per minute. In the absence of any extraneous variables, the response rate would approach the same terminal rate regardless of the starting point. In addition, behavior would be attracted to the same point continuously. The response rate would remain at approximately 50 responses per minute for each interval and each session.

In limit cycles, the attractor is slightly more complex. It is described as lying on a closed curve. Trajectories may approach many points on the curve or only a small subset of points. In a Period 2 limit cycle, the behavior of the system alternates between two points. This is very similar to the description of FI dynamics offered by Herrnstein and Morse (1958), who suggested that the number of responses emitted in an interval, or the duration of the postreinforcement pause (Shull, 1971), alternates between a high and a low value. In a Period 3 limit cycle, the system cycles between three points on the curve. This is similar to Dews' (1970) description of FI dynamics. He suggested that measures of responding cycle between a low, a medium, and a higher value. In a Period 4 limit cycle, the system alternates between four points. There can be an infinite variety of limit cycles, but the central feature is that they are periodic. The behavior of the system will come back to and repeat earlier values. As in the case with point attractors, regardless of where the behavior of the system starts, it will be drawn onto the curve.

Toroidal flow involves a still more complex motion. With a point attractor, behavior was drawn to a one-dimensional figure. With a limit cycle, behavior was drawn to a curve, a two-dimensional figure. With toroidal flow, the motion is drawn to the surface of a torus, which is essentially a three-dimensional, doughnut-shaped figure. Behavior is drawn around the surface of the torus with either a periodic or a quasi-periodic orbit. An FI schedule could be governed by this type of attractor. Behavior is pulled around the torus. Sometimes it is low on the torus, such as during the postreinforcement pause. Sometimes it is at a moderate level on the torus, while response rate is increasing. Sometimes it is high, such as during terminal responding. The doughnut is thick. On some orbits the behavior of the system reaches the outside edge of the doughnut (i.e., many responses), on some orbits it reaches only the inside ring closest to the hole (i.e., few responses), and on many orbits it reaches an intermediate level.

Strange attractors draw behavior into orbits in which motion is neither periodic nor quasi-periodic, but for which orbits are nonetheless confined to a low-dimensional surface. Motion on a strange attractor is often chaotic in the sense that it is impossible to forecast the system's long-term behavior in the presence of even the smallest amount of observational error. In the three simpler attractors, the behavior of the system approaches a specific point, a curve, or a surface. This occurs regardless of the starting point. With a strange attractor, the starting point is vitally important in determining the future behavior of the system. All future trajectories are determined by the initial conditions. If two trajectories were started at the same time, but with slightly different initial positions, they would quickly diverge. Small differences in the initial conditions produce very great ones in the final phenomenon. This is why it is difficult to predict long-term behavior: If the initial estimates of parameters of the system are just slightly inaccurate, the error will increase exponentially, so that the state of the system is essentially unknown after a very short time.

If prediction is impossible, then the chaotic system can resemble a stochastic system (a system subject to random external forces). However the source of the irregularity is quite different. With nonlinear phenomena, the irregularity is part of the intrinsic dynamics of the system and is not the result of unpredictable outside influences. Lowe and Wearden (1981) proposed a stochastic mechanism to account for the dynamics of FI schedules. They argued that FI periodicities are controlled by known independent variables that are also affected by random periodic inputs. If the phenomenon were governed by a strange attractor, the Lowe and Wearden model could accurately mimic the distribution of performance across many intervals, but would not be able to predict performance during any one interval.

The final type of attractor produces turbulence. Here the attractor is no longer low-dimensional. Motion is highly erratic. If behavior in an FI schedule were turbulent, then predicting individual responses would be impossible. One could say only that there is a distribution of positions and velocities to which the individual responses as a whole converge. Thus, the kind of analysis offered by Lowe and Wearden (1983) would be the most detailed type of analysis possible.

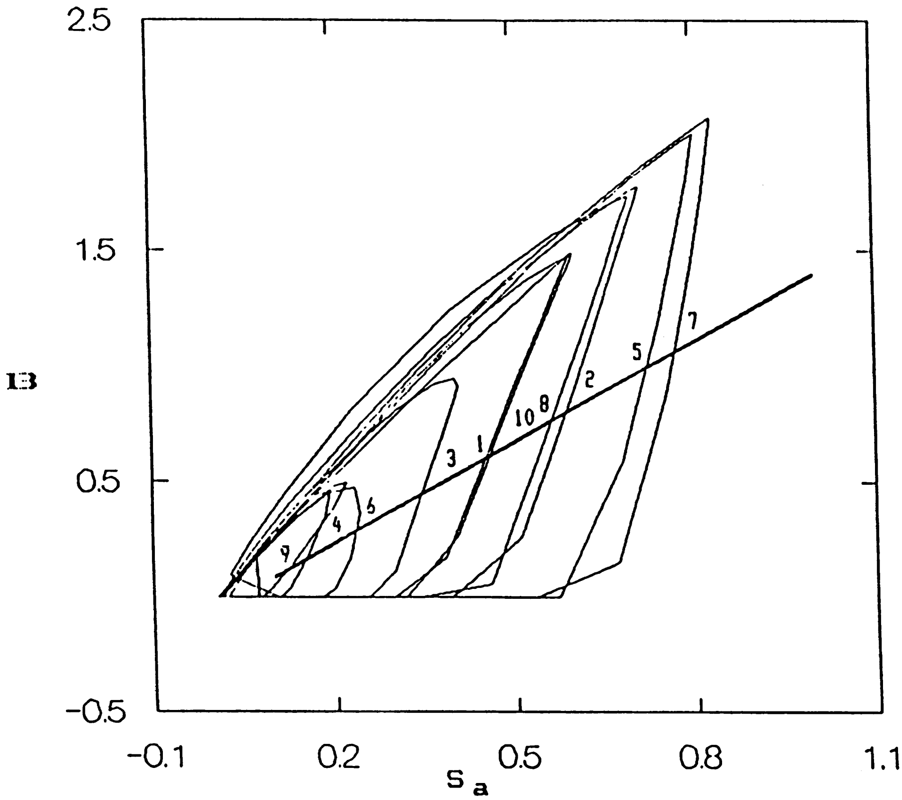
The properties of the attractor are not necessarily discernible from the phase portrait. To examine complex attractors, it is often useful to produce Poincaré sections from the phase portraits and return maps from the Poincaré sections.

A Poincaré section is a means of simplifying a complicated system. It is constructed by viewing the phase space diagram in such a way that the motion is observed periodically. Imagine placing a piece of paper through the phase portrait in Figure 2. Orbits traveling through the phase space will cross the paper at some point x,y on the two dimensional paper sheet. The Poincaré section records these points.

Figure 3 displays the phase portrait from the hypothetical system, the position of the slicing plane, and the Poincaré section. The Poincaré section is a two-dimensional graph. It contains two pieces of information: (a) the position on the x,y plane of each point of intersection and (b) the order in which the points crossed the plane. The upper panel displays a two-dimensional projection of the same three-dimensional phase portrait displayed in Figure 2. The lower panel displays the Poincaré section constructed by the intersection of positively directed trajectories with the plane. The position of the plane is depicted in the upper panel by the heavy intersecting line. The intersecting plane must be positioned in phase space so that it transects all orbits. In this particular case, there exists an infinite number of plane positions that can satisfy this condition. Any of these alternatives should provide the same information. Here, the intersecting plane was perpendicular to the x,y plane such that $y = 1.5x$. (The position of the intersecting plane was selected in this example for convenience.)

Follow the path of Orbit 1 in the upper panel of Figure 3. As responding increases and reaches its maximum rate, it will pass through the intersection plane. The position of the intersection will be recorded as dot Number 1 on the Poincaré section. The same can be seen for all orbits.

