THE RELATION BETWEEN THE GENERALIZED MATCHING LAW AND SIGNAL-DETECTION THEORY

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The generalized matching law can be applied to a signal-detection matrix to give two equations. The first relates responding in the presence of the stimulus to the reinforcements for the responses, and the second relates responding in the absence of the stimulus to the reinforcements for the responses. Evidence for stimulus discrimination is given by biases that are opposite in sign in the two equations. As the logarithmic ratio and z proportion transformations are similar, the combination of the absolute values of the two logarithmic biases gives a measure equivalent to the signal-detection measures d' and η. The two equations can also be combined to eliminate the biases caused by the signalling stimuli and to produce a generalized matching-law statement relating overall performance to the obtained reinforcements.

Key words: generalized matching law, signal-detection theory, sensitivity, response bias

Over the past two decades, two areas of psychology have rapidly proceeded toward quantification. Signal-detection theory (Green and Swets, 1966) has been concerned with measuring the control of stimuli over behavior, while the matching law (Herrnstein, 1970), and more latterly the generalized matching law (Baum, 1974), describe how behavior is distributed over choice alternatives according to the reinforcement for the choice alternatives. In this paper we shall show the relation between these two fields of endeavor and, more specifically, show how applications of the generalized matching law can provide a measure of both discrimination and overall response bias. We shall proceed by discussing the generalized matching law before moving to an analysis of its relation to signal-detection theory.

The Generalized Matching Law

The usual statement of the generalized matching law (Baum, 1974) is:

$$\frac{P_1}{P_2} = c\left(\frac{R_1}{R_2}\right)^a \qquad (1),$$

where P_1 and P_2 are the numbers of responses emitted to each of two keys or key colors, and

R_1 and R_2 are the numbers of reinforcements obtained on them. The parameter c is known as bias. Bias refers to the tendency for the animal to emit proportionally more responses on one key than on the other when equal reinforcements are obtained, and to maintain this proportional preference over and above the behavior changes associated with reinforcement ratios greater or less than 1.0. The parameter a describes the sensitivity of the ratio allocation of responses to changes in the ratio of obtained reinforcements. If $a = 1$, then apart from bias, the response ratio matches the reinforcement ratio; if $a = 0$, the response ratio is insensitive to changes in the reinforcement ratio; if $a = \infty$, responses are exclusively to the key on which the higher reinforcement rate is available (Lander and Irwin, 1968).

The usual way of estimating the values of a and c in Equation 1 is to fit a straight line by the method of least squares to the logarithmic form of the equation:

$$\log (P_1/P_2) = a \log (R_1/R_2) + \log c \qquad (2).$$

Usual value of a for concurrent variable-interval schedules is between 0.8 and 1.0 (Lobb and Davison, 1975; Myers and Myers, 1977). This value is similar to that found as a Stevens Law exponent for estimation of time by animals (Catania, 1970) and humans (Eisler, 1976).

Signal-detection analyses are more often carried out using logistic transformations, rather

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than logarithmic transformations. The measure log $(x/1 - x)$, where x is a probability or a proportion, is the logistic transformation (Bush, 1963; Fisher and Yates, 1963). Now, Equation 2 can be rewritten, defining p as the proportion of responses occurring on key 1, and r as the proportion of reinforcements obtained on key 1:

$$\log (p/1 - p) = a \log (r/1 - r) + \log c,$$

or as:

$$\text{logit } C = a \text{ logit } E + \log c,$$

where C is proportional choice and E is proportional environmental effects. In other words, the logarithmic transformation of the ratio of two numbers is the logistic transformation.

Bush (1963) demonstrated a close relation between the logistic transformation and the z- or cumulative-normal transformation used extensively by Green and Swets (1966) in their signal-detection analyses. Dusoir (1975) and Ogilvie and Creelman (1968) also point out that the results of a logistic analysis of signal-detection data are difficult to discriminate from a z-transform analysis. Thus, the different data transformations (log, z) used in the generalized matching law and signal-detection analyses lead to no fundamental difference in the results of the analyses, except insofar as the units of the derived measures differ.

The Generalized Matching Law Applied to Signal Detection

In signal-detection experiments, a stimulus is presented on some choice trials. If this stimulus is discriminated, and the animal emits the correct response, reinforcement is obtained. Correct responses consist either of reporting the presence of the stimulus (hit) or its absence (correct rejection). Incorrect responses consist of either reporting the stimulus as present when it was absent (false alarm) or of reporting it absent when it was present (false rejection). In the usual generalized matching-law experiment, only two responses are counted. Because of the added stimulus dimension in signal-detection experiments, four response classes are defined and recorded. The two responses (yes, no) and the two stimulus conditions (present, absent) define the columns and rows of a 2 × 2 matrix. This matrix is shown in Figure 1 with the frequencies (*not* the probabilities) of each event shown in each cell.

