

AN ANALYTIC COMPARISON OF HERRNSTEIN'S EQUATIONS AND A MULTIVARIATE RATE EQUATION

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Herrnstein's equations are approximations of the multivariate rate equation at ordinary rates of reinforcement and responding. The rate equation is the result of a linear system analysis of variable-interval performance. Rate equation matching is more comprehensive than ordinary matching because it predicts and specifies the nature of concurrent bias, and predicts a tendency toward undermatching, which is sometimes observed in concurrent situations. The rate equation contradicts one feature of Herrnstein's hyperbola, viz., the theoretically required constancy of k . According to the rate equation, Herrnstein's k should vary directly with parameters of reinforcement such as amount or immediacy. Because of this prediction, the rate equation asserts that the conceptual framework of matching does not apply to single alternative responding. The issue of the constancy of k provides empirical grounds for distinguishing between Herrnstein's account and a linear system analysis of single alternative variable-interval responding.

Key words: matching, quantitative law of effect, linear systems, mathematics, bias, undermatching, value, variable-interval schedules

In an earlier paper, McDowell and Kessel (1979) showed that a linear system analysis of variable-interval (VI) responding yields a rate equation that is as accurate as Herrnstein's (1970) hyperbola in describing the relationship between response and reinforcement rates on VI schedules. The rate equation was also shown to predict a form of Herrnstein's hyperbola and to predict forms of matching in concurrent situations. The purpose of the present paper is to examine in more detail the similarities and differences between Herrnstein's equations and the multivariate rate equation.

THE LINEAR SYSTEM ANALYSIS

For a linear system analysis of VI responding, the organism is conceived of as a system (describable in principle by a linear differential equation) that transforms a reinforcement input into a response output. The reinforce-

ment input is said to "drive" the system which in response produces a behavioral output.¹

The input function is designated $R(t)$, or reinforcement as a function of time, and the output function is designated $B(t)$, or behavior (i.e., operant responding) as a function of time. Each function is written mathematically as a train of rectangular pulses spaced irregularly along a time axis. The height of each reinforcement pulse is designated A_R (reinforcement pulse amplitude), and the width of each reinforcement pulse is designated w . The height of each response pulse is designated A_B (response pulse amplitude), and the width of each response pulse is designated w^* .

The linear system analysis produces an equation that describes the relationship between the input and the output functions. This "rate" equation expresses the average response rate, R_{out} , on a VI schedule in terms of the average reinforcement rate, R_{in} , and the amplitudes and widths of the reinforcement and response pulses:

$$R_{out} = \left\{ \ln \left[1 + \frac{A_B(1 - e^{-w^*})}{\gamma A_R(1 - e^{-w})} (e^{1/R_{in} + w} - 1) \right] - w^* \right\}^{-1}. \quad (1)$$

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¹Although the expression, "linear system analysis," will be used here, it is worth noting that the approach described in this paragraph is also applicable to non-linear systems.

The factor, γ , is a scalar constant that is characteristic of the system. A detailed derivation of Equation 1 is given in McDowell and Kessel (1979). The equation states that response rate varies as a joint function of reinforcement rate and reinforcement and response pulse amplitudes and widths.

The amplitudes and widths of Equation 1 may be interpreted with reference to the ordinate dimensions of $R(t)$ and $B(t)$. The ordinates of these two functions are value-like quantities. Anything that increases the value of the reinforcer can be considered to increase the size of A_R . For example, the reinforcement pulse that corresponds to the delivery of a small amount of a .30 M sucrose solution can be considered to have a higher amplitude than the pulse that corresponds to the delivery of the same amount of a .03 M sucrose solution. In general, if increasing the size of a reinforcement parameter (other than rate) results in higher response rates in a single alternative situation, then increases in that parameter can be considered to increase the amplitude of the pulses to which the reinforcer deliveries correspond. Two obvious examples of this type of parameter are amount and immediacy, since increases in either parameter are known to produce higher response rates in single alternative situations (de Villiers, 1977; de Villiers & Herrnstein, 1976). Response pulse amplitude, A_B , can be interpreted in a similar way, except that higher amplitude response pulses may correspond to more highly aversive responses. Increases in response parameters such as the force requirement of a leverpress, for example, can be expected to increase the amplitude of the pulse to which the leverpress corresponds.

The width parameters, w and w^* , are the durations of the reinforcement and response pulses. These durations do not necessarily coincide with the durations of the reinforcements and responses themselves (cf., e.g., Killean, 1975). In most operant conditioning experiments, however, it seems reasonable to suppose that since the durations of the actual events are short, the durations of the pulses will also be small.

The scalar constant, γ , is actually a representation of the transfer function, $G(s)$. The product of $G(s)$ and the Laplace transformation of the input is the Laplace transformation of the output. The actual output (as a func-

tion of time) is obtained by finding the inverse Laplace transformation of this result. If a differential equation describing an organism exists, then the transfer function can be calculated and must remain invariant for that organism under all conditions where the differential equation holds. Since an appropriate differential equation has not been written, however, the transfer function cannot be expressed explicitly. This means that it can appear only as a factor (viz., γ) in Equation 1. Assuming that an organism can, in principle, be described by a linear differential equation, γ must remain invariant under all conditions where the differential equation is expected to hold. In practice, this means that γ must remain invariant across changes in the parameters of reinforcement and responding. On the other hand, surgical or pharmacological insult to the organism may cause the value of γ to change.

HERRNSTEIN'S EQUATIONS AS APPROXIMATIONS OF THE RATE EQUATION

Although several somewhat complicated forms of Herrnstein's equations can be produced by rearranging Equation 1 (McDowell & Kessel, 1979), the equations in their familiar forms can be produced in a fairly straightforward manner.

