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IS MATCHING COMPATIBLE WITH REINFORCEMENT MAXIMIZATION ON CONCURRENT VARIABLE INTERVAL, VARIABLE RATIO?[1]

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Four pigeons on concurrent variable interval, variable ratio approximated the matching relationship with biases toward the variable interval when time spent responding was the measure of behavior and toward the variable ratio when frequency of pecking was the measure of behavior. The local rates of responding were consistently higher on the variable ratio, even when there was overall preference for the variable interval. Matching on concurrent variable interval, variable ratio was shown to be incompatible with maximization of total reinforcement, given the observed local rates of responding and rates of alternation between the schedules. Furthermore, it was shown that the subjects were losing reinforcements at a rate of about 60 per hour by matching rather than maximizing.

Key words: concurrent VI VR, matching, maximization, reinforcement, pigeons

In psychology, biology, and economics it is often assumed that behavioral adaptation entails optimization of some variable or set of variables, such as reinforcement rate, fitness, or wealth (Lea, 1978). In operant psychology one version of this view has motivated theories of concurrent schedule performance. It is, for example, argued that the matching relationship between choice proportions and reinforcement proportions is the result of some process that maximizes reinforcement rate (e.g., Mackintosh, 1974; Shimp, 1975). According to one version of maximization theory, the subject is said to choose between simultaneously available reinforcement schedules so as to produce the greatest reinforcement rate, summing across both schedules (e.g., Rachlin, Green, Kagel, & Battallio, 1976). This account is also implied in Baum's (1973) feedback theory of schedule performance and in recent ethological work on the economics of behavior (e.g., Rapport & Turner, 1977). As we show, performance in a concurrent variable-interval variable-ratio (*conc* VI VR) schedule provides a convenient test of the overall reinforcement rate maximizing hypothesis.

Performance on concurrent schedules generally conforms to the matching relation (de Villiers, 1977) described by the equation (Baum, 1974),

$$\frac{B_1}{B_2} = a\left[\frac{R_1}{R_2}\right]^b, \qquad (1)$$

where B_i is a measure of performance at alternative i, say, response rate or time spent responding, R_i is the obtained reinforcement rate at alternative i, and a and b are empirical constants which both equal 1.0 in the normative form of the relationship. A log transformation of Equation 1 produces the linear equation:

$$\log(B_1/B_2) = b \log(R_1/R_2) + \log a. \qquad (2)$$

Baum (1974) suggested that a measures bias, which he defined as the amount of preference not accounted for by the explicitly measured reinforcement, and that b measures the relative value of units of R_1 and R_2 as discriminated by the subject. Equations 1 and 2 will serve as vehicles for fitting the present results.

A common approach to the maximization hypothesis is to show that results described by Equation 1 can also be described by an optimization analysis of some sort. However, another approach is to devise procedures that prohibit simultaneous matching and reinforcement maximization and to see which principle, if either, controls performance. As we demon-

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strate below, the *conc* VI VR is just such a procedure and is therefore particularly useful in the ongoing assessment of the role of maximization in the matching relation.

METHOD

Subjects

Four White Carneaux pigeons, maintained at 80% of their free-feeding weights, served. The birds had previously participated in other experiments involving concurrent schedules.

Apparatus

A standard two-key operant conditioning chamber (31 cm by 31 cm by 33 cm) housed the subjects during sessions. The response keys (Gerbrands) were 23 cm from the floor, 16 cm apart, and required a force of more than .15 *N* to be operated. Each effective response produced a brief auditory feedback click. The left key was transilluminated white, the right key could be transilluminated red or green. Reinforcement consisted of 3.2-sec access to the grain hopper, which was illuminated during reinforcement and was located midway between the two response keys and 9 cm from the floor. The experimental chamber was enclosed in a sound attenuating box and lit by two 28-V dc lamps. White noise was piped into the box to mask extraneous sounds.

Procedure

Reinforcement was scheduled by a changeover-key concurrent procedure (Findley, 1958). Responses at the right key, designated the main key, were occasionally reinforced with grain. A single response at the left key, designated the changeover key, alternated the key color and available schedule at the main key. When the main key was red, responses were reinforced on the VR schedule; when the main key was green, responses were reinforced on the VI schedule. The two schedules, however, were concurrent in the sense that the VI timer ran during the period when the subject had access to the VR schedule. While the subject was working on the VR schedule, a reinforcer could therefore be pending on the VI schedule. The changeover key was always illuminated white, and a changeover response initiated a 1.5-sec changeover delay (COD), which imposed a minimum interval of 1.5 sec between a schedule alternation and a reinforced response.

Table 1

Order of schedule parameters and number of sessions in force.

Schedule		
VI in seconds (green key)	*VR in responses (red key)*	*Sessions*
VI 30	VR 30	78
VI 15	VR 30	51
VI 40	VR 30	35
VI 40	VR 45	53
VI 40	VR 60	69
VI 30	VR 30	118

The intervals on both the VI and VR schedules (Fleshler & Hoffman, 1962) were approximately exponentially distributed for time and responses respectively. In the course of the study the interval lengths were varied systematically. Table 1 lists the nominal mean values, their order, and the number of sessions the subjects were exposed to each pair. Sessions were terminated after 60 reinforcers.

RESULTS

Table 2 summarizes the main results, based on data averaged from the last 10 sessions of each condition. When the birds approached exclusive preference for the VI schedule and rarely sampled the VR, the obtained ratio values (column 2) sometimes differed substantially from the scheduled ones or were indeterminate. If indeterminate, the table shows a blank. Overall session time, exclusive of reinforcements, is given by the sum of the times in column 3.

Figures 1 and 2 show the logarithm of the response (column 4) and time (column 3) ratios as a function of the logarithm of the reinforcement (column 6) ratios for each subject, omitting conditions with indeterminate values of VR. The ratios were calculated from the numbers in Table 2, and the data are displayed so that the measures for the VI schedule are divided by the measures for the VR schedule. The solid lines were fitted to the data by the method of least squares, and the dashed lines indicate the values predicted by Equation 2 with *a* and *b* equal to 1.0, which is the logarithm of the normative matching relationship. The equation for the best fitting line and the proportion of variance accounted for accompanies each set of data.