Figure 4 displays a return map created from the Poincaré section. The return map is constructed by plotting the ordinates from successive intersecting points against each other; that is, $x(n)$ versus $x(n + 1)$, where n represents the other of transection. For instance, Point (1,2) was created by graphing the ordinate from Point 1 in the Poincaré section against the ordinate from Point 2 in the Poin-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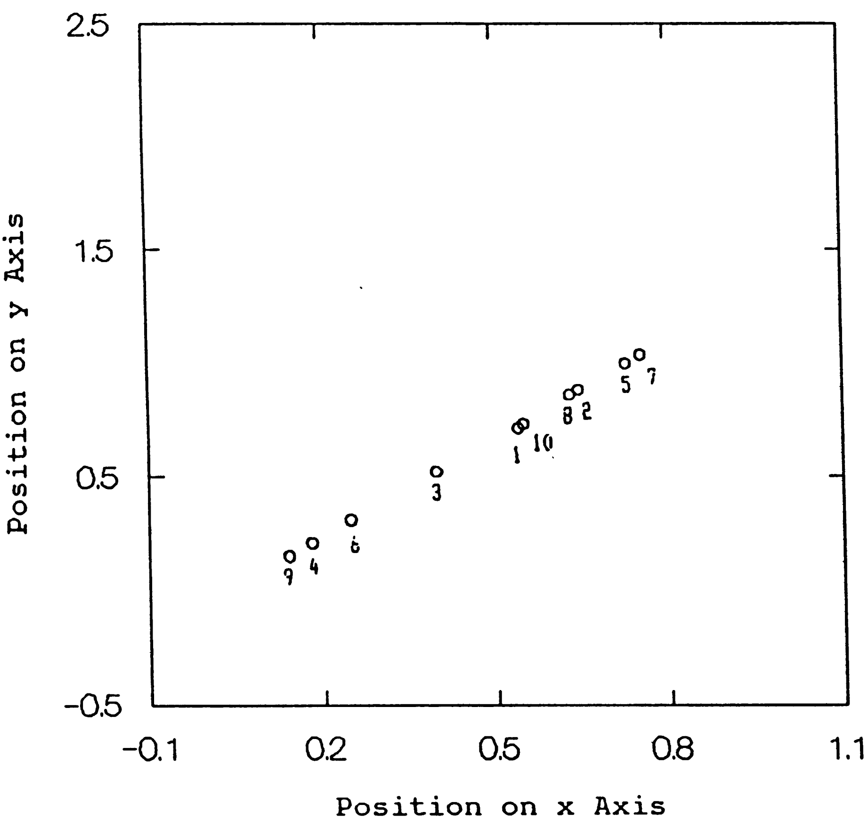
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caré section. The return map is important because it can display the properties of the attractor. If a point attractor governs the system, the map will display points converging to a central position. If a limit cycle is controlling behavior, a series of points will appear. If a strange attractor governs the system, an odd but describable shape will appear. If the system is turbulent, the map will produce a mass of seemingly random points. Figure 4 displays a roughly circular form. Thus, the hypothetical system appears to be governed by a strange attractor.

Information about the attractor can be useful for two reasons. First, it can enable prediction. This is especially true with point attractors and limit cycles. Further, the predictive power is derived without identifying the underlying mechanisms. Second, identifying properties of the attractor reveals the global organization of the system.

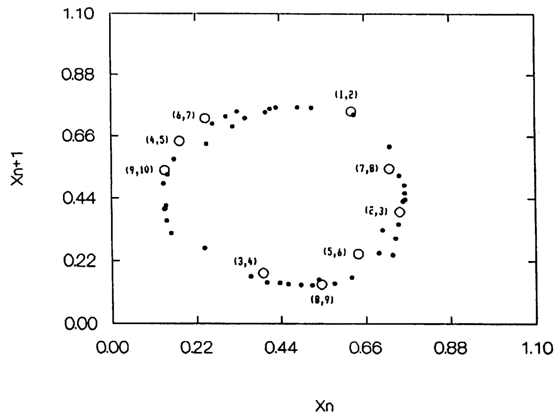

Fig. 4. Return map for the hypothetical system. The map is constructed by plotting the successive ordinates of the points of intersection from the Poincaré section. Open symbols are from points of intersection displayed in lower panel of Figure 3, closed symbols are from the full simulation.

RECONSTRUCTING THE ATTRACTOR IN THE ABSENCE OF INFINITE KNOWLEDGE

To construct a true phase portrait, the position of each variable must be known in infinite precision through time. However, the behavior analyst usually does not even know all of the variables involved in a particular setting, much less the ways in which they change over time. Takens (1981) showed that the properties of the attractor governing the behavior of an n-dimensional system could be determined if numerous measurements of the strength of any one of the state variables $x(t)$ exist. In this method, an m-dimensional phase portrait is constructed by plotting $x(t)$ versus $x(t + T)$ versus $x(t + 2T)$ versus . . . versus $x[t + (m - 1)T]$, where T is some time lag. The phase space constructed in this manner will be controlled by the same attractor as governs the n-dimensional phase space for almost every variable $x(t)$ and almost every time lag T.

An approximation of the phase portrait constructed from response rates generated by the hypothetical system is presented in Figure 5. It is apparent that this phase portrait is visually similar to that produced by the hypothetical system. Both have bounded, closed trajectories. A Poincaré section and return map were constructed from the phase portrait (Figure 5). Again, the Poincaré section and the return map constructed from the approximation are very similar to those constructed from the state variables.

RECONSTRUCTING THE ATTRACTOR WITH PIGEON DATA

It should be obvious that FI performance is not governed by a point attractor. It has far too much within- and between-interval variability. The dynamic effects are orderly enough to force the conclusion that they must be produced by the system and not by random extraneous events. The best candidates are the multidimensional attractors, toroidal flow, strange attractors, and turbulence. Techniques outlined above for examining the properties of

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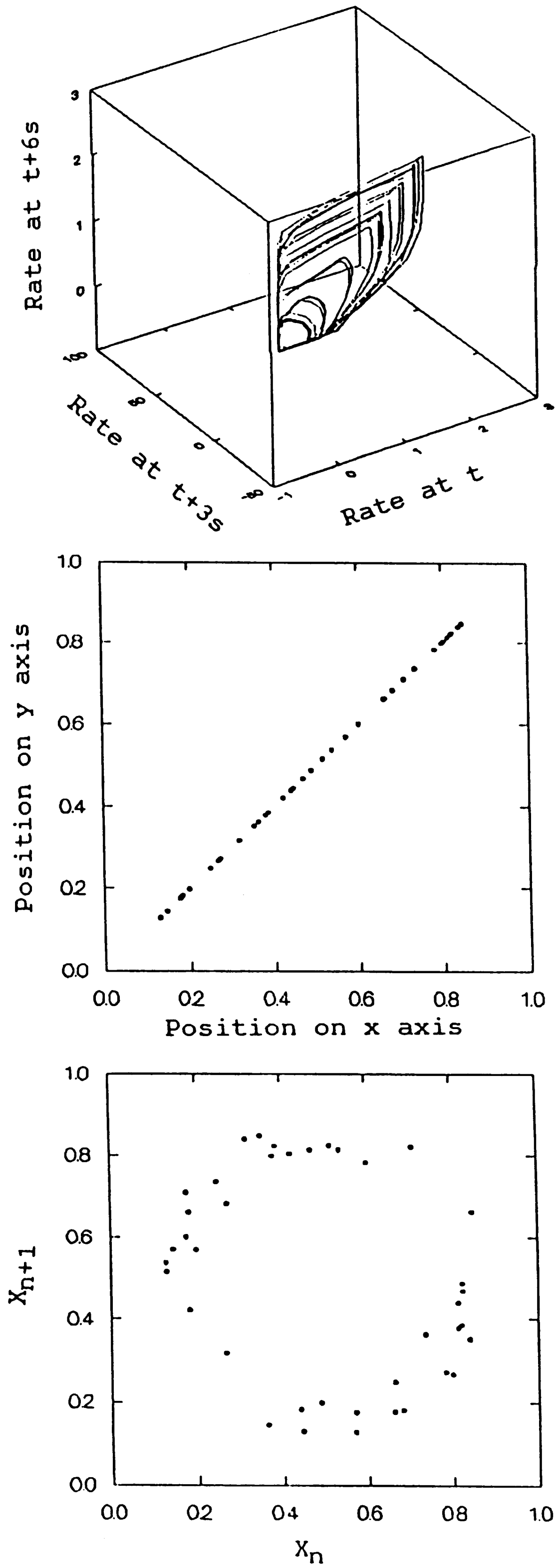
Fig. 3. Phase portrait and Poincaré section of the hypothetical system. A Poincaré section can be used to simplify phase portraits. The phase portrait is transected by a slicing plane. Orbits traveling through phase space cross the slicing plane. The Poincaré section records the location and order of transection. The upper panel displays a two-dimensional projection of the three-dimensional phase portrait in Figure 2. The heavy diagonal line shows the approximate location of the slicing plane. The lower panel displays the Poincaré section.



the attractor were employed to determine whether the behavior of a pigeon responding to an FI schedule of reinforcement could be described by a low-dimensional attractor.

METHOD

Subjects

Four White Carneau pigeons were maintained at 80% of their free-feeding weights. All subjects were experimentally naive.