Herrnstein's Hyperbola

Consider the following rearrangement of Equation 1:

$$e^{1/R_{out}} = me^{1/R_{in}} + b, \quad (2)$$

where $m = ae^{w-w^*}$, $b = (1-a)e^{-w^*}$ and $a = [A_B(1 - e^{-w^*})]/[\gamma A_R(1 - e^{-w})]$. The complete derivation of Equation 2 is given in Appendix A.

Consider also the series expansion of $e^{1/x}$, which is

$$e^{1/x} = 1 + 1/x + 1/(2!x^2) + 1/(3!x^3) + \dots + 1/[(n-1)!x^{n-1}] + \dots$$

Since the second and higher order terms of this infinite series contribute relatively little to the sum of the series (provided x is not too small), the following approximation holds:

$$e^{1/x} \cong 1 + 1/x. \quad (3)$$

Figure 1 shows that the discrepancy between

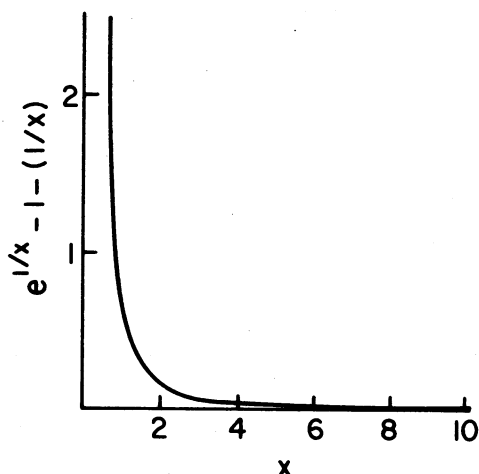


Fig. 1. A plot of the difference between $e^{1/x}$, or $e^{1/x} - 1$, and its series expansion approximation, as x increases.

$e^{1/x}$ and $1 + 1/x$ rapidly approaches zero as x increases. At $x = 10$, for example, the discrepancy is on the order of 10^{-3} . This means that for rates of responding and reinforcement that are typically studied in the laboratory, the approximation given in Equation 3 (where x is a rate of reinforcement or responding) is quite accurate.

Substituting Equation 3 into both sides of Equation 2 gives

$$\begin{aligned}
 1 + 1/R_{out} &= m(1 + 1/R_{in}) + b \\
 1/R_{out} &= m(1/R_{in}) + (b + m - 1) \\
 R_{out} &= \frac{[1/(b + m - 1)]R_{in}}{R_{in} + m/(b + m - 1)} \quad (4)
 \end{aligned}$$

Letting $1/(b + m - 1) = k$ and $m/(b + m - 1) = r_e$, Equation 4 becomes

$$R_{out} = \frac{kR_{in}}{R_{in} + r_e} \quad (5)$$

which is Herrnstein's hyperbola. In other words, Herrnstein's hyperbola is an approximation of the rate equation, obtained when an approximation of the series expansion of $e^{1/x}$ (as given in Equation 3) is substituted for $e^{1/x}$ in the rate equation. When $1 + 1/x$ accurately approximates $e^{1/x}$, i.e., at ordinary or typical rates of reinforcement and responding, the rate equation and Herrnstein's hyperbola are nearly indistinguishable.² It is not surprising, then, that the rate equation and Herrnstein's hyperbola provide equally good fits to data. But the identity of the two equations

²Rates are assumed to be expressed in the customary units of reinforcements/hour and responses/minute.

also means that any ordinary data that can be described by Herrnstein's hyperbola (e.g., de Villiers, 1977; de Villiers & Herrnstein, 1976) can be equally well described by the rate equation. Moreover, laborious comparisons of the two equations' fits to data are not necessary inasmuch as their equivalence can be demonstrated analytically.

The Matching Equation

Familiar forms of the matching equation can also be produced from the rate equation. The form of proportional ratio matching (de Villiers, 1977) predicted by the rate equation is

$$\frac{e^{1/R_{10UT}} - 1}{e^{1/R_{20UT}} - 1} = \left[\frac{a_1}{a_2} \right] \frac{e^{1/R_{1IN}} - 1}{e^{1/R_{2IN}} - 1} \quad (6)$$

where $a_1 = [A_{1B}(1 - e^{-w^*})]/[A_{1R}(1 - e^{-w_1})]$ and $a_2 = [A_{2B}(1 - e^{-w^*})]/[A_{2R}(1 - e^{-w_2})]$. The numerical subscripts refer to the two response alternatives. The complete derivation of Equation 6 is given in Appendix B.³ The factor a_1/a_2 is the concurrent bias (Baum, 1974). Evidently, when the response pulses are identical for the two alternatives ($A_{1B} = A_{2B}$ and $w^*_1 = w^*_2$), and the reinforcement pulses are also identical ($A_{1R} = A_{2R}$ and $w_1 = w_2$), $a_1/a_2 = 1$, i.e., there is no bias and Equation 6 reduces to ordinary matching.