Table 2

Summary of the Results: 10-Session Averages

	1	2	3		4		5		6		7
	Schedule parameters[a]	*Obtained VR ratio (resps/ reinfs)*	*Time (min.)*[b]		*Responses*[c]		*Local resp. rate*[c] *(resps./min)*		*Reinforce-ments*		
Subject			*VI* (T_1)	*VR* (T_2)	*VI* (H_1)	*VR* (H_2)	*VI*	*VR*	*VI* (R_1)	*VR* (R_2)	*Changeovers*
3	VI 30 VR 30	32	5.96	9.45	529	1,088	89	115	25	35	121
	VI 15 VR 30	—	15.85	—	940	—	59	—	60	—	—
	VI 40 VR 30	31	8.79	9.49	678	1,089	77	115	24	36	83
	VI 40 VR 45	42	20.42	6.80	1,390	874	68	129	39	21	61
	VI 40 VR 60	65	32.22	3.90	2,116	413	66	106	53	7	72
	VI 30 VR 30	31	16.82	5.45	1,188	633	71	116	40	20	49
83	VI 30 VR 30	31	21.71	4.61	1,285	410	59	89	47	13	68
	VI 15 VR 30	—	16.24	—	627	—	39	—	60	—	—
	VI 40 VR 30	33	36.49	2.49	1,441	130	40	52	56	4	28
	VI 40 VR 45	40	37.41	1.50	1,564	121	42	81	57	3	13
	VI 40 VR 60	—	40.86	—	1,871	—	46	—	60	—	—
	VI 30 VR 30	31	17.67	5.45	894	639	51	117	40	20	33
365	VI 30 VR 30	31	4.84	7.18	415	1,245	86	173	20	40	66
	VI 15 VR 30	31	5.09	4.23	565	869	111	205	32	28	51
	VI 40 VR 30	31	2.60	7.98	223	1,529	86	192	11	49	37
	VI 40 VR 45	41	6.93	10.89	594	1,545	86	142	22	38	87
	VI 40 VR 60	63	24.57	9.38	2,024	741	82	79	48	12	106
	VI 30 VR 30	31	4.63	7.42	372	1,270	80	171	19	41	58
473	VI 30 VR 30	35	29.77	1.10	1,308	130	44	118	56	4	12
	VI 15 VR 30	—	15.71	—	705	—	45	—	60	—	—
	VI 40 VR 30	33	32.70	1.96	1,441	239	44	122	52	8	28
	VI 40 VR 45	43	39.09	0.44	1,655	30	42	68	59	1	9
	VI 40 VR 60	—	39.17	—	1,850	—	47	—	60	—	—
	VI 30 VR 30	34	27.47	2.10	1,393	225	51	107	53	7	23

[a] Nominal values.

[b] Time exclusive of reinforcement time.

[c] Responding to main key.

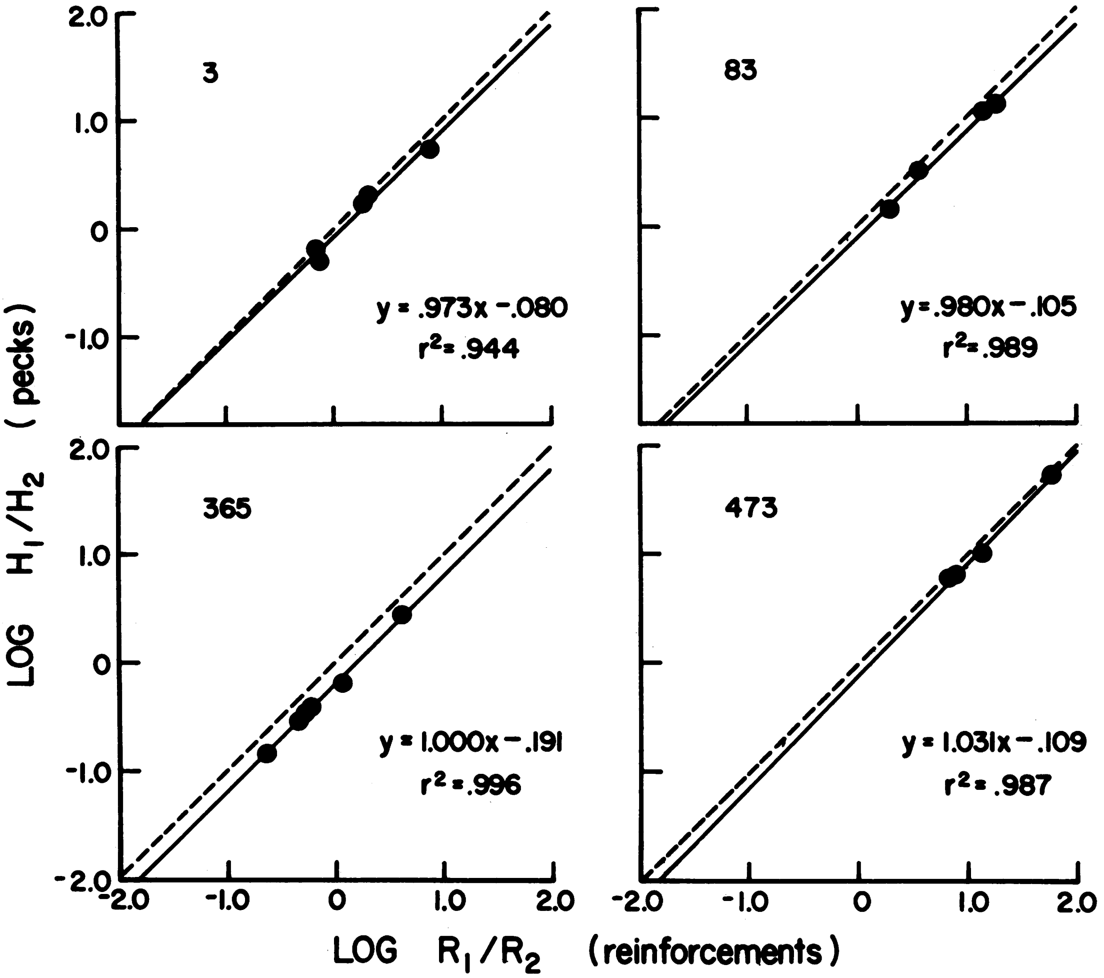

Fig. 1. Interval pecks over ratio pecks (H_1/H_2) as a function of interval reinforcements over ratio reinforcements (R_1/R_2) for each subject. Solid lines show best fits of Equation 2 as given along with proportion of variance accounted for (r^2); dashed lines show normative matching relationship.

For response ratios, the slopes fitted to the individual subjects approached 1.0 closely, as predicted by the matching equation with the parameter $b = 1.0$. The intercepts are negative, which indicates that the subjects responded more on the VR schedule than predicted by the reinforcement ratios and that the parameter a was less than 1.0. For time ratios, the slopes fell below 1.0 for three of the four subjects. The intercepts were positive for all subjects, indicating $a > 1.0$. The subjects therefore spent more time responding on the VI schedule than predicted by the reinforcement ratios.

The parameter values for individual subjects are suspect because of the small number and narrow range of reinforcement ratios, barely covering one log unit for each subject. Pooling data is one way to ameliorate this problem, for the independent variable would then cover approximately 2.5 log units with 19 values of R_1/R_2. Figures 3 and 4 show the best fits of Equation 2 to the pooled data, as well as proportions of variance accounted for, for response and time measures, respectively.[3]

Although the equation is being fitted to the data from several subjects, pooling differs from averaging. The individual data points are not lost in a pool, but still appear as more or less dispersed around fitted functions. Because the

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[3] The functions for pooled data differ slightly from Baum's (1974) fits to these results because of a small rounding error, now corrected, in the calculation of reinforcement frequencies.