Apparatus

The experimental chamber was 36 cm long, 32 cm wide, and 35 cm high. The walls and floor were lined with unpainted aluminum. Two 1-W white lights, each located in an upper corner of the response panel, provided general illumination. The panel also contained three 1.9-cm-diameter response keys (Gerbrands), each 21 cm above the floor and 8 cm apart. The center key could be transilluminated by two 1-W red lights. The keys were activated by a force of at least 0.18 N. An aperture (5 cm square) beneath the center key and 9 cm from the floor allowed occasional access to Purina® Pigeon Checkers, the birds' standard diet. During the 4-s feeder cycles, the aperture was illuminated by a 1-W white light, and the key and houselights were turned off. Continuous white noise helped mask extraneous sounds. Experimental events were programmed and recorded by an experimental controller (Walters & Palya, 1984).

Procedure

Sessions were conducted 6 days per week. Each session ended after 50 food deliveries. After key pecking had been established via autoshaping, subjects were exposed to an FI 20-s schedule. The experimental condition lasted 20 sessions. The number of responses emitted during 1-s bins was recorded.

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Fig. 5. The reconstructed phase portrait, a Poincaré section, and return map for the hypothetical system. The upper panel displays response rates measured at three different time lags (lag = 3 s). The center panel displays a Poincaré section created from the phase portrait. The lower panel shows the return map constructed from the Poincaré section. The return map is similar to the map displayed in Figure 4 and suggests that responding is controlled by a strange attractor.

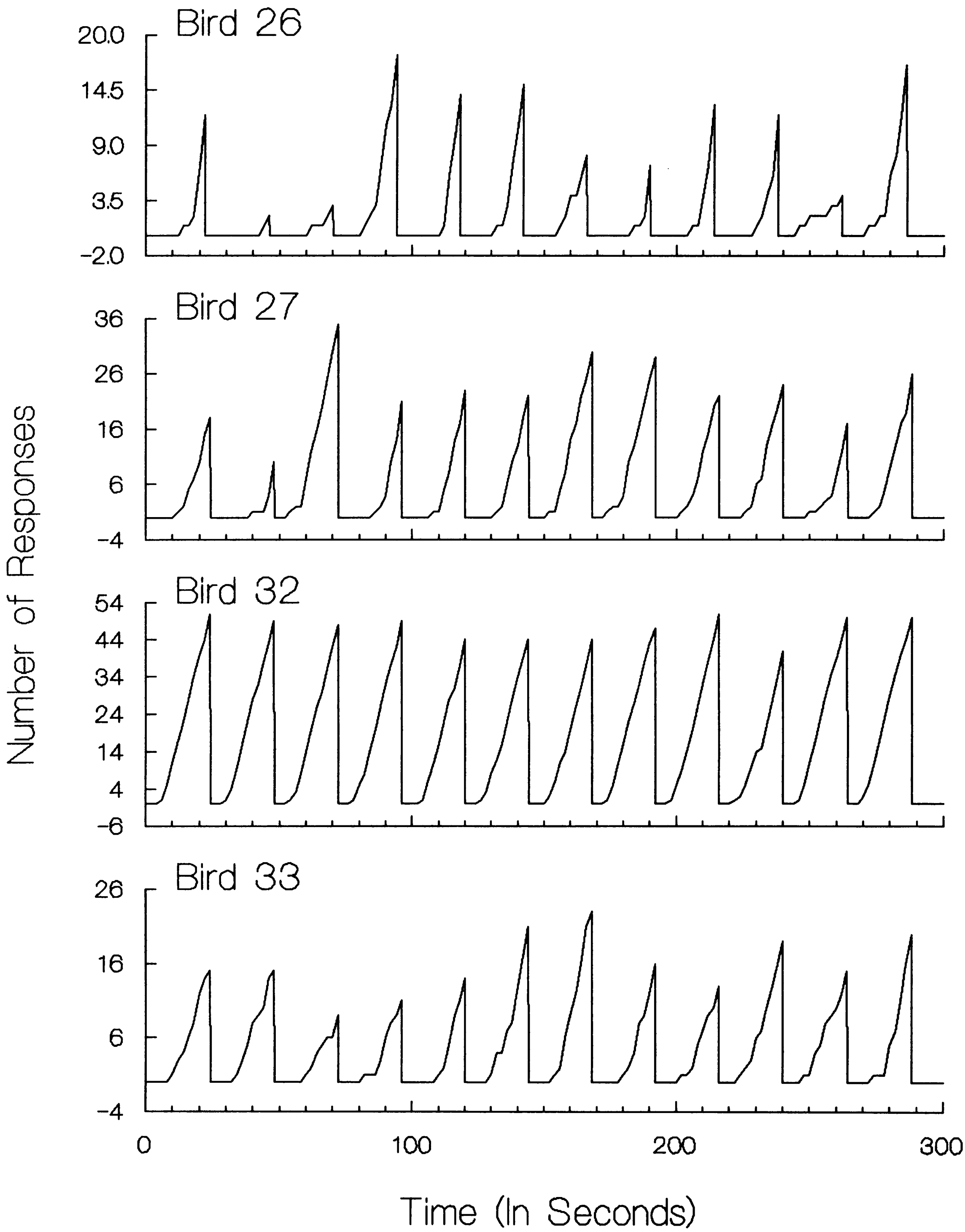

Fig. 6. Sample cumulative records of responding. The records display responding during Intervals 24 through 35 from the 20th session for each subject.

Table 1
Index of curvature, runs tests, and autocorrelations.

			Runs test (*Z* score)		Correlation coefficient	
Subject	Session	Index of curvature	Response number	PRP	Response number	PRP
26	16	0.541	4.6	3.2	.17	.11
	17	0.514	3.9	4.6	.19	.05
	18	0.588	4.0	3.7	.07	.22
	19	0.524	5.0	4.2	.02	−.01
	20	0.494	3.2	2.9	.39	.22
27	16	0.291	5.2	3.7	−.05	−.01
	17	0.351	2.3	3.2	.20	.09
	18	0.355	3.2	3.3	−.01	.04
	19	0.394	2.4	2.6	.03	−.02
	20	0.401	2.7	2.8	.25	.12
32	16	0.138	2.8	2.8	−.09	.24
	17	0.118	5.7	4.1	.17	.05
	18	0.149	4.9	3.3	−.02	.04
	19	0.126	5.1	5.5	−.06	−.06
	20	0.232	2.1	3.0	.22	−.03
33	16	0.178	4.9	5.7	−.15	.14
	17	0.242	2.9	3.2	.07	−.03
	18	0.166	3.2	2.7	.17	.02
	19	0.207	3.0	3.0	−.09	.04
	20	0.232	4.5	5.3	.31	−.11

Results and Discussion

The credibility of the chaos analysis depends upon the similarity of the obtained FI performance and that of typical FI performance. All data analyses examined performance during the last five sessions of the condition. Figure 6 displays representative segments of the cumulative record from each subject. The pattern can be described as a pause followed by a gradually increasing response rate that reaches a maximal value near the end of the interval. The records display several other features that are typical of FI responding (Ferster & Skinner, 1957). For example, there is occasional rough grain, obvious renewed pausing after responding had begun. Degenerate intervals appeared; responding decelerated late in the interval. Knees were seen, characterized by decelerating response rates early in the interval followed by renewed rapid responding. Nothing in these cumulative records seems to indicate atypical FI performance.

Quantitative measures of variability also indicate that the current experiment produced typical dynamic effects. An index of curvature was computed for each subject for each of the last five sessions. Complete results are displayed in Table 1. Values ranged from 0.118 to 0.588, with a median value of 0.267. Runs tests and Lag 1 autocorrelations were conducted for both postreinforcement pause and response number. These results are also displayed in Table 1. All runs tests were significant ($Z = 2.1–5.7$, $p < .05$). The correlation coefficients ranged from +.39 to −.11. Twenty-five of the 40 coefficients were positive; however, none of the coefficients were large (median coefficient = .04). All values were similar to those found in previous reports (Dews, 1970; Gentry & Marr, 1982; Gentry et al., 1983; Palya & Bevins, 1990; Shull, 1971; Wearden, 1979; Wearden & Lowe, 1983). It appears that these data are representative of FI performance and show the characteristic dynamic effects.

To proceed with the analysis, it is necessary to convert the data from response counts in each second to response rates. The approximation of the phase portrait requires multiple determinations of a single continuous variable. Unfortunately, the traditional conditioning procedures do not provide this kind of measurement. The number of responses per bin is not a continuous variable. Interresponse time is a continuous variable, but it cannot be measured at regular intervals. Both the number of responses per bin and interresponse times can be converted into momentary response rates. The response rate during any second t was computed as the average of the response rates during seconds t, $t - 1$, and $t + 1$. This smoothing routine was conducted twice.