The approximation given in Equation 3 of the series expansion of $e^{1/x}$ is again applicable. Subtracting 1 from both sides of Equation 3 gives

$$e^{1/x} - 1 \cong 1/x.$$

Figure 1 shows that the discrepancy between $e^{1/x} - 1$ and $1/x$ rapidly approaches zero as x increases. This means that $1/x$ is a good approximation of $e^{1/x} - 1$ at rates of responding and reinforcement that are typically studied in the laboratory. Substituting this approximation into both sides of Equation 6 gives

$$\begin{aligned}
 \frac{1/R_{10UT}}{1/R_{20UT}} &= \left[\frac{a_1}{a_2} \right] \frac{1/R_{1IN}}{1/R_{2IN}} \\
 \frac{R_{20UT}}{R_{10UT}} &= \left[\frac{a_1}{a_2} \right] \frac{R_{2IN}}{R_{1IN}},
 \end{aligned}$$

or

$$\frac{R_{10UT}}{R_{20UT}} = \left[\frac{a_2}{a_1} \right] \frac{R_{1IN}}{R_{2IN}} \quad (7)$$

³As noted in Appendix B, the uppercase subscripts on the rates in Equation 6 indicate that w and w^* are assumed to be negligible.

which is the familiar form of proportional ratio matching. In other words, the familiar forms of matching are approximations of the forms predicted by the rate equation. Again, it is not necessary to test the predicted forms against data because their equivalence to familiar matching can be demonstrated analytically. Evidently, any ordinary data that can be described by the matching law can be equally well described by the forms of matching predicted by the rate equation.

DIFFERENCES BETWEEN HERRNSTEIN'S EQUATIONS AND THE RATE EQUATION

Although Herrnstein's equations and the rate equation are nearly indistinguishable in terms of their ability to describe data, they differ structurally in several important respects.

Herrnstein's Hyperbola

Herrnstein's hyperbola (Equation 5) is written with two fitting parameters, k and r_e . The parameter k is the y -asymptote of the hyperbola, i.e., the maximum possible rate of responding. This maximum rate is obtained when $r_e = 0$. The parameter r_e governs the rapidity with which the function approaches its asymptote. When r_e is small, R_{in} dominates the denominator on the right hand side of Equation 5 and the function approaches its asymptote quickly (i.e., response rate is nearly asymptotic even at low rates of reinforcement). For larger values of r_e , the function approaches its asymptote more slowly.

In a theoretical paper, Herrnstein (1974) argued that the value of k should depend only on response form. That is, for a given response (such as key pecking), k should remain invariant across changes in the parameters (such as amount) of reinforcement. This invariance is required by the fact that k , in Herrnstein's

hyperbola, is equivalent to $\sum_{i=0}^n R_i$ (where R_i is the rate of responding on the i^{th} response alternative) in the matching law (Herrnstein, 1974). Because of this equivalence, k has been described as the total amount of behavior that the organism can exhibit, scaled in units commensurate with those of the observed response.

It can be shown that the multivariate rate

equation entails a different interpretation of Herrnstein's k , and that the theoretically required invariance of k is not in agreement with predictions made by the rate equation.

Equations 4 and 5 show that Herrnstein's k may be expressed in terms of the parameters of the rate equation as follows:

$$k = 1/(b + m - 1),$$

where $m = ae^{w-w^*}$, $b = (1-a)e^{-w^*}$, and $a = [A_B(1 - e^{-w^*})]/[\gamma A_R(1 - e^{-w})]$. According to the rate equation, then, the value of Herrnstein's k depends on characteristics of the reinforcer (A_R and w) as well as on characteristics of the response (A_B and w^*). Obviously, the latter dependency is in agreement with Herrnstein's interpretation of k while the former is not. With regard to reinforcement, the rate equation predicts that k will vary with reinforcement pulse amplitude and width as

$$k = [c(e^w/A_R) + d]^{-1}, \quad (8)$$

where $c = e^{-w^*}(A_B/\gamma)(1 - e^{-w^*})$ and $d = (e^{-w^*} - 1)$, and both are held constant across variations in A_R and w . The complete derivation of Equation 8 is given in Appendix C.

Besides gamma, the constants c and d include only quantities that characterize the response pulses. To say that c and d must be held constant across variations in A_R and w means that the response form must not change. Thus, under conditions where Herrnstein would expect k to remain invariant, the rate equation predicts that k will vary directly with A_R , and inversely with w . Since A_R is a value-like quantity that depends on parameters of reinforcement such as amount or immediacy, Equation 8 predicts that increases in the amount or immediacy of reinforcement will produce increases in the value of k (given a constant or negligible w).

Data bearing on the empirical validity of Equation 8 are scarce. However, de Villiers (1977), in a review of the evidence for Herrnstein's equations, noted that data on the constancy of k were equivocal. For example, he cited three studies that reported a different value of k for each of two levels of reinforcement magnitude. In two of these three studies (Campbell & Kraeling, 1953; Keesey, 1964; the exception was Schrier, 1965), the change in k was in the direction predicted by Equation 8, i.e., the higher magnitude of reinforcement was associated with the higher value of k .

In addition to these studies, Bradshaw, Szabadi, and Bevan (1978) recently reported clear variations of k with changes in reinforcement magnitude. These investigators studied leverpressing supported by VI reinforcement in four rats. Reinforcement consisted of .083-min access to a sucrose solution. Each rat was exposed to a series of VI schedules using a .00 M sucrose solution as the reinforcer, a series of schedules using a .05 M sucrose solution, and a series using a .32 M sucrose solution. Three values of k were obtained for each rat, one for each level of sucrose concentration. For all four rats, k was found to be a monotonically increasing function of sucrose concentration, as predicted by Equation 8.

Although Equation 8 contradicts Herrnstein's theoretical requirement that k remain invariant, data supporting the constancy of k are not necessarily inconsistent with the equation. Equation 8 specifies at least two conditions under which k will, or will appear to, remain invariant across variations in parameters of reinforcement such as amount or immediacy.