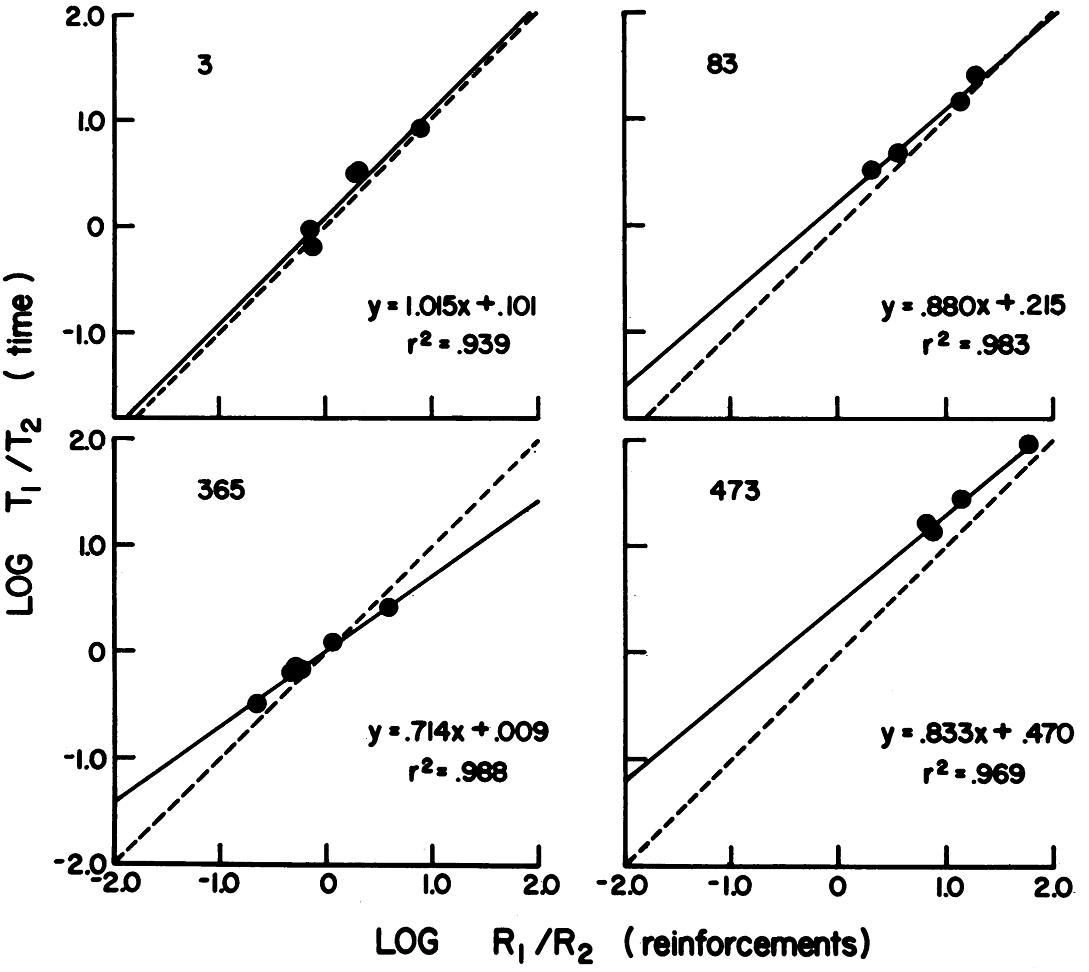

Fig. 2. Same as Figure 1, except that interval time-spent-responding over ratio time-spent-responding (T_1/T_2) is on the y-axis.

variance accounted for by the pooled functions in Figures 3 and 4 is not substantially lower than that by the individual functions in Figures 1 and 2, we may conclude that interindividual differences are a relatively small source of variance. The slopes for both pooled time and response ratios approximated 1.0 closely, while the intercepts showed the same pattern as did the individual intercepts, with $a < 1.0$ for responses and $a > 1.0$ for time spent responding. The line of lower slope in Figure 4 is explained in the Discussion.

Table 3 assesses the day-to-day variability for individual subjects. It presents sample standard deviations (using $\sqrt{n-1}$) of the daily performance for the 10 sessions summarized in Table 2 and Figures 1 to 4. Note that, for convenience in calculation, the measures of performance used here were relative frequencies [e.g., $H_1/(H_1 + H_2)$] rather than the ratios (e.g., H_1/H_2) in Figures 1 to 4. The means of the standard deviations are shown to facilitate comparisons. In general, the standard deviations ranged between .05 to .1, comparable to other studies of the matching law. For three of the four subjects, relative time spent responding was slightly less variable than relative responding. However, neither measure was markedly more variable than relative reinforcement itself.

DISCUSSION

Figures 1 to 4 show that the generalized matching law (Equation 1) described the distri-