The smoothing routine preserves the global characteristics of responding, but it does alter the data somewhat. The procedure yields a continuously changing response rate that begins near zero and increases and decreases smoothly to the peak response rate, rather than increasing and decreasing irregularly over the course of the interval. Knees and degenerate intervals are preserved, but graininess is removed. The peak response rate is decreased in some intervals because the zero response rate from the hopper cycle is averaged into the peak rates. The period of not responding is also shortened because bins with small response rates are averaged into the zero rate intervals close to the end of the postreinforcement pause.

The smoothing procedure was conducted for

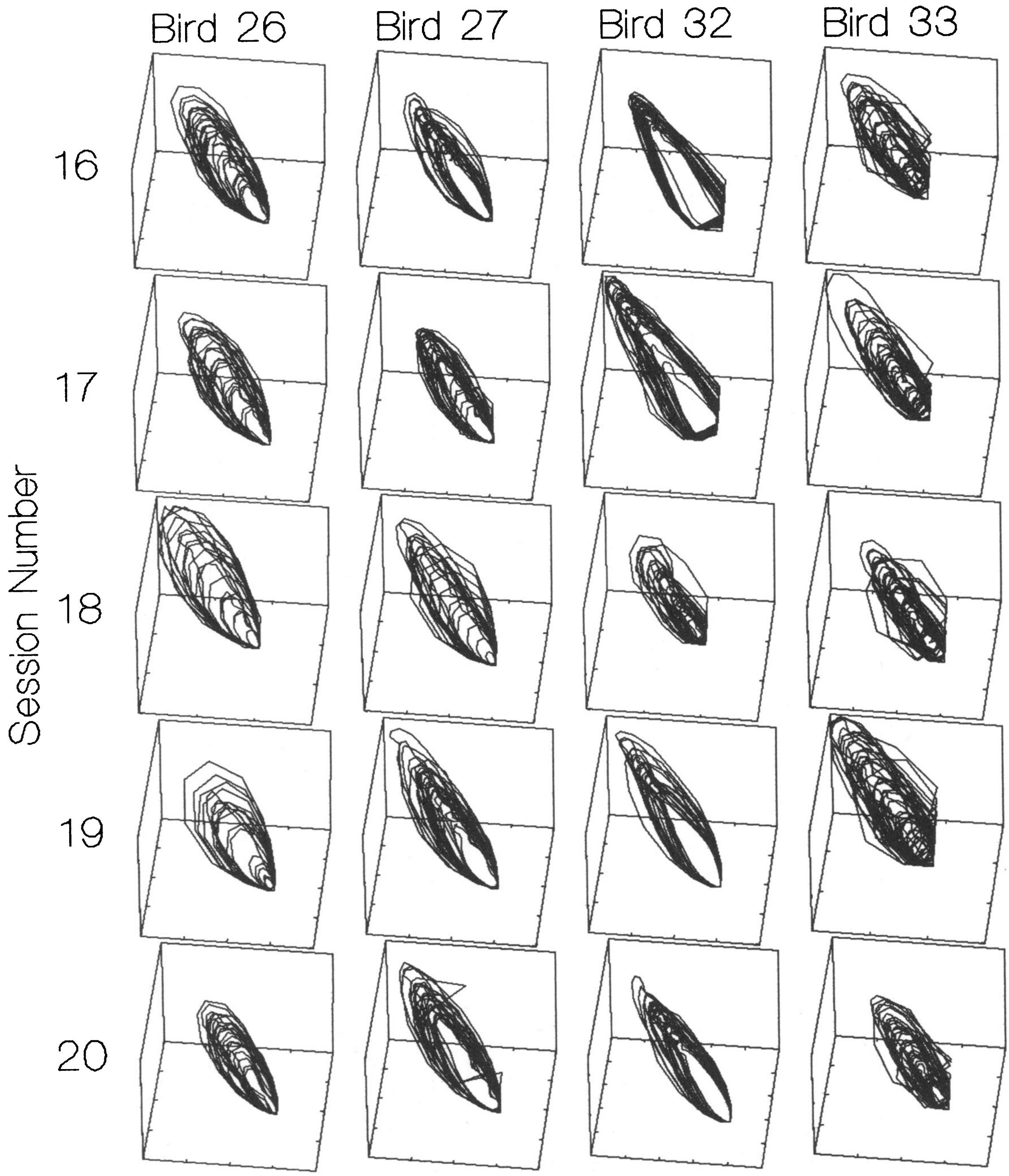

Fig. 7. Reconstructed phase portraits. Phase portraits were constructed using response rates for each second. Only the behavior from Intervals 15 to 39 are displayed. The axes represent response rate at time t, response rate at time $t + 3$ s, and response rate at time $t + 6$ s.

each of the last five sessions for each subject. This produced 20 lists of about 1,200 determinations of the response rate. All further analyses were conducted with these data. Takens' (1981) method was used to reconstruct the phase space and examine properties of the attractor. Because Takens' method involves plotting $x(t)$ versus $x(t + T)$ versus $x(t + 2T)$

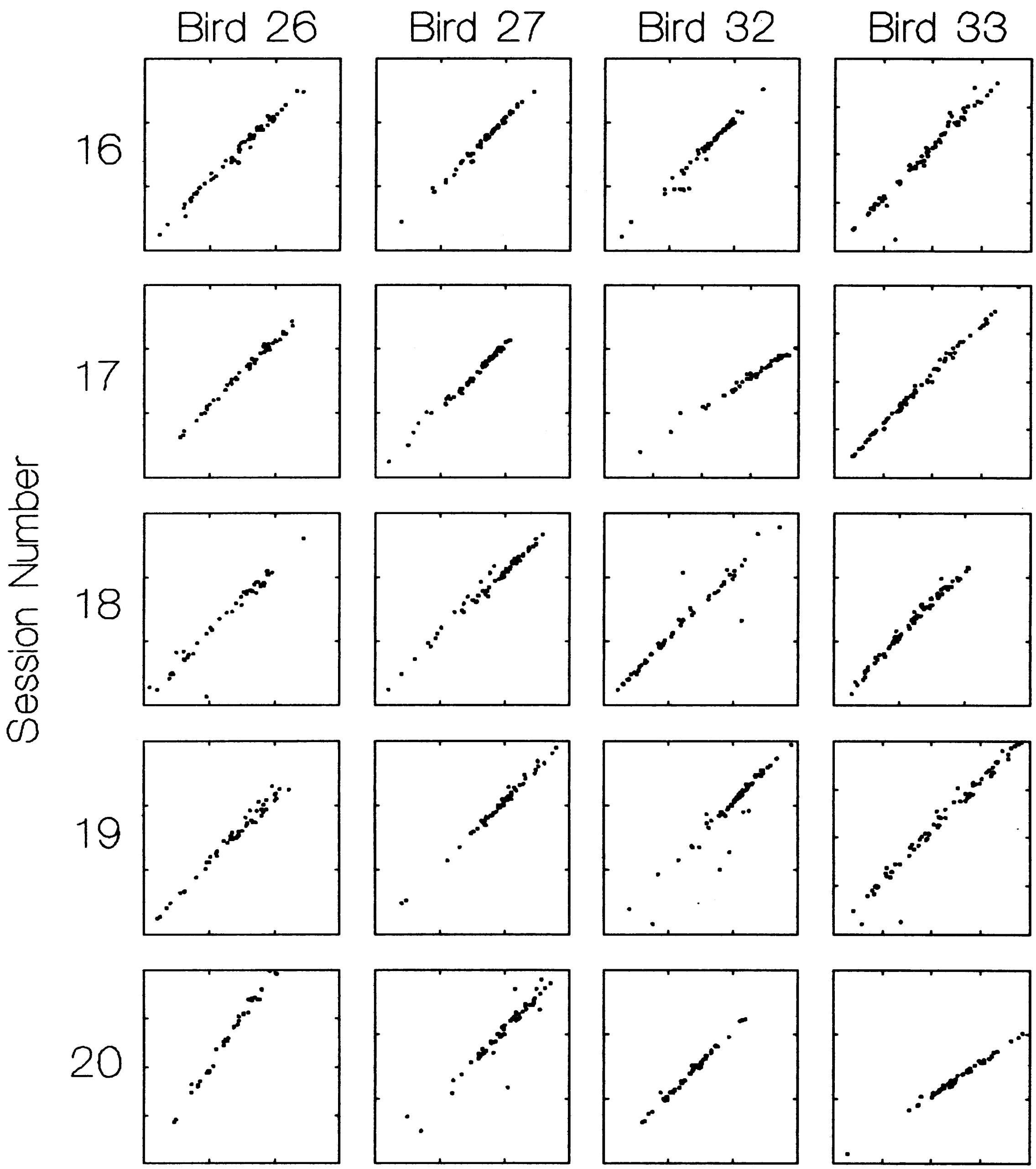

Fig. 8. Poincaré sections. All Poincaré sections were created by passing a plane through the center of the orbits displayed in Figure 7 and recording the position of points of intersection.