Obviously, if the range of variation of A_R is small, obtained values of k can be expected to be nearly equal. Equation 8 predicts that a plot of $1/k$ against e^w/A_R will be a straight line. When the slope of this line is shallow, even moderately large variations of A_R will produce nearly equal k s. To prevent misinterpreting a small positive slope as supporting the invariance of k , it would be necessary to determine values of k for a fairly large number of A_R values (e.g., 5 or 6) so that the small positive slope would appear as a clear trend in the data. It is worth noting that, of the five studies interpreted by de Villiers (1977) as supporting the constancy of k , four involved comparisons of only two k values and the fifth compared only three. On the other hand, in at least one of these studies (Kraeling, 1961), k remained approximately invariant over a fairly wide range of reinforcement magnitudes (5 to 125 cc of a sucrose solution).

The second condition under which k should remain approximately invariant is when $A_B(1 - e^{-w^*})e^{-w^*} \ll \gamma$. As can be seen from Equation 8, when this inequality holds, the slope of the plot of $1/k$ against e^w/A_R will be nearly equal to zero. If γ is a moderate number (i.e., not small), w^* is small (brief duration response pulse), and A_B corresponds

to the "effortfulness" or aversiveness of the response, then one possible interpretation of this inequality is that it specifies a situation where responding requires little effort. That is, when responding requires little effort, A_B is small, the inequality holds, and k would be expected to remain invariant across changes in A_R . For example, k might be expected to remain approximately invariant across changes in A_R for pigeons pecking keys, but less so for rats pressing levers (as in Bradshaw et al.'s 1978 experiment) and perhaps even less so for pigeons pressing treadles. Of the studies interpreted by de Villiers (1977) as supporting the constancy of k , only one involved lever pressing in rats. Three involved rats running in alleys. Since running is a naturally occurring high frequency behavior in rats, it might be considered less aversive than leverpressing and hence more conducive to the stability of k . The remaining study involved rats swimming in a cold water tank, which is more difficult to interpret. By contrast, of the four studies that do not support the constancy of k (three reviewed by de Villiers, 1977, plus Bradshaw et al., 1978), three involved lever pressing (2 in rats, 1 in monkeys) and one involved rats running in alleys.

There are other conditions under which k should remain invariant, e.g., when $e^w \cong A_{R_i}$ or $e^w/A_{R_i} = k'$ for all i , where i is a level of reinforcement magnitude or immediacy and k' is an arbitrary constant. However, these conditions are difficult to interpret in real terms.

The Matching Equation

One important difference between the rate equation and the matching law is that the former predicts and specifies the nature of concurrent bias, whereas the latter can account for bias only with the aid of post hoc addenda (Baum, 1974; Baum & Rachlin, 1969; Killeen, 1972; Rachlin, 1971). Equation 7 shows that the rate equation predicts that bias will be given by

$$\frac{A_{1B}(1 - e^{-w_1})}{A_{2B}(1 - e^{-w_2})} \cdot \frac{A_{2B}(1 - e^{-w_2^*})}{A_{1B}(1 - e^{-w_1^*})}$$

McDowell & Kessel (1979) suggested that reinforcement pulse amplitude and width be consolidated into a single quantity, P_R , where $P_R \equiv A_R(1 - e^{-w})$. The quantity P_R (reinforcer "power") corresponds to the value of the rein-

forcer (apart from any value contributed by reinforcement rate).⁴ A similar quantity, P_B , can be defined for the response pulses, viz., $P_B \equiv A_B(1 - e^{-w^*})$. The quantity P_B corresponds to the aversiveness of the response. Substituting these definitions into the above expression for bias gives

$$\frac{P_{1B}}{P_{2B}} \cdot \frac{P_{2B}}{P_{1B}}$$

Consequently, the approximation of biased matching predicted by the rate equation may be written (from Equation 7)

$$\frac{R_{1OUT}}{R_{2OUT}} = \left[\frac{P_{1B}}{P_{2B}} \cdot \frac{P_{2B}}{P_{1B}} \right] \frac{R_{1IN}}{R_{2IN}} \quad (9)$$

Equation 9 predicts that an increase in the value of the first alternative's reinforcer will produce a bias in favor of the first alternative, and that an increase in the aversiveness of the first alternative's response will produce a bias in favor of the second alternative. Thus Equation 9 accounts for three of the four sources of bias documented by Baum (1974), viz., response bias, a discrepancy between scheduled and obtained reinforcement, and qualitatively different reinforcers. The first source of bias is reflected in different values of P_B ; the latter two sources of bias are reflected in different values of P_B . Equation 9 cannot account for bias produced by qualitatively different schedules of reinforcement inasmuch as the rate equation applies only to VI schedules. It is interesting to note that Baum (1974) produced a form of Equation 9 by argument. Here, the equation is shown to be a formal consequence of a linear system analysis of VI responding.

A second difference between the rate equation and the matching law is that the former predicts a tendency toward undermatching in all concurrent situations. Baum (1974), de Villiers (1977), and Myers and Myers (1977) have noted this tendency in data from concurrent schedules. The rate equation predicts ordinary matching only when reinforcement pulse width (w) is negligible (see Appendix B). When w is not negligible, rate equation matching should be written (from Appendix B)

$$\frac{R_{1OUT}}{R_{2OUT}} = \frac{e^{1/R_{1IN} + w} - 1}{e^{1/R_{2IN} + w} - 1} \quad (10)$$

For simplicity, the series expansion approximation of $e^{1/x} - 1$ is used for the response rates in this equation, the concurrent bias is assumed to equal 1 (no bias), and reinforcement pulse widths (w) are assumed to be equal for the two alternatives. The quantity, $1/R_{1IN} + w$, is the average time between the initiation of adjacent reinforcement pulses (McDowell & Kessel, 1979). When w is negligible, $1/R_{1IN}$ is a good estimate of this time. When w is not negligible, however, $1/R_{1IN}$ underestimates the average time between the initiation of adjacent reinforcement pulses by a constant amount at all reinforcement rates. In the latter case, a plot of R_{1OUT}/R_{2OUT} against R_{1IN}/R_{2IN} will show undermatching.