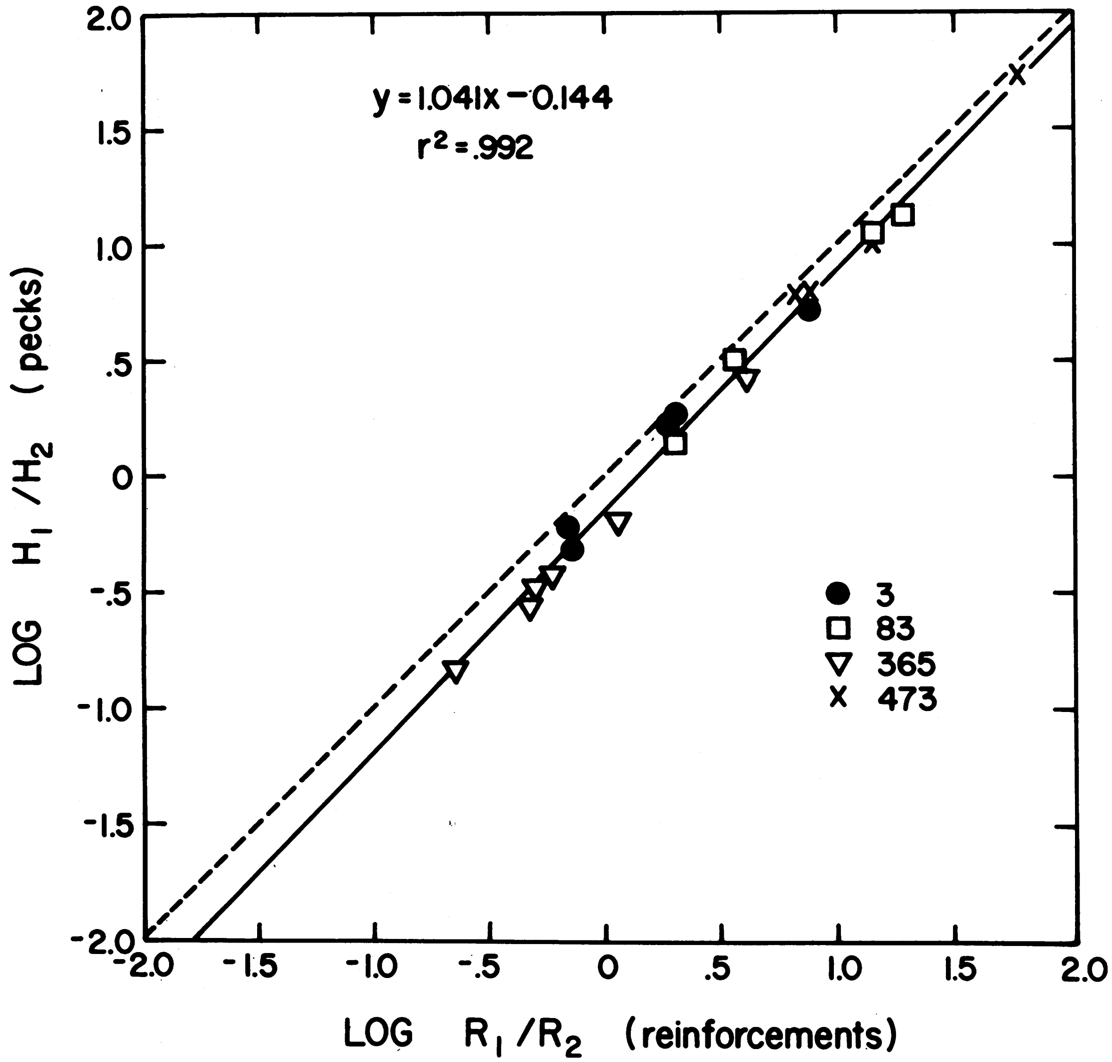

Fig. 3. Equation 2 fitted to H_1/H_2 vs. R_1/R_2 for all subjects pooled.

Table 3

Sample Standard Deviations of Daily Relative Reinforcement, Responses, and Time

Schedule parameters		*Subjects*											
		3			365			83			473		
VI	*VR*	$\frac{R_1^{a}}{R_1+R_2}$	$\frac{H_1^{b}}{H_1+H_2}$	$\frac{T_1^{c}}{T_1+T_2}$	$\frac{R_1}{R_1+R_2}$	$\frac{H_1}{H_1+H_2}$	$\frac{T_1}{T_1+T_2}$	$\frac{R_1}{R_1+R_2}$	$\frac{H_1}{H_1+H_2}$	$\frac{T_1}{T_1+T_2}$	$\frac{R_1}{R_1+R_2}$	$\frac{H_1}{H_1+H_2}$	$\frac{T_1}{T_1+T_2}$
30	30	.029	.029	.030	.030	.029	.070	.073	.102	.063	.093	.121	.054
15	30	—	—	—	.046	.048	.063	—	—	—	—	—	—
40	30	.088	.129	.152	.032	.031	.073	.093	.104	.068	.062	.066	.026
40	45	.045	.062	.054	.062	.058	.094	.069	.078	.048	.021	.022	.008
40	60	.064	.089	.066	.115	.160	.184	—	—	—	—	—	—
30	30	.161	.177	.150	.025	.028	.036	.048	.080	.047	.073	.081	.041
Mean:		(.078)	(.097)	(.090)	(.052)	(.059)	(.087)	(.071)	(.091)	(.057)	(.062)	(.073)	(.032)

[a] *SDs* for relative frequency of reinforcement.
[b] *SDs* for relative frequency of responding.
[c] *SDs* for relative time spent responding.

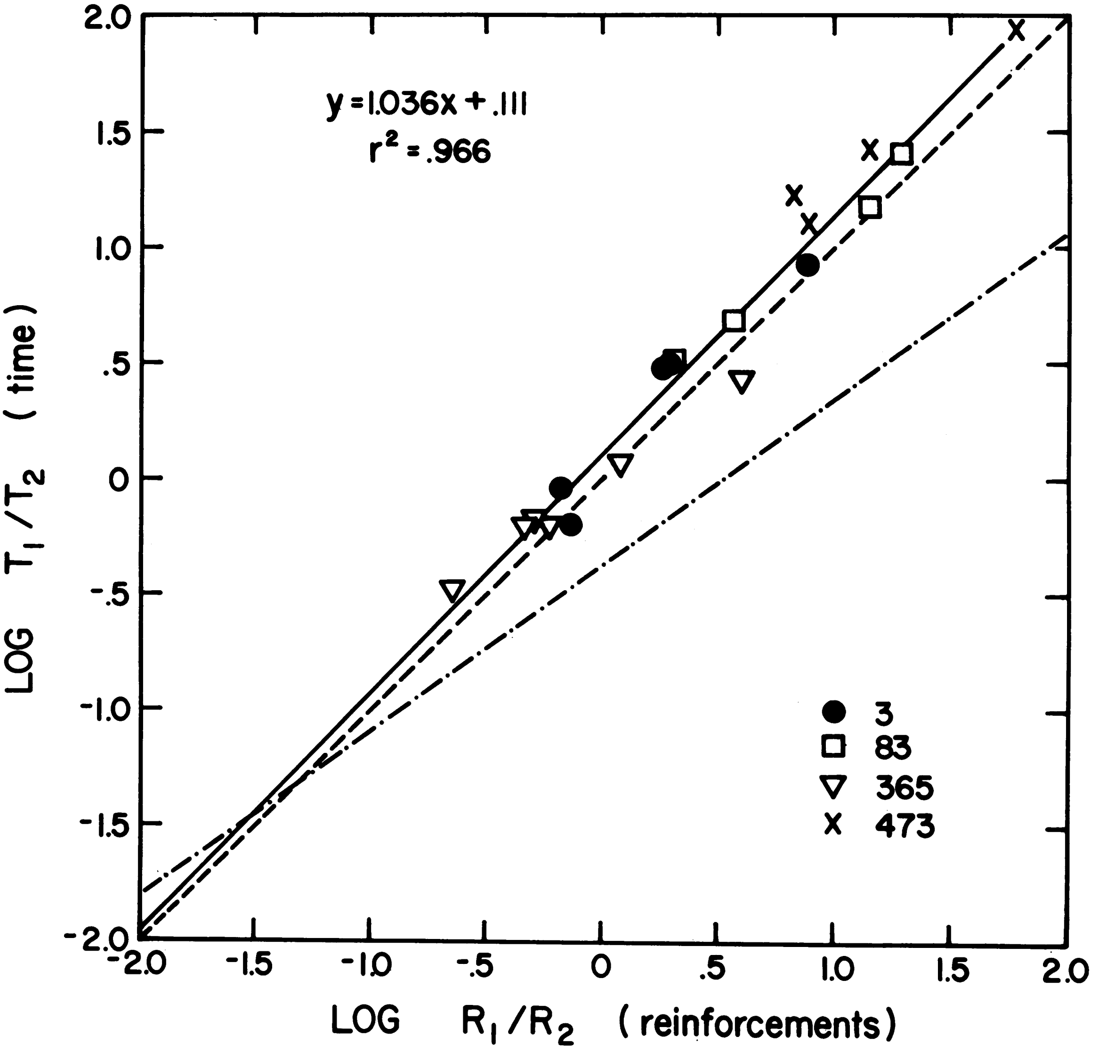

Fig. 4. Equation 2 fitted to T_1/T_2 vs. R_1/R_2 for all subjects pooled. Line of lowest slope and negative intercept explained in Discussion.

bution of responses and time spent responding for four pigeons on six different *conc* VI VR schedules, using either individual or pooled data. The amount of variance accounted for was above .96 for eight of the 10 fitted functions and above .939 in all cases. According to the interpretation that *b* of Equation 2 (the slope) scaled the relative value of the two reinforcers, pooled peck and time ratios (Figures 3 and 4) suggest that the pigeons were indifferent to whether the reinforcers came from the VI or the VR, since *b* approximated 1.0.

