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ON THE EXPONENT IN THE "GENERALIZED" MATCHING EQUATION

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A power function equation between ratios of behavior and ratios of reinforcement rates has been called a generalized form of Herrnstein's (1961) matching law, even without a formal relationship having been shown between the two equations. The present work uses a functional relationship to prove that when ratios of reinforcement are not equivalent to ratios of behavior, and the transform leading to this inequality is consistent for every pair of reinforcement rates, the result is a power function relationship between response and reinforcement ratios. The label "generalized matching equation" for the power function equation is thus validated formally.

Key words: matching equation, power function, reinforcement ratios

Several recent theoretical articles have offered derivations of a well known and empirically validated description of choice behavior known as the matching law (Herrnstein, 1979; McDowell, 1980; Myerson & Miezin, 1980; Rachlin, 1978; Staddon, 1977; Staddon & Motheral, 1979). This relationship asserts that organisms match the proportion of total responses emitted (or of time allocated) for an alternative to the proportion of total reinforcement obtained from that alternative:

$$\frac{B_1}{\sum_i B_i} = \frac{R_1}{\sum_i R_i};\ i=1,\ldots,n$$

or equivalently

$$\frac{B_1}{R_1} = \frac{B_2}{R_2} = \ldots = \frac{B_n}{R_n} \tag{1}$$

where B is the measure of behavior allocation and R is the measure of reinforcement rate.

Many authors (e.g., Baum, 1974, 1979; Davison & Hunter, 1976; de Villiers, 1977; Staddon, 1977) have noted that behavior often deviates from that predicted by Equation 1 in systematic ways. Often animals undermatch or respond disproportionately less to schedules with high payoff rates. These and other deviations are often captured by a power function form referred to as a generalized matching equation (e.g., Wearden, 1980),

$$\frac{B_i}{B_j} = b\left(\frac{R_i}{R_j}\right)^a \tag{2}$$

where a and b are the parameters of a power function (see Baum, 1974; Lander & Irwin, 1968; Staddon, 1968).

One might wonder why Equation 2 is referred to as a generalized matching equation. It is more general in that it fits data much better than Equation 1 in many circumstances, and it reduces to Equation 1 with appropriate parameter values. It is nonetheless incorrect to call Equation 2 a generalization of Equation 1 for these reasons alone. There has been no formal relationship shown between these equations, and were it not for the ability of the second to fit data that the first cannot, one would not suspect that Equation 2 was the *proper* generalization of Equation 1.

The parameter b (of Equation 2) has been suggested as a bias term to account for differential preferences that an organism might have for one or more of the alternatives (Baum, 1974). Data from studies involving qualitatively different reinforcers support these notions and suggest that biases are indeed multiplicative, and thus justify to some extent the inclusion of this parameter as a preference "fudge factor" (Miller, 1976). There have, however, been few satisfactory accounts for the appearance of the exponent term, a (but see Baum, 1979; Wearden, 1980).

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The present derivation demonstrates that this exponent indeed arises quite naturally as a true generalization of the matching law.

Consider Equation 1 in the form

$$\frac{B_i}{B_j}=\frac{R_i}{R_j}\ ;\ \forall\ i,j,$$

where all B and R are positive. Suppose now that the mechanism which determines the behavioral output for a given set of response rates does not perfectly conform to the matching relation, so that instead of the above equation

$$\frac{B_i}{B_j}=\mathrm{f}\left(\frac{R_i}{R_j}\right)\ ;\ \forall\ i,j, \tag{3}$$

describes behavior allocation ratios. The function f is any arbitrary function describing how the ratio of responses might vary from the ratio of reinforcement rates, but is the same function for all possible pairs, *i,j*. Thus, if some mechanism results in a deviation from matching for a specific pair, *i,j* (e.g., due to a response ceiling) this mechanism results in a similar deviation for every other pair *i,k; j,k;* etc. This function f might be any arbitrary transform of the ratio: linear or nonlinear, but it is the same for all pairs, *i,j*.