The ability of Equation 10 to account for undermatching is illustrated in Figure 2. The data are from Trevett, Davison, and Williams' 1972 study of concurrent VI VI responding in pigeons. This study was selected because, according to de Villiers (1977), it provides the strongest evidence for systematic undermatching on concurrent schedules. The left-hand panels of the figure show Trevett et al.'s data plotted in the customary logarithmic coordinates (Baum & Rachlin, 1969). The dashed lines were fitted by the method of least squares, and the slope (m) of the fitted line is given for each plot. Substantial undermatching was observed in three of the four pigeons. The right-hand panels show fits of Equation 10 (in logarithmic coordinates) to these data. Values of w were estimated by an iterative least-squares program that incremented w from a small value until the slope of the regression line was 1 when rounded to two decimal places. The estimated value of w (in hours), and the slope (m) of the regression line are given for each plot. Solid diagonals in each panel represent the locus of perfect unbiased matching.

Figure 2 shows that, although the behavior of Trevett et al.'s pigeons does not conform to ordinary matching, it does conform to the form of matching predicted by the rate equation when w is taken into account in the manner specified by Equation 10.⁵ Estimated val-

⁴The term *power* was suggested by Herrnstein (1971). It may ultimately prove to be a poor name for this parameter, however. Power ordinarily refers to the time rate at which work is done (joules/sec). This is obviously not its meaning in the present context.

⁵In apparent agreement with de Villiers (1977), ordinary or "true" matching is here understood to mean proportional ratio matching, where bias is permitted but under- and overmatching are not.

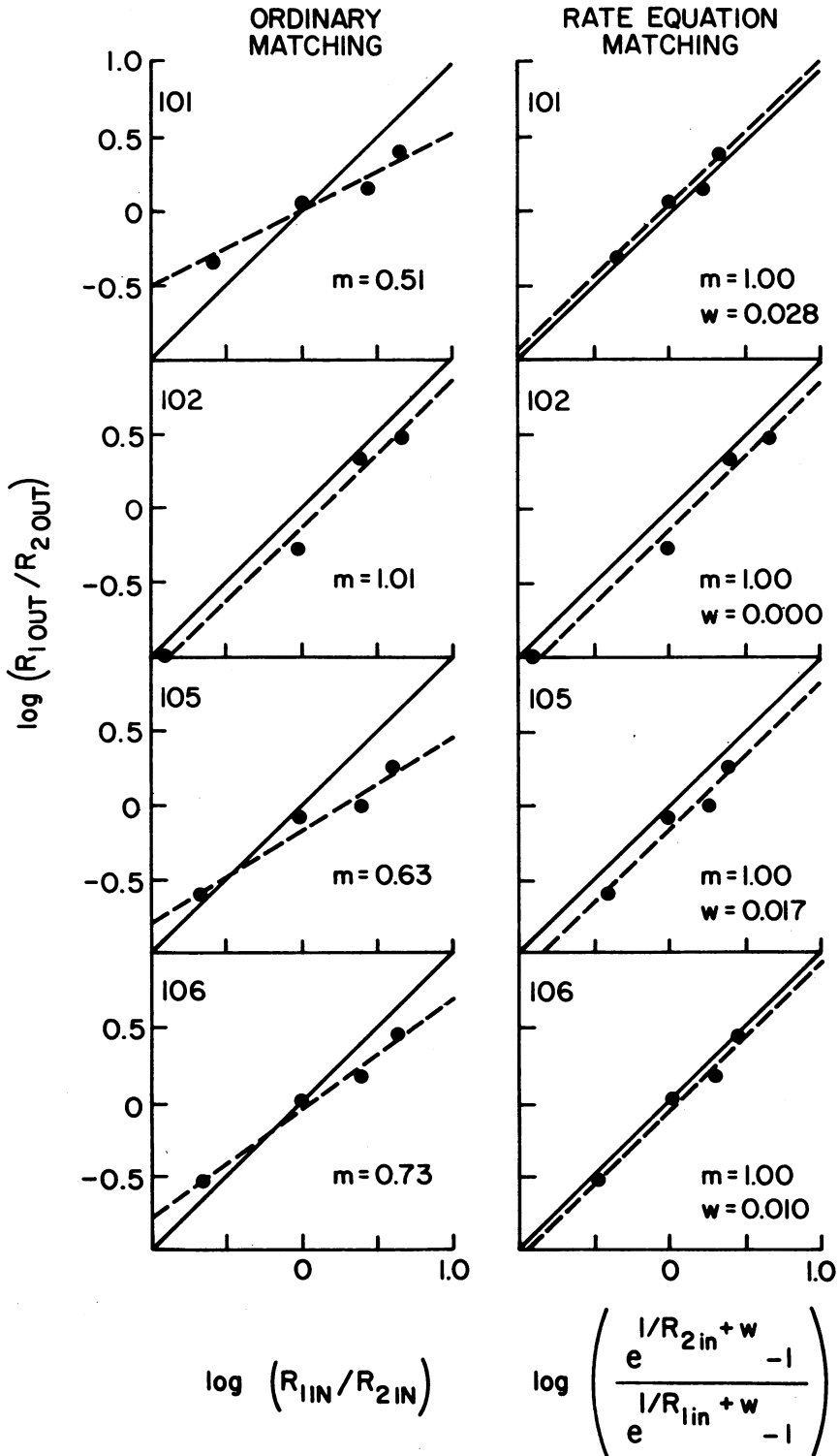


Fig. 2. Ordinary matching (left-hand panels) and rate equation matching (right-hand panels), for Trevett et al.'s (1972) four pigeons. Dashed lines were fitted by the method of least squares. The slope (m) of the fitted line is given in each panel.