The intercepts of Equation 2, log *a*, were negative for peck ratios and positive for time ratios. This means that the reinforcement ratios overestimated by a constant proportion the frequency of pecking on the VI schedule and underestimated by a constant proportion the amount of time spent responding on the VI schedule. Specifically, the best fit of Equation 1 to the pooled response data is given by multiplying the reinforcement ratios by .718, and the best fit to the pooled time data is given by multiplying the reinforcement ratios by 1.291. Although both these biases fall within the systematic relation expressed by the generalized matching law, Equation 1, it is not obvious why the response measure should be biased (in Baum's sense of bias, viz. Baum, 1974) toward the ratio schedules while the time measure should be biased toward the interval. It may help to clarify the relationship of these biases to local response rates on the two schedules.

As is standard procedure for concurrent schedules, Equation 2 has been fitted to measures of behavior and reinforcement over the entire session. To obtain local response rates, we must divide responses to an alternative by time spent working at it. However, responses to each alternative and time spent working there have already been accounted for by Equation 1. Figures 3 and 4 illustrated the following two relationships (in log coordinates):

$$\frac{H_1}{H_2} = a_H \left[\frac{R_1}{R_2}\right]^{b_H} = .718 \left[\frac{R_1}{R_2}\right]^{1.041}, \qquad (3)$$

and

$$\frac{T_1}{T_2} = a_T \left[\frac{R_1}{R_2}\right]^{b_T} = 1.291 \left[\frac{R_1}{R_2}\right]^{1.036}. \qquad (4)$$

Equation 3 is for pecks; Equation 4, for time spent responding; subscripts 1 and 2 refer to interval and ratio responding, while H and T refer to responses and time spent responding as measures of behavior, respectively. Dividing Equation 3 by Equation 4 yields the following:

$$\frac{\dfrac{H_1}{T_1}}{\dfrac{H_2}{T_2}} = \frac{a_H}{a_T} \left[\frac{R_1}{R_2}\right]^{b_H - b_T} = .556 \left[\frac{R_1}{R_2}\right]^{.005}. \qquad (5)$$

The term on the left is the ratio of local rates

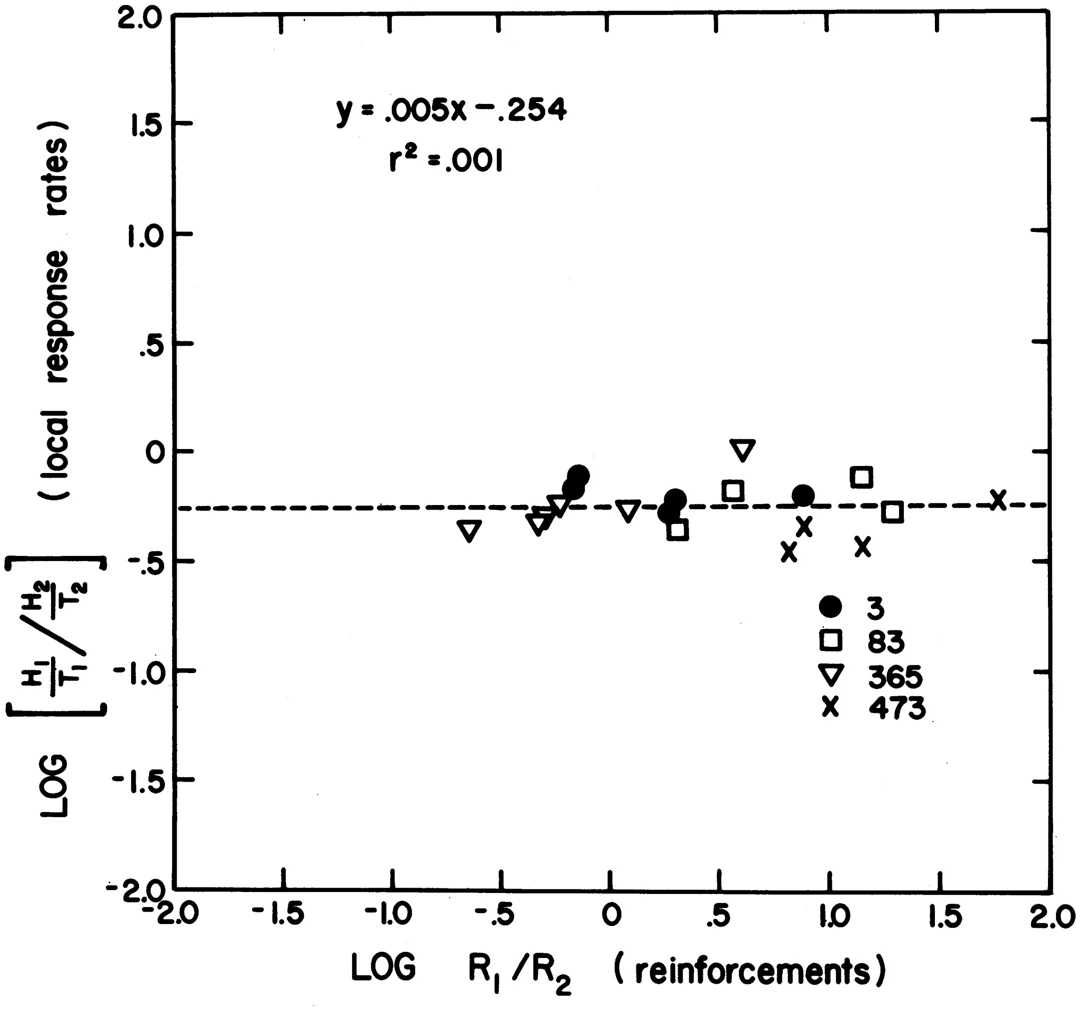

Fig. 5. Quotient of local response rates on interval and ratio schedules as a function of R_1/R_2. Equation 5 fitted to all subjects pooled.

of pecking on interval and ratio schedules. The variable on the right, R_1/R_2, is the ratio of overall reinforcement frequency from interval and ratio schedules.