(where T is some time lag), the first practical problem involved selecting an appropriate time lag. Takens suggested that, in principle, any lag should suffice. A lag of 3 s was selected through trial and error. The lag was increased until the figure produced an organization that could be detected visually.

Figure 7 displays phase portraits for each subject. Behavior in phase space is organized. In general, response rate begins at a near-zero level, increases to a maximum rate, and then decreases. Very small orbits represent knees or intervals in which little responding occurred. Between-interval variability can also

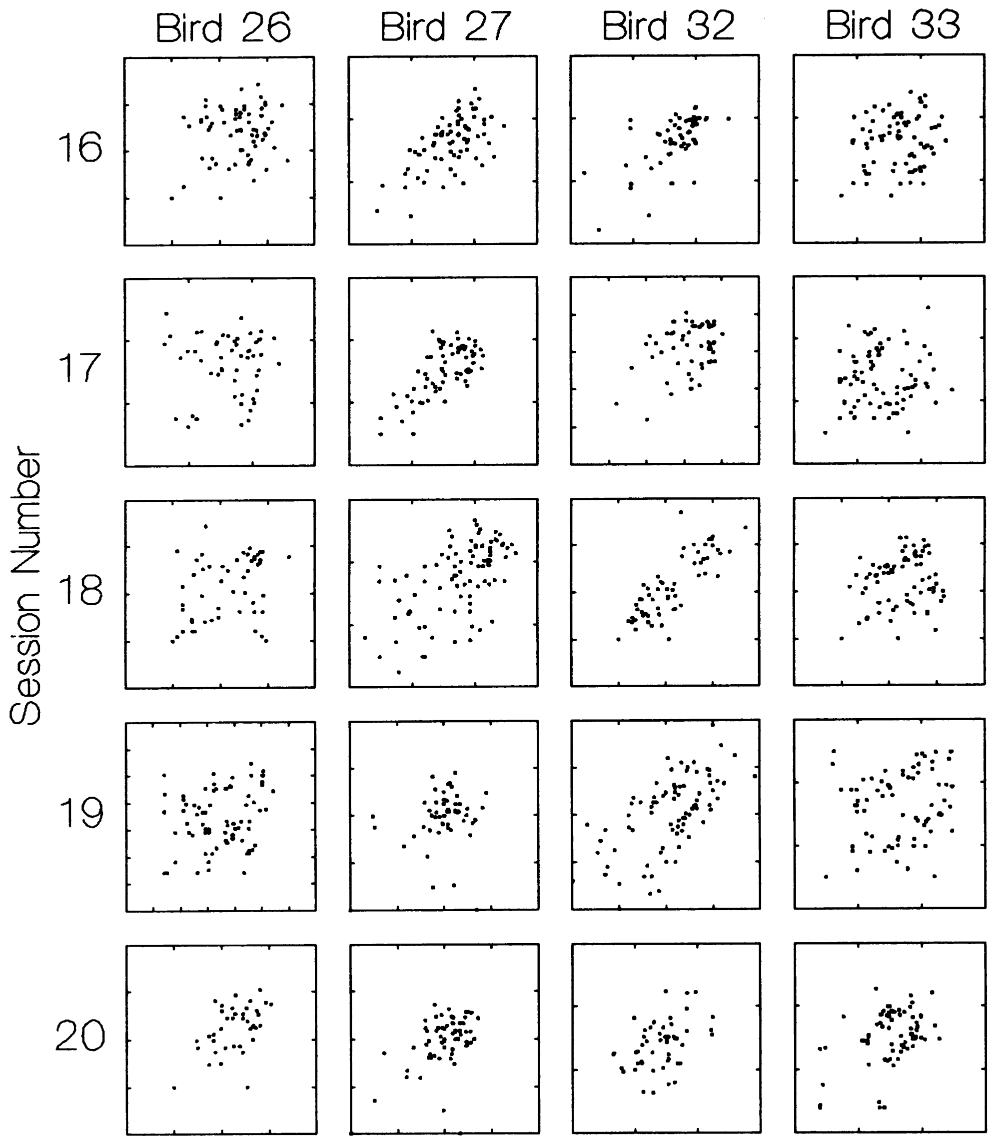

Fig. 9. Return maps. All return maps were constructed from the Poincaré sections shown in Figure 8.

be observed. Responding of Birds 27 and 32 showed much less variability. This can be detected by noting that all orbits follow a very similar path. This can be seen clearest for Bird 32, Session 16. Birds 26 and 33 had more intrasession variability. Orbits reached many different maximum response rates before decreasing. As a result, trajectories occupy many positions within the bounded shape. Behavior within each interval might be described by phase portraits, but the maximum value of responding in the next interval cannot be obtained from these figures. Properties of the attractor assumed to govern the organization were examined by constructing Poincaré sections and return maps.

Poincaré sections were created by passing a plane through Figure 7 as described above. Figure 8 displays Poincaré sections for all subjects. The Poincaré sections show a series of points falling along the diagonal. The diagram simply displays the points of intersection. The Poincaré section becomes more interesting when a return map is constructed by plotting the ordinates from successive intersecting points against each other (Figure 9). If the system is controlled by a low-dimensional attractor, the return map will display some structure or organization. If no organization is captured or if behavior is controlled by random inputs, then the return map should produce a mass of dots occupying all possible spaces on the return map. Figure 10 displays a return map constructed from random behavior.

The return maps are not as orderly as could be hoped. However, they show more organization than would be expected to occur from random processes. The maps, in general, share two features: (a) Points tend to cluster along the diagonal, and (b) there tends to be an absence or a lower concentration of points in the center of the figure. Both of these traits are well illustrated in the return map constructed for Bird 32, Session 19.

THE RETURN MAP AS A PREDICTIVE SCHEME

The appearance of order in the map affords the opportunity to use the map as a predictive scheme to forecast future behavior. Thus, the Poincaré section and map can serve two functions: (a) The map gives information about the underlying system, and (b) it can provide an empirical prediction scheme even in the absence of knowledge of the governing equations. The utility of these maps for predicting response rates in FI schedules was evaluated.

Prediction was based on the assumption that the return maps captured the motion of the response rate as it travels through phase space. Although the maps did not show a clear pattern, they do show an approximately linear relationship with an open center. This is similar to an ellipse. Therefore, it will be assumed that the motion of response rates is approximately elliptical. Thus, response rate should increase within each interval according to an ellipse. Likewise, between-interval variability should be described by an ellipse.

If the shape of the attractor has been described and the present behavior is known, then future behavior can be predicted. Four different types of predictions were made from the return maps. If the response rate in any given second were specified, then (a) a priori and (b) a posteriori predictions of response rate in the next second were generated. These models predict response rate in the next second as a function of performance in the current second. If the behavior over an entire interval was provided, then (c) a priori and (d) a posteriori predictions of response rate in each second of the next interval were produced. These models predict response rates over the entire interval as a joint function of the passage of time in the current interval and of the strength of responding in the previous interval. In the a posteriori models, constants for the ellipse were estimated from the same session. In the a priori models, constants for the ellipse were estimated from the immediately preceding session. The values of all constants are included in Appendix 2.

Table 2 displays the percentage of variance accounted for (VAC) by these analyses. As can be seen, the predictions provided a close quantitative fit for both a priori and a posteriori analyses when predicting response rates during the next second. The technique was considerably less successful when predictions were made for responding across the entire next interval. This is characteristic of strange attractors (Abraham & Shaw, 1983; Rossler, 1976; Shaw, 1981). Prediction is more successful over short time frames because of the indeterminacy problem; small errors in estimation become amplified over time. A posteriori predictions for the next second ranged from 89% to 98% VAC. The a priori account predicted between 83% and 98% of the variance. When predictions were made for the entire next interfood interval, the a posteriori model accounted for between 50% and 76% of the data variance. The a priori account predicted between 41% and 72% VAC.