ues of w ranged from 1.07 sec (P102) to 100.44 sec (P101) with a median value of 49.14 sec. This fairly long median value of w should be viewed with caution. The fits in Figure 2 show that Equation 10 can account for undermatching when appropriate values for w are selected, but these values are not necessarily accurate estimates of w . Because statistical procedures attribute all of the undermatching to a non-zero w , undermatching that is also caused by other factors (e.g., poor discriminability between the two schedules) will result in artificially inflated estimates of w . In the case of Trevett et al.'s data, for example, de Villiers (1977) argued that prior exposure to concurrent VI FI schedules may have contributed to the observed undermatching. The fairly lengthy values of w obtained here seem to support this argument. It is important to recognize, however, that Equation 10 predicts a tendency toward undermatching in all concurrent situations, whether or not the undermatching is aggravated by other factors. This is because w , although it may be very small, can never equal zero.

DISCUSSION

Herrnstein's equations are approximations of the equation produced by a linear system analysis of VI responding. The rate equation provides a more complete account of matching on concurrent schedules because it predicts and specifies the nature of concurrent bias, and predicts a tendency toward undermatching, which is sometimes observed in concurrent situations. Of course, Herrnstein's matching law can be made to account for bias and undermatching by inserting two parameters, c and n , as follows:

$$\frac{R_{1OVT}}{R_{2OVT}} = c \left(\frac{R_{1IV}}{R_{2IV}} \right)^n,$$

$n \leq 1$. This power function version of matching has been discussed by Baum (1974), de Villiers (1977), and Staddon (1968, 1972), among others. Rate equation matching (from Equations 7 and 10),

$$\frac{R_{1OVT}}{R_{2OVT}} = \left[\frac{a_2}{a_1} \right] \frac{e^{\frac{1}{2} R_{2IV}^{1+n} w} - 1}{e^{\frac{1}{2} R_{1IV}^{1+n} w} - 1},$$

although not a power function, also involves two parameters, viz., a_2/a_1 and w and, as noted in the previous section, accounts for bias and

predicts a tendency toward undermatching in all concurrent situations. The advantage of rate equation matching is that the parameters it entails are not post hoc addenda, as are c and n in the power function version of matching, but are formal consequences of the linear system analysis.

Clearly, the rate equation does not contradict the matching law; it is merely more comprehensive than ordinary matching. The rate equation does, however, contradict one feature of Herrnstein's hyperbola, viz., the theoretically required constancy of k . This contradiction has important consequences for the conceptual framework of matching in single alternative responding. The relevant feature of this framework is the idea, implied by Herrnstein's hyperbola, that single alternative response rates are determined by relative, not absolute, rates of reinforcement. According to the rate equation, however, the idea of reinforcement relativity in single alternative responding is both unnecessary (in order to describe the data) and incorrect.

In order to understand why this is so, it may be helpful to examine the apparent reasoning in Herrnstein's 1970 paper where his hyperbola is developed and discussed at some length. According to Herrnstein (and others), it would be convenient if the relationship between response and reinforcement rates were a direct proportionality given by

$$R_{OVT} = kR_{IN}. \quad (11)$$

This is of course a straight line with 0 intercept. It permits response rates to increase without bound as reinforcement rates get large. Unfortunately, the data show that response rates are bounded by a y -asymptote > 0 . As far as describing the data is concerned, then, the problem is to make Equation 11 bend downward. Herrnstein accomplishes this by borrowing a property from concurrent schedules and rewriting Equation 11 as

$$R_{OVT} = \frac{kR_{IN}}{R_{IN} + \tau_e}.$$

The property borrowed from concurrent schedules is that response rate in one component varies inversely with reinforcement rate obtained in the other. Herrnstein accordingly speculates that there may be an additional source of reinforcement in the single alternative situation, viz., τ_e , with which response rate

varies inversely. With regard to describing the data, the only importance of this speculation is that it makes Equation 11 bend downward. Conceptually, however, it introduces the framework of matching into the single alternative situation. That is, according to the hyperbola, single alternative responding depends on a relative, not an absolute, reinforcement rate.

But the rate equation also bends downward in such a way as to provide a good description of the data, and without involving any sources of reinforcement other than the one in direct contact with the behavior. Evidently, then, the conceptual framework of matching is not necessary in order to describe the data. In addition, however, the rate equation asserts that the conceptual framework of matching in single alternative responding is incorrect. Herrnstein (1974) showed that the matching principle formally requires k to remain invariant across changes in parameters of reinforcement such as amount or immediacy. His argument can be summarized as follows.