Since the logarithm of Equation 5 is linear, we have plotted the relevant data in Figure 5 in logarithmic form. The line shows Equation 5 fitted to the present data from pooled individual subjects. Had the exponents in Equations 3 and 4 been equal, the exponent in Equation 5 would have been zero and would therefore have plotted as a horizontal line, indicating that the local rate of interval responding was a constant fraction (.56) of the local rate of ratio responding. The small exponent (.005) in Equation 5 results in negligible positive slope in Figure 5, far too small to be detected in the data, given its variability. Variance accounted for is also negligible (.001) inasmuch as there is virtually no correlation between the plotted variables. Local response rate on the variable interval is lower than that on the variable ratio for every condition but one, and their quotient varies around the predicted logarithmic value of −.25 (the logarithm of .56). Fitting Equation 5 to the subjects individually results in low positive slopes for three of them and a low negative slope for Subject 3. The best fitting individual quotients of local rates of responding range from .27 to .66 and average at .51, which is close to the best fitting quotient for the pooled data.

Rachlin (1973) has noted that in concurrent VI VI, matching is approximated by equal proportional adjustments in both responses and time spent responding. Consequently, the local response rates to the alternatives remain equal to each other. Figure 5 generalizes this principle. Here local response rates are unequal, but their ratio remains virtually invariant over the range of R_1/R_2 and of T_1/T_2. Whether the subject is spending more time at the interval or ratio or getting more reinforcement from one schedule or the other, the local rate of interval pecking remains approximately 56% of that of ratio pecking, pooled over subjects.

Rachlin noted further that multiple schedules differ from concurrent schedules in that by definition, they do not allow the subject to redistribute time. When the subject matches or, more commonly, undermatches on multiple schedules, responses are redistributed, but not time.[4] Consequently, the local response rates as usually measured must change as the components receive varying proportions of the total responding. However, with equal reinforcement rates from the two components, a *mult* VI VR should produce response rates in the same ratio as found here with local rates of responding because at equal reinforcement any difference must be due to schedule effects alone.

The relevant experiment was done by Zuriff (1970). In a parametric study of *mult* VI VR, Zuriff found the usual undermatching for multiple schedules and the usual elevation of ratio responding relative to interval responding. When the two components reinforced equally, the average interval response rate was approximately .59 of the average ratio response rate (estimated from Figure 1 in Zuriff, 1970), close to the value found here (.56).

Another relevant set of data was reported in Herrnstein (1964), describing a concurrent chain procedure in which the first links were variable-interval schedules and one second link was a variable interval whereas the other was a variable ratio. The two second links therefore constituted a *mult* VI VR with frequencies of reinforcement varied over six pairs of values. The original data have been reanalyzed using a least squares fit of Equation 5 over the four subjects to obtain the value of a_H/a_T, which is the estimated quotient of interval to ratio rates of responding when reinforcements are equal. The best fitting value was .57, again close to the present result. The data also showed the typical undermatching for multiple schedules.

From these three studies, representing three somewhat different procedures, we may conclude that, at least for pigeons, variable-interval and variable-ratio response rates bear a strongly determined relationship to each other to the degree that the former is about 55% to 60% of the latter independent of the overall allocation of reinforcement. We have not included an analysis of Bacotti's (1977) recent study of *conc* VI FR because the fixed-ratio schedule may or may not introduce additional influences on local rates of responding. Given the consistency of the finding with VR and VI, it seems reasonable to conclude that the difference in local rate results from differences in the

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[4] Bouzas and Baum (1976) and White (1978) have, however, presented evidence for redistributions of time spent responding in multiple schedules.

ratio and interval contingencies as such rather than from a particular interaction between the schedules in any given procedure.

In *conc* VI VR, the familiar rate difference on interval and ratio schedules arises in the quotient of the biases in the matching equations for responses and time spent responding, not their absolute values (see Equations 3, 4, and 5). Consequently, since the quotient is about .55 to .6 (interval over ratio), the bias toward the VI for time spent responding must be almost but not quite twice that for responses. However, the absolute values of the response and time biases may favor the interval, the ratio, or be symmetrical about 1.0. In our data, the biases—.718 and 1.291 for responses and time spent responding, respectively—were symmetrical about 1.0, but in principle they need not be. In Bacotti's experiment (1977), for example, subjects also responded at higher local rates on the ratio than on the interval schedule, but they were biased toward interval schedule for both responses and time spent responding.

The fact that subjects can respond at higher local rates on ratio schedules even when showing a bias toward the interval schedule probably reflects a difference in response topography. The differential reinforcement for low rates of responding built into interval schedules, but not into ratio schedules, has been widely noted (Anger, 1956; Ferster & Skinner, 1957). If, as a result, the behavior sustained by interval schedules was composed of relatively more time not in the act of pecking than that sustained by ratio schedules, the quotient of local rates would lie below 1.0 and would be independent of the overall distribution of reinforcements. Figure 5 confirms both inferences. The schedule contingency, affecting response topography, is therefore evident in the local response rates, even while the overall levels of pecking and time spent responding are controlled by the molar matching principle. Similar conclusions are suggested by findings in concurrent and multiple VI VI with differing responses (Beautrais & Davison, 1977; Davison & Ferguson, 1978).

Having shown that *conc* VI VR approximates matching, we now turn to the question of reinforcement maximization. Rachlin et al. (1976) mapped the expected overall reinforcement rate on *conc* VI VI as a function of the overall division of time between the alternatives. Given their assumptions about performance, maximizing approximated matching for some schedule values. We apply a similar analysis here to see how the expected reinforcement rate varies as a function of the division of time between the two alternatives on *conc* VI VR.

Assume that switching from one alternative to the other conforms to a Poisson process. This means that the intervals between switches are exponentially distributed and that the conditional probability of a switch at time t since the last switch is stationary, i.e., independent of time and constant. Several findings support this assumption, and we will show later that the present results support it further. In a discrete-trial choice procedure (Herrnstein, 1971) in which reinforcers were arranged for pigeons on a *conc* VI VI schedule, the probability of switching from one alternative to the other appeared stationary (the data are displayed in de Villiers, 1977). In a continuous *conc* VI VI procedure in which pigeons served as subjects (Heyman, 1979), analysis showed that the changeover probabilities were generally stationary. A study of the temporal pattern of pecking colored hat pins by chicks (Machlis, 1977) found that the intervals between pecks were adequately described by an exponential distribution.

The probability density function for switching at time t since the last switch for a Poisson process is $u_i e^{-u_i t}$, the exponential distribution, where the u's refer to the reciprocals of the mean interchangeover times between the two alternatives. For u_1 and u_2, corresponding to the two alternatives, we have substituted the equivalent parameters p and I, where p is the proportion of time spent at schedule 1 (identical to $T_1/(T_1 + T_2)$ in the nomenclature of Equation 4) and I is an empirical quantity defined below. It is easy to show that $p = u_2/(u_1 + u_2)$, which says, in effect, that the proportion of time spent at a schedule is equal to the relative rate of switching away from the other schedule. We have defined I as a quantity (Heyman, Note 1) equal to one-half the harmonic mean of the interchangeover times, $1/(u_1 + u_2)$. Therefore, $u_1 = (1 - p)/I$, and $u_2 = p/I$. Integrating the expression $\int_0^\infty u_i e^{-u_i t} t \, dt$ shows that the mean interchangeover time is $I/(1 - p)$ at alternative 1 and I/p at alternative 2. The quantity I, then, scales the overall tendency to switch between the alternatives.