RESPONSE

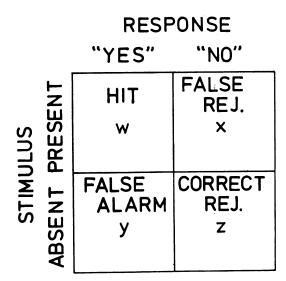|	"YES"	"NO"
STIMULUS PRESENT	HIT w	FALSE REJ. x
ABSENT	FALSE ALARM y	CORRECT REJ. z

Fig. 1. The matrix of stimulus and response events from a signal-detection experiment. The *numbers* of events occurring in each cell are shown by the letters w, x, y, and z.

The generalized matching law relates the ratio of frequencies of two responses to the ratio of frequencies of reinforcement for them. In the typical signal-detection experiment, every correct response is reinforced, so the numbers w and z are the numbers of reinforcements as well as the numbers of responses. The generalized matching law predicts that the ratio of *yes* to *no* responses will be controlled by the reinforcement ratio (w/z), whether or not the signalling stimuli are presented. If the response ratios were analyzed without regard to the presence or absence of the signalling stimuli, we would obtain:

$$\log\left(\frac{w + y}{x + z}\right) = a \log\left(\frac{w}{z}\right) + \log c \qquad (3),$$

where c is a bias (Baum, 1974) toward emitting one of the two responses.

If the subject fails to discriminate the presence from the absence of the stimulus, the same overall response bias (log c in Equation 3) should be shown in the performance when the stimulus is present and when it is absent. However, if the subject's behavior is controlled by the presence and absence of the stimulus, the behavior will be biased toward saying *yes* when the stimulus is present, and toward saying *no* when it is absent. Assuming, therefore, that (1) the emission of *yes* and *no* responses is controlled by the reinforcements for these responses; (2) that the performance may, for

extraneous reasons, be biased toward one or the other response; and (3) that the presence or absence of a signalling stimulus will bias performance toward the response *yes* in the stimulus and toward *no* in its absence, we can write two equations. In the presence of the stimulus,

$$\log\left(\frac{w}{x}\right) = a_{r_1} \log\left(\frac{w}{z}\right) + \log c + \log d \qquad (4),$$

and in the absence of the stimulus,

$$\log\left(\frac{y}{z}\right) = a_{r_2} \log\left(\frac{w}{z}\right) + \log c - \log d \qquad (5),$$

where log d is the bias caused by the signalling stimulus, and a_r is the sensitivity to reinforcement. The stimulus bias term is negative in the second equation because the bias in the absence of the stimulus is toward saying *no,* while Equation 5 has *yes* responses as the numerator.

Preserving the form of the matrix in Figure 1, Equations 4 and 5 can be generalized to the case in which reinforcements are intermittently scheduled for responses. The appropriate design here is a multiple (concurrent variable-interval extinction) (concurrent extinction variable-interval) schedule in which the component currently providing reinforcement is signalled. If P refers to responses and R to reinforcements, subscripted as in Figure 1,

$$\log\left(\frac{P_w}{P_x}\right) = a_{r_1} \log\left(\frac{R_w}{R_z}\right) + \log c + \log d \qquad (6)$$

and

$$\log\left(\frac{P_y}{P_z}\right) = a_{r_2} \log\left(\frac{R_w}{R_z}\right) + \log c - \log d \qquad (7).$$

We shall further assume that the bias (log d) caused by the signalling stimuli is a function of the ratio of the physical stimuli (S_1/S_2) signalling the availability of reinforcement, and that under conditions where stimulus differences are manipulated these would change behavior with a sensitivity a_s. Thus,

$$\log\left(\frac{P_w}{P_x}\right) = a_{r_1} \log\left(\frac{R_w}{R_z}\right) + a_{s_1} \log\left(\frac{S_1}{S_2}\right) + \log c \qquad (8).$$

and

$$\log\left(\frac{P_y}{P_z}\right) = a_{r_2} \log\left(\frac{R_w}{R_z}\right) - a_{s_2} \log\left(\frac{S_1}{S_2}\right) + \log c \qquad (9)$$

The stimulus aspect of these equations is consistent with the generalized matching-law ap-

proach and is, in fact, a statement of Stevens Law (Stevens, 1957).