Consider this rearrangement of the hyperbola:

$$\frac{R_{OUT}}{k} = \frac{R_{IN}}{R_{IN} + r_e} \quad (12)$$

Equation 12 entails the matching principle only if k is the total amount of behavior that the organism can exhibit, viz., $R_{OUT} + R_e$, where R_e is the responding associated with the extraneous reinforcement, r_e . That is, if $k = R_{OUT} + R_e$, then Equation 12 becomes

$$\frac{R_{OUT}}{R_{OUT} + R_e} = \frac{R_{IN}}{R_{IN} + r_e} \quad (13)$$

which is matching. Now, given matching, the denominator of the response proportion cannot change, no matter what happens to reinforcement because, by definition, it exhausts the plenum of responses (see, in addition to Herrnstein, 1974, de Villiers & Herrnstein, 1976). As one example, if R_{OUT} decreases because of an increase in r_e , the difference in the denominator on the left is made up for by an increase in R_e .

If k is found to vary with changes in the parameters of reinforcement, however, when k cannot be equal to the total amount of behavior. But if k is not equal to the total amount of behavior, then Equation 13 does not follow from Equation 12, and if this is true, then Equation 12 (Herrnstein's hyperbola) does not entail matching.

Since the rate equation predicts that k will vary with A_R according to Equation 8, it asserts that k is not equal to the total amount of behavior. In other words, the rate equation says that the conceptual framework of matching does not apply to single alternative responding.

The issue of the constancy of k obviously provides clear empirical grounds for distinguishing between Herrnstein's account and a linear system analysis of single alternative VI responding. It is worth noting that both accounts have attractive and unattractive features. The major advantages of Herrnstein's account are its simplicity, and its appealing conceptual treatment of all behavior as choice behavior (Herrnstein, 1970; Rachlin, 1976). In a linear system analysis, the conceptual unity of Herrnstein's account is lost (although a form of matching is retained for concurrent VI VI performance), and the result of the analysis, viz., Equation 1, is cumbersome. The major advantages of the linear system analysis, however, are its comprehensiveness and its promise of generality. Since the analysis involves a purely formal method, it may be applicable to a variety of situations where equilibrium or steady state behavior is supported by reinforcement.

SOME GENERAL REMARKS ON THE LINEAR SYSTEM ANALYSIS

Interpreting the Parameters of the Rate Equation

The conceptual interpretation of the parameters of Equation 1 that is used here will be satisfactory for many purposes. Ultimately, however, the parameters of the equation will require physical definition. This may be a problem in the case of pulse amplitudes and widths. Consider, as an example, a situation in which the amount of reinforcement for pigeons' keypecking is increased by increasing the duration of access to grain. It is not clear whether this increase should be reflected in a larger pulse amplitude or in a larger pulse width. Another difficulty arises in attempting to specify how various reinforcer characteristics (such as amount and immediacy) combine to determine A_R .

McDowell and Kessel (1979) suggested one possible solution to these and similar problems. They chose to leave the amplitudes and

widths of Equation 1 physically undefined and to define new "power" parameters in terms of the original parameters. These power parameters and their definitions are discussed above (see also footnote 4). The conceptual advantage of this approach is that responding and reinforcement are completely described by only two quantities each, namely, rate of occurrence and power. The power parameters subsume all reinforcer and response characteristics that, when considered separately, would make physical definition difficult. An important practical advantage of the approach is that it is easy to specify the measurement operations whereby the actual values of the power parameters are determined. As suggested by McDowell and Kessel, measurement may be accomplished by matching-based reinforcer and response scaling (cf. Baum, 1974; Herrnstein, 1971; Miller, 1976). The units on power may simply be defined, as in the case of force (newtons), work (joules), heat (kilocalories), and so on.

Consider Equation 1 where the original parameters are replaced by power parameters:

$$R_{OVR} = \{ \ln[1 + (P_B/\gamma P_R)(e^{1/P_R} - 1)] \}^{-1}. \quad (14)$$

The uppercase subscripts on the rates indicate that reinforcement and response pulse widths are assumed small in this equation, as seems reasonable for most operant conditioning experiments. Say that P_R , expressed perhaps in thordikes, has been determined for three different reinforcers, and that P_B , expressed perhaps in watsons, has been determined for three different responses. Note that these values would be determined in concurrent situations with equal VI schedules on the two operanda. The value of γ would be assessed by picking one of the responses, one of the reinforcers and an arbitrary VI.⁶ Once these quantities are known, the rate of responding that would be produced by any VI input rate, and for any combination of reinforcer and response, can be calculated from Equation 14.

Restriction of the Rate Equation to VI Performance

With regard to possible further applications of the mathematics, it may be worthwhile to

clarify the sense in which Equation 1 is restricted to VI performance and to indicate some of the features of other types of behavior that must be taken into account if the analysis is to be extended to other cases.

Whenever the differential equation that describes the system is not known, one must begin the analysis by writing appropriate expressions for $R(t)$, the input or reinforcement function, and $B(t)$, the output or response function. Once these functions have been written, the rest of the analysis follows routinely (assuming that the system is linear). To the extent that $R(t)$ and $B(t)$ accurately describe the reinforcement input and the response output, the analysis can be expected to yield an equation that accurately describes the relationship among characteristics of the original functions. If $R(t)$ and $B(t)$ describe the input and output only poorly, however, then it is unlikely that the resulting equation will be successful. One important inadequacy in $R(t)$ or $B(t)$ occurs when some feature of the actual input or output is ignored when the functions are written. If periodicity in the input, for example, is not taken into account in $R(t)$, then the final equation may fail to describe input-output relationships accurately.