For example, with $p = 1/2$ and $I = 10$ sec, the changeover rate is 3 per min, while with the same p but $I = 20$ sec, the changeover rate is 1.5 per min.

When the scheduled reinforcement times are exponentially distributed, as is approximately the case for VI schedules based on Fleshler and Hoffman's (1962) list, stationary changeover probabilities imply (Heyman, Note 1; Heyman & Luce, in press) that the expected reinforcement rate on the VI alternative of a concurrent schedule is:

$$R = \frac{p}{\mathrm{VI} + r_1} + \frac{1-p}{\mathrm{VI} + \frac{I}{p}}, \tag{6}$$

where VI is the mean scheduled interreinforcement time, p is the proportion of time spent responding at the VI, I is defined as before, and r_1 is some function of the interresponse time distribution for VI responding while the VI schedule is being attended. The first quotient is the expected reinforcement rate for reinforcers that set up while the schedule is attended. The parameter r_1 may closely approach zero when reinforcement depends on a continuous response, such as standing on a platform (Baum & Rachlin, 1969; Brownstein & Pliskoff, 1968), and it is the average interresponse time when the interresponse times are exponentially distributed, as is approximately the case for pigeons pecking on VI schedules (Blough & Blough, 1968; Catania & Reynolds, 1968). The second quotient is the expected reinforcement rate for reinforcers that set up while the VI alternative is unattended. The expression I/p gives the average absence time from the VI schedule.

The expected reinforcement rate on the VR alternative is simply $(1 - p)/\mathrm{VR}r_2$, where VR is the average ratio requirement and r_2 is the mean interresponse time on the VR schedule while it is being attended. The expected overall reinforcement rate (reinforcement rate feedback function) in a *conc* VI VR schedule, then, is

$$R = \frac{p}{\mathrm{VI} + r_1} + \frac{1-p}{\mathrm{VI} + \frac{I}{p}} + \frac{1-p}{\mathrm{VR}r_2}. \tag{7}$$

To test the adequacy of Equation 7, we predicted the reinforcement rates for this study. Values for VI, VR, p, I, r_1, and r_2 were taken or calculated from Tables 1 and 2 and inserted into Equation 7. The parameters r_1 and r_2 were set equal to the respective average interresponse times and I was determined from the equation $[2p(1 - p)]/I = COs/T$, where COs is the number of changeovers in a session and T is the session time. Table 4 compares the predictions and the actual obtained reinforcement rates. The predicted values account for .998 of the variance in obtained reinforcement rates. We may therefore take Equation 7 to be a close approximation to the relationship between programmed and obtained reinforcement rates on a *conc* VI VR schedule, which, in turn, serves as indirect validation for the assumption of stationarity in changeover probabilities that it is based on.[5]

Table 4

Obtained and predicted reinforcement rates (Equation 7)

Schedule parameters	*Subject*	*Reinforcements/Hour*	
		Obtained	*Predicted*
VI 30/VR 30	3	234	236
	83	137	143
	365	300	297
	473	117	121
VI 15/VR 30	3	227	225
	83	215	217
	365	386	371
	473	215	221
VI 40/VR 30	3	197	194
	83	92	92
	365	340	340
	473	104	99
VI 40/VR 45	3	132	129
	83	93	91
	365	202	201
	473	91	91
VI 40/VR 60	3	100	98
	83	88	87
	365	106	105
	473	89	90
VI 30/VR 30	3	162	164
	83	156	158
	365	299	297
	473	122	127

By varying p (proportion of time spent responding at the VI) in Equation 7, it is possible to determine the division of time that would maximize reinforcement rate for the obtained local response rates and I values. Maximum

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[5] It also justifies omitting a term for the COD from the feedback function. Including the COD complicates the algebraic representation of the feedback function considerably without materially changing its value.

reinforcement rates were established by inserting programmed values of VI and VR into Equation 7 along with the obtained performance parameters, I, r_1 and r_2, and varying p. Where R was maximized, we extracted the corresponding predicted VI and VR reinforcement rates. By this procedure, we can determine the values of R_1 and R_2 (see Equations 3 and 4) at which the subject would have been maximizing overall reinforcement rate.

Figure 4 and Equation 4, it should be recalled, showed that the observed relationship (in nonlogarithmic form) between time spent responding and obtained reinforcements was:

$$\frac{T_1}{T_2} = 1.291 \left[\frac{R_1}{R_2}\right]^{1.036}. \qquad (4)$$

Given the definition of p as $T_1/(T_1 + T_2)$, it is obvious that $p/(1 - p) = T_1/T_2$. To maximize overall reinforcement rate according to Equation 7, the best fitting function would have to have been as follows:

$$\left[\frac{p}{1-p}\right]_{\max} = \left[\frac{T_1}{T_2}\right]_{\max} = .413 \left[\frac{R_1}{R_2}\right]^{.716}. \qquad (8)$$

Equation 8 plots as the right-most line in Figure 4, clearly not a fit to the data points for any of the subjects. To maximize reinforcement rate, the subjects should have been biased toward the VR, not the VI, and they should have undermatched. The intercept in logarithmic coordinates should have been $-.384$ to maximize total reinforcements, not $+.111$ as it actually was.

We tried to improve the case for maximization by taking account of a possible bias for VI reinforcements acting in opposition to the predicted bias for the VR. Since VI reinforcements are earned at lower local rates of responding than VR reinforcements, and since it is not unreasonable to suppose that subjects prefer lesser local rates of work, other things equal, a VI preference may in fact be present. The best available estimate of the size of this preference would be the bias term in Equation 4, which is 1.291. We have therefore recalculated the maximum possible reinforcement rates according to Equation 7, but weighting all VI reinforcements by a factor of 1.291. Once again, we can then estimate the best fitting function for $p/(1 - p)$, under the adjusted maximization formula, with the following result:

$$\left[\frac{p}{1-p}\right]'_{\max} = \left[\frac{T_1}{T_2}\right]'_{\max} = .579 \left[\frac{R_1}{R_2}\right]^{.742}. \qquad (9)$$

Equation 9 moves us negligibly closer to the observed data. Even when VI reinforcements are given extra weight, maximization of reinforcement still requires a sizable bias toward the VR and significant undermatching, neither of which were observed for the pooled data. Some of the individual functions did show undermatching (see Figure 2), but all of them also showed a VI bias. Furthermore, if performance had maximized reinforcement, the variance accounted for by Equation 9 would have been only .644. The actual variance accounted for by Equation 4 was .966.

The extent of the departure from reinforcement maximization is illustrated in Figure 6. The abscissa gives the logarithms of the observed values of T_1/T_2 and the ordinate gives logarithms of the values of T_1/T_2 that would have maximized reinforcement rate, with VI reinforcements adjusted as in Equation 9 (the unadjusted values would be displaced downward slightly). Overall reinforcement maximization requires the data points to approximate the 45° diagonal, a tendency shown by no subject. The downward displacement from the 45° line shows again that maximization required a VR bias, rather than the observed VI bias. For every observed value of T_1/T_2, maximization called for a shift towards the VR, hence a lower predicted value of T_1/T_2. The discrepancy between observed and predicted values grew larger with increasing T_1/T_2. The low slope of the best fitting function indicates that maximization would have constrained T_1/T_2 within a far narrower range of values than was observed. In fact, if instead of matching, the subjects had spent a constant proportion of their time on the VR, they would have earned more food, since the fitted function is almost horizontal.