Although it is true that the nonlinear dynamical analysis was able to account for a large percentage of the variance, it is not clear that the procedure is more successful than typical accounts. The largest source of variance came from within-interval dynamics. The smoothing procedure resulted in response rates that continuously increased and decreased and introduced a weak second-to-second sequential dependency. It is possible that any continuous function might provide a good quantitative fit. The ellipse contained two fitting parameters whose values were estimated from the return maps. It seems reasonable to compare the results obtained by the nonlinear dynamic analysis to traditional models that contain at least two free parameters. The values of the free parameters were estimated from session data and not from the return maps. Five commonly used functions served as control models:

a linear function

$$R_{n+1} = aR_n + b, \tag{8}$$

an exponential function

$$R_{n+1} = aR_n^b, \tag{9}$$

a sine function

$$R_{n+1} = a \sin(bR_n), \tag{10}$$

and the quadratic functions

$$R_{n+1} = aR_n^2 + bR_n, \tag{11}$$

and

$$R_{n+1} = aR_n^2 + bR_n + c \tag{12}$$

where a, b, and c are constants.

Four versions of each model were employed: a priori and a posteriori models that predicted response rate in the next second as a function of response rate in the present second and a priori and a posteriori models that predicted response rate during each second over the entire next interval as a function of the passage of time during the interval. Constants for the a priori model for each of the models were estimated from the average performance from the previous session. Constants for the a posteriori models were estimated using the average performance from the same session. The values of all constants are included in Appendix 2.

Table 2 displays the VAC by the control models. All models provide a reasonably close quantitative fit when predicting response rates in the next second. Predictions accounted for between 55.8% and 94.6% of the data variance. These values are comparable to those produced by the nonlinear dynamics analysis. These results suggest that response rates exhibited a robust second-to-second dependency that any increasing function could capture. The control models were less successful when predicting response rates across the entire next interval. The VAC ranged from 19.7% to 64.6%. The control accounts were more successful for Birds 27 and 32 because these subjects exhibited much less interval-to-interval variability.

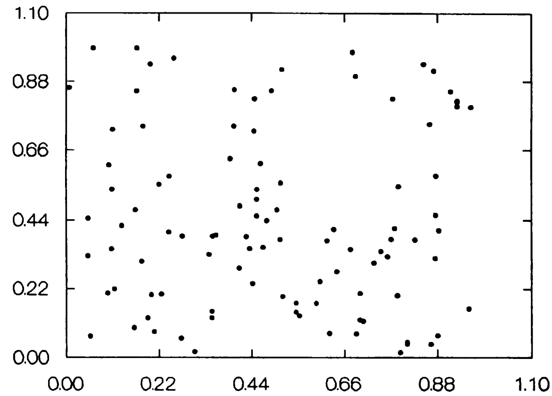

Fig. 10. Return map of random behavior. This figure could be produced by a stochastic system or by randomly ordering the behavioral output from a large sequence of FI intervals generated by a deterministic system. Under these conditions, an approximation of the phase portrait, a Poincaré section, and this return map could be constructed. No order can be detected. Points appear in equal densities across the entire graph.

All control models were able to account for within-interval variability, but none could describe between-interval dynamics as effectively as the nonlinear dynamical model. The control models were more successful when less between-interval variability was exhibited. In contrast, the nonlinear dynamical systems analysis could predict performance with equal success regardless of the amount of between-interval variability. This suggests that the account may have captured some of the interval-to-interval dynamics.

GENERAL DISCUSSION

The dynamical phenomena discussed above—toroidal flow, strange attractors, turbulence, chaotic motion, indeterminacy, and diverging trajectories—are different from the concepts familiar to behavior analysts. This paper hopes to (a) introduce techniques that might prove useful in identifying attractors, (b) suggest the possibility that strange attractors might underlie behavioral systems, and (c) suggest a utility in examining them. Although more evidence is needed to conclude that strange attractors govern behavioral systems, finding evidence for strange attractors could force a reevaluation of some of the basic concepts in behavior analysis.

Table 2
Percentage of variance accounted for by models.

		Chaos analysis		Linear		Exponential	
Subject	Session	Next second	Next interval	Next second	Next interval	Next second	Next interval
26	16	93.4 (91.4)	66.8 (65.6)	91.9 (86.9)	42.7 (42.5)	92.2 (86.8)	26.9 (25.8)
	17	94.3 (90.6)	71.8 (72.2)	92.2 (87.8)	46.9 (46.9)	92.9 (87.6)	26.4 (24.7)
	18	92.1 (90.4)	64.8 (68.2)	91.9 (86.9)	26.8 (26.4)	92.5 (87.6)	20.3 (20.2)
	19	92.3 (90.6)	67.9 (62.1)	90.7 (85.7)	33.7 (32.3)	90.8 (86.5)	17.8 (15.9)
	20	94.0 (82.8)	69.9 (71.1)	92.5 (87.5)	24.9 (24.6)	93.1 (87.0)	20.9 (20.4)
	Grouped	92.0 (90.1)	70.8 (67.9)	92.9 (87.9)	23.2 (22.8)	93.1 (88.0)	16.6 (16.1)
27	16	97.8 (95.5)	55.5 (62.2)	94.0 (87.0)	54.0 (57.1)	93.0 (88.5)	58.1 (57.7)
	17	97.8 (97.0)	50.4 (41.8)	89.8 (84.8)	55.9 (52.9)	92.6 (85.6)	58.0 (57.0)
	18	93.6 (96.3)	76.0 (45.3)	93.0 (88.0)	19.5 (19.4)	94.6 (89.5)	13.9 (12.9)
	19	97.5 (97.7)	60.0 (52.3)	92.4 (88.6)	56.6 (55.2)	93.2 (88.5)	55.9 (54.1)
	20	95.9 (96.0)	70.1 (59.9)	92.2 (87.5)	64.7 (64.6)	91.4 (86.5)	57.1 (56.6)
	Grouped	97.2 (96.5)	65.5 (50.6)	93.1 (87.9)	47.2 (47.9)	93.0 (87.0)	42.7 (42.7)
32	16	95.0 (95.1)	66.3 (51.2)	68.2 (68.9)	41.6 (40.5)	74.0 (59.1)	59.6 (59.1)
	17	95.9 (95.9)	72.4 (64.1)	72.1 (58.7)	46.6 (43.2)	75.7 (65.6)	50.4 (50.0)
	18	93.9 (89.5)	68.2 (72.2)	78.1 (66.7)	30.5 (28.7)	77.9 (66.2)	24.2 (23.8)
	19	96.2 (95.5)	67.7 (68.1)	75.7 (72.1)	42.3 (41.2)	88.6 (73.1)	58.7 (52.2)
	20	97.8 (97.2)	65.2 (71.8)	76.5 (69.1)	45.9 (44.6)	89.5 (76.6)	59.1 (57.6)
	Grouped	96.4 (96.0)	72.7 (70.3)	77.0 (63.5)	43.2 (43.1)	88.0 (71.2)	53.6 (53.6)
33	16	89.1 (85.9)	63.1 (70.5)	83.4 (75.6)	22.5 (24.6)	80.7 (75.1)	22.0 (21.9)
	17	90.0 (87.0)	64.1 (67.2)	85.1 (79.8)	24.6 (20.5)	78.5 (76.7)	22.5 (21.1)
	18	91.8 (89.0)	57.8 (64.9)	80.8 (79.5)	23.4 (22.4)	76.6 (60.4)	25.3 (25.2)
	19	88.2 (89.0)	64.8 (62.8)	84.5 (75.5)	29.0 (21.8)	78.4 (73.1)	27.3 (26.7)
	20	90.2 (89.1)	64.7 (61.9)	82.2 (77.9)	22.2 (20.1)	81.1 (75.4)	22.1 (21.3)
	Grouped	89.9 (88.4)	63.5 (67.2)	83.3 (78.9)	24.9 (23.1)	82.9 (76.1)	23.6 (22.8)

Note. Values are for a posteriori predictions; a priori predictions are in parentheses.

Behavior analysis assumes that phenomena change linearly and reach equilibrium. In a nonlinear world, equilibrium and stable performance are rare and/or ephemeral events. If behavior is a nonlinear phenomenon, it would imply that the effects that are currently identified as stable are illusory. The stable behavior typically observed is obtained by averaging out the dynamic effects. Currently, conditions are conducted until steady-state behavior is obtained. Yet, even in what is now labeled steady-state behavior, performance fluctuates. Experiments that conduct conditions for hundreds of sessions continue to show large irregular intersession modulations in performance (Cumming & Schoenfeld, 1958; Palya, 1992).