If Equation 9 is subtracted from Equation 8, assuming that the sensitivities to reinforcement a_r and to stimuli a_s are the same in the presence and in the absence of the stimuli, we obtain, after some rearrangement:

$$\log\left(\frac{P_w}{P_x}\right) = \log\left(\frac{P_y}{P_z}\right) + 2a_s \log\left(\frac{S_1}{S_2}\right) \qquad (10).$$

This equation is an isosensitivity curve that shows the covariation of correct responses in the stimulus with incorrect responses in its absence when reinforcement rates are varied but the stimuli are kept constant. It is analogous to Green and Swets' (1966) isosensitivity equation which, in the present notation is:

$$z\left(\frac{P_w}{P_w + P_x}\right) = a\, z\left(\frac{P_y}{P_y + P_z}\right) + z(d') \qquad (11),$$

or, cast into logs,

$$\log\left(\frac{P_w}{P_x}\right) = a \log\left(\frac{P_y}{P_z}\right) + \log(d').$$

Bush, Luce, and Rose (1964) and Luce (1963) derived a similar equation which, if S_1 and S_2 are symmetrically similar, can be written in the present notation:[2]

$$\frac{P_x}{P_w} \cdot \frac{P_y}{P_z} = \eta^2.$$

Rearranging and taking logs:

$$\log\left(\frac{P_w}{P_x}\right) = \log\left(\frac{P_y}{P_z}\right) - 2 \log \eta \qquad (12).$$

Since η is defined as falling between 0 and 1, the term $-2 \log \eta$ will be positive. Equation 12 is therefore also analogous to Equation 10.

It is of particular interest that essentially the same equation describing the isosensitivity relation in signal detection can be derived from considerations of asymptotic schedule performance (the generalized matching law), from considerations of theoretical stimulus distributions (Green and Swets, 1966), and from the asymptotic behavior of linear operator models for learning (Bush *et al.,* 1964). From the point of view of the generalized matching law, interest centers on the way in which the measure of discriminability is independent of both response bias (log c) and the ratio of reinforcements obtained for the

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[2]Many thanks to Tony Nevin who pointed out that a similar approach had been taken by Luce and his co-workers.

two responses, given the above assumptions. If, for constant stimuli, the discrimination term in Equation 10 was not constant, an analysis using Equations 8 and 9 would help show which assumption was not met. Summarizing Equation 10, we can say that the signal-detection measure of discriminability is independent of both response bias and reinforcements, and that the measure is a combination of two generalized matching law biases, bias for saying *yes* in the stimulus and for saying *no* in its absence.

Equations 8 and 9 can also be added to produce the equivalent of an isobias function:

$$\log\left(\frac{P_w}{P_x}\right) + \log\left(\frac{P_y}{P_z}\right) = 2a_r \log\left(\frac{R_w}{R_z}\right) + 2\log c \quad (13).$$

Bush *et al.* (1964) gave a similar isobias function derived from their linear operator model. They stated, in the present terminology,

$$\log\left(\frac{P_w}{P_x}\right) + \log\left(\frac{P_y}{P_z}\right) = -2\log b.$$

Again, *b* is bounded by 0 and 1, so the term −2 log b is positive. Luce (1963) related *b* to the payoffs and instructions in the experiment, and thus it is equivalent to the reinforcement term in Equation 13. The difference between Luce's equation and Equation 13 is the provision in the latter for a term measuring response bias caused by extraneous factors.

The right-hand side of Equation 13 is the standard generalized matching-law combination. The left of the equation, while it measures the behavior of saying *yes* relative to saying *no*, is different from the simple summation of *yesses* and *nos* in Equation 3. It is, rather, the concatenation of the *yes* to *no* ratios in the presence and the absence of the stimulus according to the same rules given by Baum and Rachlin (1969) for the combination of behavior-effecting independent variables. This suggests that the application of the generalized matching law to the signal-detection matrix using Equation 3 would be incorrect, and that Equation 13 should be used.

An interesting possibility occurs here. If the existence of different controlling stimuli at different times requires the summation of the log ratios, rather than the log of the sum of responses, for a generalized matching-law fit, our usual measures of responding in concurrent schedules would be in error if there were discriminable periods of different reinforcement rates. Some VI schedules may provide predictably different reinforcement rates immediately after reinforcement, compared with those rates longer times after reinforcement. Some may even specify no reinforcement availability immediately after a reinforcement. If performance is measured simply as sums of responses, discriminable periods like this will give an undermatching relation to the ratio of reinforcements. Baum (personal communication) has wondered why this laboratory more often finds undermatching in concurrent schedules than his own laboratory. The reason could be that we use arithmetic variable-interval schedules in which the shortest interval is directly related to the average interval. As a result, the possibly discriminable periods of no reinforcement after a reinforcement has been obtained would get relatively longer with greater differences in reinforcement rates for the two responses. This explanation of undermatching in terms of an unwanted discrimination appears to conflict with Baum's (1974) explanation in terms of lack of discriminability between components.