It remains to be shown how much reinforcement the subjects lost by matching instead of maximizing. Table 5 lists the (unadjusted) maximizing and actual VI and VR reinforcement rates. The subjects would have increased their reinforcement rates by 30% by shifting to the maximizing values, a change of the order of about 60 reinforcements per hour. The failure to approximate reinforcement maximization cannot credibly be attributed to a mere

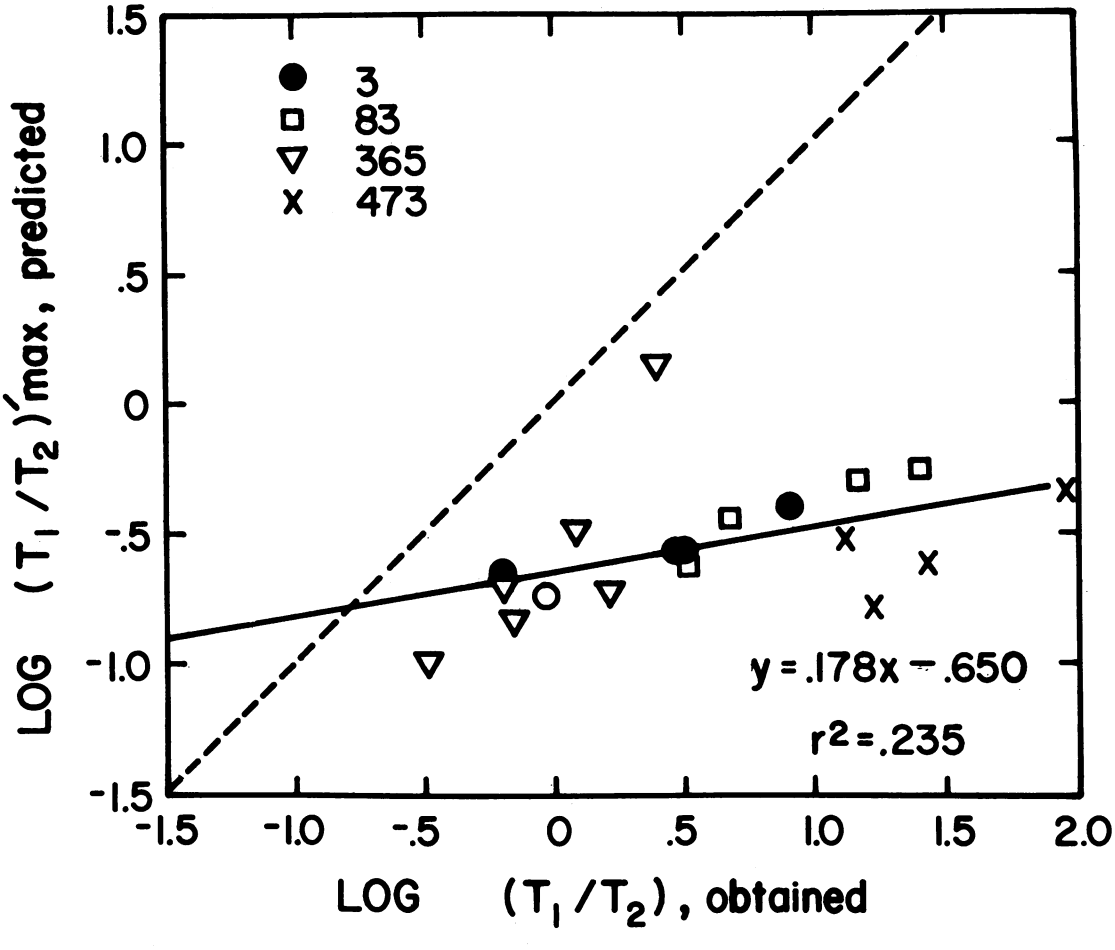

Fig. 6. For each observed value of T_1/T_2 (on the x-axis), the y-axis gives the value that would have maximized overall, adjusted reinforcement rate under the prevailing schedules of reinforcement. If subjects had been maximizing, the points would have approximated the dashed line. Solid line is best fitting linear relation between coordinates for all subjects pooled.

lack of discriminability inasmuch as pigeons are often sensitive to much smaller changes in overall rates of reinforcement.

To summarize the main results, the ratio of local VI and VR response rates was independent of reinforcement rates even while overall responses and time spent responding followed the matching relation. Moreover, the value of the ratio of local rates, .56, was nearly the same as that found in other studies which have compared VI and VR schedule performance. This may indicate two independent levels of reinforcement effects in concurrent schedules. First, the topography of responding at an alternative is determined by the reinforcement contingency there. Second the distribution of reinforcements between the alternatives determines the allocation of the two response topographies.

The generalized matching equation accounted for over .95 of the variance in response and time ratios. Predictions based on maximization of overall reinforcement rate failed by a large margin to predict the amount of time spent responding at the schedules. In fact, performance failed to approximate overall maximization even qualitatively. Subjects consistently showed a bias toward the VI for time spent responding, whereas maximization would have required a bias toward the VR. These findings extend previous work suggesting that matching in *conc* VI VI schedules may not be explainable as overall maximization of the reinforcement delivered by the concurrent

Table 5

Reinforcements Actually Obtained and Predicted by Maximization

		VI		*VR*		*Total*	
Schedule	*Subject*	*Predicted*	*Obtained*	*Predicted*	*Obtained*	*Predicted*	*Obtained*
VI 30/VR 30	3	72	99	198	135	270	234
	83	69	107	142	30	211	137
	365	48	100	316	200	364	300
	473	54	109	165	7	219	116
VI 15/VR 30	3	—	—	—	—	—	—
	83	—	—	—	—	—	—
	365	103	208	316	108	419	316
	473	—	—	—	—	—	—
VI 40/VR 30	3	42	80	204	117	246	197
	83	59	86	69	6	128	92
	365	25	63	364	277	389	340
	473	36	91	220	13	256	104
VI 40/VR 45	3	42	86	146	46	188	132
	83	53	88	79	6	132	94
	365	50	75	163	127	213	202
	473	65	90	71	1	136	91
VI 40/VR 60	3	63	89	82	11	145	100
	83	—	—	—	—	—	—
	365	66	85	55	21	121	106
	473	—	—	—	—	—	—
VI 30/VR 30	3	47	108	200	54	247	162
	83	32	103	209	53	241	156
	365	45	96	312	203	357	299
	473	51	107	180	15	231	122

schedules (Heyman, Note 1; Heyman & Luce, in press). In *conc* VI VR, the discrepancy between matching and maximization is typically greater than in *conc* VI VI, but matching appears to be about equally approximated in both procedures.

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