A second assumption made by behavior analysts involves the research approach. The traditional approach has been an essentially inductive process. The theorist independently examines a set of independent variables (e.g., reinforcer frequency, reinforcer magnitude, reinforcer delay, or reinforcer deprivation). After the effects of these variables have been identified, it is hoped that unifying principles can be found to account for a large number of the effects. Although this approach has produced notable success for some goals and some types of behavior, it has not led to a unified theory of behavior.

If behavior is a nonlinear phenomenon, then the traditional approach cannot succeed. If behavior is controlled by interacting variables, the fact that they simultaneously undergo sustained motion on a complex orbit through phase space means that the signs and magnitudes of interaction change continuously throughout the orbit. That is, the effect of any one variable will depend on the simultaneous influence of several interacting variables. If one variable is repeatedly changed in a systematic fashion, the same behavioral output could occur, in each replication, only if all other variables continued to remain at the same values (with infinite precision), or changed at the same rate in each

Table 2 (*Continued*)

Sine		Quadratic (c = 0)		Quadratic	
Next second	Next interval	Next second	Next interval	Next second	Next interval
89.9 (86.8)	26.9 (25.8)	91.1 (84.2)	42.6 (41.1)	91.3 (86.2)	43.7 (43.1)
90.9 (87.6)	26.4 (24.7)	91.0 (85.9)	46.1 (45.3)	91.6 (86.1)	47.7 (47.1)
90.5 (87.6)	20.3 (20.2)	91.8 (86.2)	29.6 (27.3)	91.8 (86.2)	29.8 (27.7)
89.4 (86.5)	17.8 (15.9)	89.9 (74.1)	33.7 (30.1)	90.3 (85.2)	34.1 (33.3)
92.3 (87.0)	20.9 (20.4)	91.6 (82.6)	26.6 (25.1)	92.7 (85.3)	27.6 (25.6)
89.9 (88.0)	16.6 (16.1)	90.5 (85.8)	24.4 (23.6)	92.0 (88.3)	25.5 (24.3)
93.8 (88.5)	58.1 (57.7)	93.3 (88.2)	58.1 (56.7)	93.9 (88.4)	59.3 (57.1)
92.4 (85.6)	58.0 (57.0)	92.7 (86.8)	54.4 (53.6)	93.3 (86.4)	55.6 (54.3)
93.5 (89.5)	13.9 (12.9)	93.6 (85.3)	21.1 (20.3)	94.1 (89.3)	21.5 (20.7)
91.9 (88.5)	55.9 (54.1)	91.0 (86.0)	55.0 (54.1)	92.8 (88.4)	56.3 (55.5)
91.4 (86.5)	57.1 (56.6)	91.2 (83.8)	60.1 (59.7)	91.7 (86.6)	62.1 (60.9)
93.0 (87.0)	42.7 (42.7)	92.6 (85.2)	51.3 (50.1)	92.8 (85.8)	52.3 (51.1)
77.3 (59.1)	59.6 (59.1)	70.3 (60.2)	41.9 (41.1)	80.8 (73.3)	42.4 (41.6)
82.1 (65.6)	50.4 (50.0)	75.2 (66.3)	43.3 (42.3)	85.0 (75.9)	44.9 (43.3)
85.1 (66.2)	24.2 (23.8)	84.1 (50.8)	29.0 (27.1)	85.1 (58.7)	29.3 (28.3)
86.1 (73.1)	58.7 (52.2)	84.9 (70.1)	47.1 (46.7)	89.3 (70.0)	47.9 (47.3)
81.0 (76.6)	59.1 (57.6)	77.0 (70.8)	45.9 (45.1)	88.3 (81.0)	47.1 (45.9)
77.1 (71.2)	53.6 (53.6)	74.3 (64.1)	46.3 (45.1)	87.3 (79.6)	46.7 (46.2)
80.4 (75.1)	22.0 (21.9)	74.5 (68.5)	26.6 (25.1)	88.1 (81.1)	26.9 (26.1)
76.4 (76.7)	22.5 (21.1)	75.3 (64.3)	23.3 (21.5)	89.8 (84.8)	23.9 (22.1)
70.8 (60.4)	25.3 (25.2)	71.5 (53.5)	20.7 (19.4)	87.6 (54.5)	21.1 (19.9)
75.5 (73.1)	27.3 (26.7)	67.2 (60.3)	26.3 (25.4)	82.9 (72.5)	26.9 (25.5)
76.1 (75.4)	22.1 (21.3)	75.1 (70.5)	19.9 (19.7)	84.1 (76.5)	21.6 (20.7)
83.5 (76.1)	23.6 (22.8)	80.5 (73.5)	26.0 (25.1)	86.4 (82.1)	26.8 (25.5)

repetition. This type of precision is logically impossible in behavioral systems. Consider the level of food deprivation as an example. For behavioral output to remain constant, experienced food deprivation must start at the same level (to infinite precision) every session, and must either remain at that level or change at the same rate within each session. Eating a half gram less or one second later during one feeder cycle could alter the system. Thus, the same behavioral output could never appear in any replication. This may explain why contradictory results are sometimes found in apparently straightforward replications (e.g., Hayes & Hayes, 1990). Subtle changes in state variables (that were not being studied and that had been experimentally controlled) may have occurred during the replication. This would alter the interaction of all other variables, which could in turn alter observed behavior.

The most disturbing implication of chaos theory is its effects on prediction and control. As previously described, exact prediction can occur, in a nonlinear system, only if the starting conditions are known perfectly. In behavioral systems, this kind of precision is impossible. The same limitations that impair prediction also impair control. Control is produced by altering starting conditions or by altering conditions in an existing system. Nonlinear dynamics implies that small changes to certain state variables can alter the behavior of the system. However, it would be impossible to predict what the effect might be. This becomes apparent when you consider that it is impossible to predict, in detail, what the system might do in the absence of intervention. It is impossible to specify how the system might change, because it was impossible to specify what it would have done before the intervention. One can only be sure that the behavior of the system must be different. The best that could be hoped for is a statistical description of likely states.

Short-term control and prediction may still be possible. Some complex multifactor systems can be describable with one-dimensional maps (Lorenz, 1963; Shaw, 1981). In these cases, the parameters associated with the map are attributes of the whole system. The parameters will not be exact measures. If they are good estimates, short-term predictions will generate small amounts of error. The estimates can then be corrected for future predictions. The dynamic analyses employed in the present paper demonstrate the advantages that this type of short-term prediction provides.

The final concern raised by nonlinear dynamical systems theory revolves around the relationship between it and current behavior-analytic practices. Embracing the methods described in the present paper does not necessarily require supplanting the more traditional approach of attempting to relate changes in rates of behavior to changes in environmental conditions. Nonlinear dynamics can more properly be viewed as a framework for understanding when the various environmental effects may exert their various effects. Nonlinear dynamics might be used to describe how the system as a whole functions and how variables interact. The traditional approach can be used to identify what the variables are, and how they can be altered. The two approaches can be complementary.

Nonlinear dynamics can also be used to suggest new avenues for traditional research. For instance, reconstructed phase portraits, and the maps obtained from them, do not reveal the underlying psychological mechanisms. On the other hand, such reconstructions may provide a criterion that could be used to guide the development and assess the validity of mechanistic models. For instance, the present analysis does not reveal a mechanism that might be responsible for the sequential dependencies. However, it might provide a description of the organization of such a system that can be used to guide the search for models or to assess mechanisms.

The present paper demonstrates how nonlinear dynamics can be used to describe attractors and predict behavior, and also suggests how it can be used to search for the psychological mechanisms that produce the behavior. If nonlinear dynamics is to prove useful in behavior analysis over a wide range of problems, a fundamental question concerning the dimension of the attractors that govern behavioral systems must be answered. Should low-dimensional motion prove to be ubiquitous, there is cause for optimism. This suggests that even the most complex behavior might stem from very simple deterministic systems. This is a much more hopeful alternative than the idea that unpredictable, variable behavior is the product of a very large number of complexly interacting variables whose effects must be understood independently and in combination. Conversely, if the dimensionality of behavioral systems is generally high, behavior analysts may have to content themselves with statistical statements about distributions of response states.