Both of these explanations could be correct. Baum (1974) suggested that the exponent of Equation 1 (sensitivity) would decrease if animals could not discriminate the components (presumably either stimulus or reinforcement aspects) of a concurrent schedule. The present account suggests that bias in the presence and absence of the stimulus varied when stimuli were changed. Why should changing the stimuli affect sensitivity in one case and not in the other? The usual concurrent schedule requires stimulus control of the choice response (it must be clear, for example, which of the two responses the animal is emitting) and control by differences in the reinforcement rates for the two responses. If stimulus control of the choice response is lost because the defining stimuli become more similar, sensitivity of the response to reinforcement must also be lost, as it will become unclear which response produced reinforcement. Signal detection requires that the discrimination between the choice responses and between the reinforcements is precise, and it adds a further stimulus (the signal) that provides information on which response will be reinforced at any time. The present paper discussed variations in the signal, while Baum (1974) discussed variations in the stimulus control of the choice response.

It is clear that in a signal-detection experiment, when the presence and the absence of the signal cannot be discriminated, the choice response can still be sensitive to reinforcement changes.

Date Analysis

Two sets of data were analyzed in Table 1 to illustrate the use of the equations developed here. The first data were reported by Green and Swets (1966, p. 90) and were obtained from a human by varying the probability of a signal while maintaining constant outcomes for correct responses. The second set of data was reported by Stubbs (1976) from an experiment in which three pigeons discriminated short- and long-duration stimuli. Three analyses were carried out. The first used Equations 4 and 5 and was done to show that opposing biases were produced in the stimulus-present and absent conditions. The data were fitted to each equation by the method of least squares, giving values of a_r and a composite additive constant ($\log c \pm \log d$). The second analysis

used Equation 10 in the same way, and values for the term $2a_s \log(S_1/S_2)$ were obtained. The value of this term is, of course, also the obtained value of $-2 \log \eta$ in Equation 12. Finally, a traditional analysis using z-transforms of response probabilities (Equation 11) was carried out, and obtained values of d' are shown in Table 1, both in the usual units (standard deviations) and, to aid comparison, in the logarithmic equivalents.

Table 1 shows that an analysis by Equations 4 and 5 accounted for 95% or more of the data variance for each subject. The sensitivities to reinforcement, a_r, were generally similar for stimulus present and stimulus absent (or short *versus* long durations), and in each case there was a strong bias toward emitting the *yes* response in the stimulus, and away from emitting this response in its absence. If the additive constants ($\log c \pm \log d$) were equal in absolute value, there would be no response bias ($\log c$). The additive constants were approximately equal for all subjects except Stubbs' Bird 2, which showed a strong bias toward reporting the short stimulus.

Table 1

Analysis of data reported by Green and Swets (1966) and by Stubbs (1976) according to Equations 4 and 5, Equation 10 and Equation 11. VAC is the percentage of data variance accounted for by the least-squares fitted lines.

1. Equations 4 and 5

Author	Stimulus	a_r	$\log c \pm \log d$	VAC
Green & Swets	Present	0.49	0.34	99
	Absent	0.48	−0.29	99
Stubbs—62	Short	0.54	0.61	98
	Long	0.71	−0.55	98
63	Short	0.83	0.64	99
	Long	0.79	−0.46	98
2	Short	0.93	0.79	99
	Long	0.41	−0.36	95

2. Equation 10

Author	Slope	$2a_r \log (S_1/S_2)$	VAC
Green & Swets	0.96	0.64	97
Stubbs—62	1.05	1.14	92
63	1.10	1.14	99
2	2.50	1.56	80

3. Equation 11

Author	Slope	d' (z-units)	d' (log units)	VAC
Green & Swets	1.00	0.87	0.62	99
Stubbs—62	1.09	1.55	1.19	86
63	1.17	1.53	1.17	97
2	2.25	1.98	1.61	76

The analysis by Equation 10, shown in Table 1, gave good fits for all subjects except Stubbs' Bird 2. The intercepts of the fitted lines gave values of $2a_s \log(S_1/S_2)$ or $-2 \log \eta$ that were much the same as the values of d', transformed into logs, obtained from the usual z-transform analysis. The data for Stubbs' Bird 2 suggest that the assumption that the sensitivity to reinforcement in the presence and absence of the signal was the same may not always be met.

Conclusion

We have shown how the measure of discriminability suggested by Green and Swets (1966) and by Luce and his coworkers (Bush, Luce, and Rose, 1964; Luce, 1959, 1963) can be derived also from applications of the generalized matching law to the signal-detection matrix. We believe that the present formulation offers some advantages over the previous models. First, it easily encompasses intermittent reinforcement. Second, it shows how response bias can enter into the isobias curve. Third, it makes the role of reinforcement in signal detection more clear, especially in Equation 13. Fourth, it quantitatively relates two areas of psychology that have been developing apart (Nevin, 1969).

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