The VI case is especially simple because the reinforcement input and response output have only one identifiable characteristic each, viz., the constant average time between (reinforcing or response) events. Since the events are spaced irregularly in time in both cases, there is no periodicity, and no other identifiable feature of reinforcement or responding that needs to be taken into account. In other words, $R(t)$ and $B(t)$, as written by McDowell and Kessel (1979) for the VI case, constitute good descriptions of the actual VI reinforcement input and response output. Equation 1 is limited to the VI case because these functions describe no other input and output accurately. Consider, for example, variable-ratio (VR) responding. Although the VR and VI cases are similar (both have irregularly spaced pulses and "stationary" mean interpulse intervals in their inputs and outputs), the former is complicated by the addition of a positive feedback loop that operates at all output rates.⁷ It is neces-

⁶Although, in principle, γ can be determined by using only one VI, a statistical estimate of γ using several VIs and curve fitting procedures would undoubtedly be more accurate.

⁷Although there is also some positive feedback in the VI case, it occurs over a small range of low output rates and is minor in comparison to the positive feed-

sary to take this loop into account when calculating a VR transfer function. One possible approach to the VR case is to treat it as VI responding, but with the addition of this feedback loop. A preliminary analysis of the VR case along these lines can be found in McDowell (1979).

For fixed-interval (FI) responding, neither $R(t)$ nor $B(t)$ for the VI case are good descriptions of the actual reinforcement input or response output. Although an FI input has a stationary interreinforcement interval, it exhibits clear periodicity, as does the FI output. The fixed-ratio case is even further removed from the VI, since it is similar to FI responding but with the addition of a feedback loop. These extra features must be taken into account when calculating transfer functions for these cases. McDowell (1979) has discussed the problems involved in the analysis of the FI case, and in the analysis of some unusual interval schedules that exhibit more complicated periodicity.

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back in the VR case. This is, however, a minority opinion. VI feedback functions have figured prominently in the work of Rachlin (1978) and Staddon and Motheral (1978), to take two examples, and have been discussed by others (e.g., Baum, 1973).

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APPENDICES

APPENDIX A

Inverting, adding w^* to, and exponentiating both sides of Equation 1 produces

$$e^{1/R_{out} + w^*} = 1 + a(e^{1/R_{in} + w} - 1),$$

where $a = [A_B(1 - e^{-w^*})]/[\gamma A_R(1 - e^{-w})]$. Rearranging this expression,

$$e^{1/R_{out} + w^*} = ae^w e^{1/R_{in}} + (1 - a),$$

and multiplying through by e^{-w^*} gives

$$e^{1/R_{out}} = ae^{w - w^*} e^{1/R_{in}} + (1 - a)e^{-w^*},$$

or

$$e^{1/R_{out}} = me^{1/R_{in}} + b,$$

where $m = ae^{w - w^*}$ and $b = (1 - a)e^{-w^*}$.

APPENDIX B

Inverting, adding w^* to, exponentiating, and subtracting 1 from both sides of Equation 1 produces

$$e^{1/R_{1out} + w^*} - 1 = (a_1/\gamma)(e^{1/R_{1in} + w} - 1)$$

and

$$e^{1/R_{2out} + w^*} - 1 = (a_2/\gamma)(e^{1/R_{2in} + w} - 1),$$

where $a_1 = [A_{1B}(1 - e^{-w^*})]/[A_{1R}(1 - e^{-w})]$ and $a_2 = [A_{2B}(1 - e^{-w^*})]/[A_{2R}(1 - e^{-w})]$. The numerical subscripts (1 and 2) refer to each of two concurrently available response alternatives. For simplicity, reinforcement pulse widths (w) are assumed to be equal for the two alternatives, and response pulse widths (w^*) are also assumed to be equal. Dividing

these two equations gives a form of proportional ratio matching:

$$\frac{e^{1/R_{1out} + w^*} - 1}{e^{1/R_{2out} + w^*} - 1} = \left[\frac{a_1}{a_2} \right] \frac{e^{1/R_{1in} + w} - 1}{e^{1/R_{2in} + w} - 1}.$$

When w and w^* are negligible (cf. McDowell & Kessel, 1979), this expression becomes

$$\frac{e^{1/R_{1OUT}} - 1}{e^{1/R_{2OUT}} - 1} = \left[\frac{a_1}{a_2} \right] \frac{e^{1/R_{1IN}} - 1}{e^{1/R_{2IN}} - 1}.$$

This is Equation 6. The uppercase subscripts on the rates indicate that w and w^* are assumed to be negligible.

APPENDIX C

From Equations 4 and 5,

$$k = 1/(b + m - 1)$$

where $m = ae^{w - w^*}$, $b = (1 - a)e^{-w^*}$, and $a = [A_B(1 - e^{-w^*})]/[\gamma A_R(1 - e^{-w})]$. Thus,

$$\begin{aligned} k &= [(1 - a)e^{-w^*} + ae^w e^{-w^*} - 1]^{-1} \\ &= [e^{-w^*} - ae^{-w^*} + ae^w e^{-w^*} - 1]^{-1} \\ &= [e^{-w^*} - ae^{-w^*}(1 - e^w) - 1]^{-1}, \end{aligned}$$

or

$$\begin{aligned} k &= \left[\frac{(e^{-w^*} - 1) - \frac{A_B(1 - e^{-w^*})e^{-w^*}(1 - e^w)}{\gamma A_B(1 - e^{-w})}}{\gamma} \right]^{-1} \\ &= \left\{ \left[\frac{A_B(1 - e^{-w^*})e^{-w^*}}{\gamma} \right] \left[\frac{e^w}{A_R} \right] + (e^{-w^*} - 1) \right\}^{-1}. \end{aligned}$$

Letting $c = e^{-w^*}(A_B/\gamma)(1 - e^{-w^*})$ and $d = (e^{-w^*} - 1)$, this expression becomes

$$k = [c(e^w/A_R) + d]^{-1}.$$