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APPENDIX 1

Constants used in the hypothetical model.

$$S_r = \frac{S_{r\max}}{1 - be^{(-cS_{r\max}t)}}, \qquad \text{(Equation 2)}$$

$b = .99 \qquad c = 1.00$

$$S_a = \frac{S_{a\max}}{1 - fe^{(gS_{a\max}t)}}, \qquad \text{(Equation 4)}$$

$f = .01 \qquad g = 1.00$

$$S_r = 1 - S_a = j\{\sin[k\pi(MI + SI + T(R)]\} + l \qquad \text{(Equation 5)}$$

$j = .35 \qquad k = 1.0 \qquad l = .5$

$$T(R) = \begin{cases} 1 & 0 + o(m+n) < R < m + o(m+n),\ o = 0, 1, 2, \ldots \\ 0 & m + o(m+n) < R < (o+1)(m+n),\ o = 0, 1, 2, \ldots \end{cases} \qquad \text{(Equation 6)}$$

$m = 1.0 \qquad n = 3.0$

$$B = (S_r - S_a)p \qquad \text{(Equation 7)}$$

$p = 2.5$

APPENDIX 2

Constants used to predict data.

Subject	Session	Ellipse next interval		Linear next second		Linear next interval		Exponential next second		Exponential next interval		Sine next second	
		a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*
26	15	−0.067	0.601	1.055	0.049	0.085	−0.351	1.125	0.849	0.003	2.231	1.662	0.765
	16	−0.025	0.616	1.076	0.049	0.099	−0.432	1.158	0.858	0.002	2.358	1.886	0.693
	17	−0.126	0.659	1.048	0.054	0.092	−0.370	1.127	0.840	0.003	2.137	1.648	0.800
	18	−0.091	0.539	1.129	0.028	0.072	−0.347	1.153	0.879	0.001	3.072	1.580	0.843
	19	−0.073	0.678	1.024	0.049	0.075	−0.312	1.063	0.776	0.004	1.932	1.184	1.148
	20	−0.189	0.432	1.052	0.040	0.069	−0.260	1.073	0.852	0.005	1.841	1.462	0.831
	Grouped	−0.064	0.596	1.066	0.044	0.081	−0.344	1.115	0.841	0.003	2.268	1.552	0.863
27	15	−0.041	0.351	0.951	0.145	0.105	−0.112	1.170	0.738	0.099	1.041	1.903	0.687
	16	−0.142	0.200	0.933	0.172	0.100	−0.092	1.171	0.728	0.183	0.799	1.858	0.688
	17	−0.174	0.213	0.957	0.114	0.094	−0.096	1.110	0.773	0.081	1.026	1.623	0.785
	18	−0.115	0.541	0.956	0.144	0.119	−0.133	1.173	0.768	0.098	1.037	1.963	0.669
	19	−0.046	0.474	0.974	0.126	0.124	−0.056	1.186	0.757	0.063	1.190	1.951	0.684
	20	−0.094	0.497	0.980	0.134	0.138	−0.307	1.210	0.766	0.062	1.230	2.260	0.572
	Grouped	−0.031	0.370	0.960	0.138	0.115	−0.140	1.170	0.758	0.097	1.056	1.931	0.680
32	15	−0.059	0.361	0.853	0.501	0.071	1.003	1.411	0.566	0.764	0.365	2.054	0.799
	16	−0.077	0.379	0.760	0.636	0.090	1.225	1.520	0.556	0.958	0.371	2.553	0.599
	17	−0.092	0.513	0.765	0.532	0.067	1.182	1.395	0.585	0.914	0.330	2.310	0.597
	18	−0.249	0.285	0.737	0.234	0.033	0.389	0.937	0.554	0.306	0.395	0.910	1.476
	19	−0.095	0.376	0.769	0.463	0.063	0.993	1.281	0.628	0.783	0.339	1.918	0.764
	20	−0.050	0.285	0.726	0.579	0.071	0.998	1.421	0.509	0.782	0.365	2.042	0.747
	Grouped	−0.060	0.358	0.751	0.489	0.065	0.957	1.311	0.566	0.749	0.360	1.947	0.837
33	15	−0.159	0.432	0.864	0.103	0.037	0.214	0.924	0.715	0.184	0.521	0.951	1.970
	16	−0.199	0.426	0.865	0.077	0.025	0.203	0.895	0.782	0.163	0.469	0.860	1.310
	17	−0.618	0.408	0.877	0.086	0.034	0.110	0.873	0.691	0.110	0.639	0.716	1.761
	18	−0.344	0.443	0.834	0.152	0.036	0.339	0.974	0.705	0.268	0.442	1.092	1.108
	19	−0.121	0.635	0.856	0.136	0.043	0.239	0.968	0.669	0.206	0.540	1.014	1.245
	20	−0.159	0.456	0.882	0.117	0.048	0.186	0.987	0.673	0.183	0.595	1.174	1.021
	Grouped	−0.146	0.438	0.863	0.114	0.037	0.215	0.939	0.704	0.186	0.537	0.971	1.289

APPENDIX 2 (*Continued*)

Sine next interval		Quadratic ($c = 0$) next second		Quadratic ($c = 0$) next interval		Quadratic next second			Quadratic next interval		
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*c*	*a*	*b*	*c*
2.104	0.075	−0.275	1.418	0.005	−0.009	−0.281	1.409	0.002	0.005	−0.019	−0.010
2.011	0.076	−0.227	1.421	0.006	−0.022	−0.230	1.428	−0.002	0.006	−0.032	0.048
1.987	0.080	−0.270	1.436	0.005	−0.010	−0.287	1.466	−0.009	0.005	−0.013	0.015
1.946	0.079	−0.249	1.416	0.005	−0.033	−0.263	1.436	−0.005	0.006	−0.057	0.126
2.010	0.081	−0.435	1.506	0.004	−0.002	−0.413	1.475	0.008	0.003	0.018	−0.103
2.034	0.077	−0.224	1.300	0.003	0.004	−0.168	1.222	0.022	0.002	0.020	−0.082
2.019	0.077	−0.281	1.416	0.005	−0.013	−0.272	1.405	0.003	0.004	−0.013	0.001
1.943	0.076	−0.252	1.449	−0.001	0.111	−0.184	1.303	0.077	−0.003	0.191	−0.398
1.896	0.076	−0.256	1.442	−0.003	0.154	−0.155	1.213	1.080	−0.006	0.225	−0.368
1.904	0.078	−0.272	1.410	−0.001	0.093	−0.210	1.290	0.048	−0.003	0.157	−0.327
1.962	0.078	−0.251	1.468	−0.001	0.116	−0.215	1.382	0.040	−0.004	0.202	−0.438
1.865	0.079	−0.268	1.504	0.001	0.089	−0.237	1.430	0.034	−0.002	0.176	−0.446
1.943	0.077	−0.209	1.443	0.002	0.091	−0.175	1.353	0.049	−0.002	0.181	−0.464
1.899	0.076	−0.251	1.453	−0.001	0.107	−0.198	1.334	0.056	−0.003	0.188	−0.409
2.976	0.083	−0.391	1.711	−0.012	0.331	−0.169	1.210	0.321	−0.012	0.359	0.098
2.857	0.084	−0.318	1.810	−0.016	0.420	−0.175	1.302	0.373	−0.015	0.408	0.059
2.688	0.088	−0.280	1.628	−0.016	0.393	−0.110	1.072	0.390	−0.016	0.403	−0.050
2.843	0.081	−0.675	1.588	−0.004	0.131	−0.120	0.868	0.209	−0.004	0.109	0.109
2.746	0.086	−0.361	1.689	−0.011	0.306	−0.276	1.460	0.130	−0.008	0.237	0.253
2.837	0.081	−0.385	1.786	−0.012	0.332	−0.127	1.038	0.460	−0.011	0.303	0.149
2.855	0.081	−0.404	1.700	−0.012	0.316	−0.162	1.148	0.312	−0.011	0.292	0.124
2.121	0.086	−0.438	1.351	−0.004	0.101	−0.156	1.011	0.086	−0.006	0.161	−0.158
2.265	0.084	−0.372	1.233	−0.004	0.099	−0.026	0.887	0.074	−0.006	0.145	0.238
2.078	0.082	−0.677	1.440	−0.002	0.076	−0.415	1.190	0.052	−0.004	0.108	−0.164
2.139	0.086	−0.390	1.365	−0.006	0.147	−0.059	0.902	0.139	−0.008	0.195	−0.245
2.157	0.079	−0.495	1.455	−0.004	0.128	−0.269	1.149	0.090	−0.006	0.177	−0.255
2.164	0.081	−0.362	1.358	−0.005	0.132	−0.192	1.099	0.087	−0.008	0.215	−0.425
2.172	0.083	−0.459	1.375	−0.004	0.116	−0.192	1.045	0.088	−0.006	0.168	−0.170