Given the above assumptions (which indeed reflect a generalization of Equation 1) it can be shown that f must be a power function.

Notice that

$$\frac{B_i}{B_j}=\frac{B_i}{B_k}\cdot\frac{B_k}{B_j}=f\left(\frac{R_i}{R_k}\right)\cdot f\left(\frac{R_k}{R_j}\right)\ ;\ \forall\ i,j,k$$

and thus,

$$f\left(\frac{R_i}{R_j}\right)=f\left(\frac{R_i}{R_k}\right)\cdot f\left(\frac{R_k}{R_j}\right)\ ;\ \forall\ i,j,k.$$

Given that each rate parameter R is positive (if R equals zero it will not be considered), there exists some real, positive number c such that $R_i = cR_k$ and a real, positive number d such that $R_k = dR_j$. This implies that $R_i = cdR_j$. (If the ratio of two R's is allowed to vary continuously then c and d are positive continuous variables. In practice only a finite number of values of these ratios are sampled, but it is assumed here that all values could be sampled, and if they were, the same f would apply.) By substitution it is evident that

$$f\left(\frac{cdR_j}{R_j}\right)=f\left(\frac{cR_k}{R_k}\right)\cdot f\left(\frac{dR_j}{R_j}\right)$$

or

$$f(cd)=f(c)\cdot f(d). \tag{4}$$

This result is a functional equation of Cauchy that has a known solution for the most general continuous case:

$$f(x)=x^a,$$

the power function (a is the arbitrary parameter of this power function; see Appendix).[1]

In other words if the animal's behavior ratios do not perfectly match reinforcement rate ratios, but deviate consistently, then it follows from Equation 4 that the function (f) of the product (cd) is equal to the product of the separate functions ($f(c)\cdot f(d)$). This simple result requires that f be a power function:

$$(cd)^a=c^a\cdot d^a;$$

no other function will do. (If f is the identity function—in the event of perfect matching—a is equal to one.)

Thus if behavior *tends* to follow the matching law but varies from this relation in some consistent way, then it follows that deviations from this law must be captured by a power function; and thus it is shown that the power function equation is formally a generalization of the matching law.

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[1] This logic is virtually identical to that used by Shepard (1978) in his argument against assumed internal power transforms from psychophysical data fit by power functions.

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APPENDIX

An elementary proof given in many introductory probability texts (e.g., Hoel, Port, & Stone, 1971, p. 127) is that if

$$g(c+d) = g(c) \cdot g(d) \qquad (1A)$$

then $g(x)$ is an exponential function, e^{ax}. (This proof is important for proving the independence property of the exponential distribution.) For application to probability distribution functions $0 < = g(x) < = 1$, this requires that $a < = 0$. If the upper bound on the function $g(x)$ is eliminated, as in the present application, a can be any real number.

Given the above relationship it is easy to prove that if

$$f(c \cdot d) = f(c) \cdot f(d)$$

then $f(x)$ must be the power function, x^a.
PROOF:

Let

$$f(c \cdot d) = f(c) \cdot f(d).$$

By substituting an equivalent form it can be shown that

$$f(e^{\ln c} \cdot e^{\ln d}) = f(e^{\ln c}) \cdot f(e^{\ln d})$$

or by simplifying that

$$f(e^{\ln c + \ln d}) = f(e^{\ln c}) \cdot f(e^{\ln d})$$

Now let

$$h(x) = f(e^x).$$

Thus

$$h(\ln c + \ln d) = h(\ln c) \cdot (\ln d).$$

From Equation 1A it is apparent that $h(x)$ is an exponential function, therefore

$$h(x) = f(e^x) = e^{ax}.$$

Finally, let

$$y = e^x.$$

Substitution results in

$$f(y) = y^a,$$

which proves that the function f must be a power function.