SIZING PARTICLES WITH A COULTER COUNTER

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ABSTRACT A theorem is presented which permits a determination of the amplitude of the signal generated by the passage of a particle of arbitrary shape through a Coulter counter. The theorem is applied to particles of two shapes, a sphere and a prolate spheroid. For the sphere the signal is directly proportional to the volume of the particle. For the spheroid the result is a complicated function of the shape. Two spheroids of the same volume but different shapes will give different signals.

INTRODUCTION

A convenient method for counting and sizing particles in suspension relies on the difference in conductivity between the particles and the suspending fluid. We have idealized in Fig. 1 a device manufactured by Coulter Electronics, Inc., Hialeah, Fla.

FIGURE 1

The conducting fluid is being pumped from left to right. As a particle enters the narrow tube the resistance between the electrodes is changed and a deflection is observed on the meter. So long as the number density of the particles in suspension multiplied by the volume of the tube is much less than one, the ability of the device to act as a counter is reliable and straightforward.

It would also be convenient if one could infer the size of the particle from the magnitude of the deflection (1). It is this question to which we address ourselves. We first derive a general theorem (valid to first order in the ratio of the size of the particle to the size of the tube) for the change in resistance due to a particle of arbitrary shape in the tube. As an application of the theorem we will consider two particle shapes, a sphere and a prolate spheroid.

ANALYSIS

As an approximate boundary value problem we seek a potential function $\phi(x, y, z)$ such that

$\nabla^2\phi =$ inside the tube and outside the particle

$\partial\phi/\partial n = 0$ on the side of the tube and on the surface of the particle. (We have assumed for simplicity that the particle is nonconducting.)

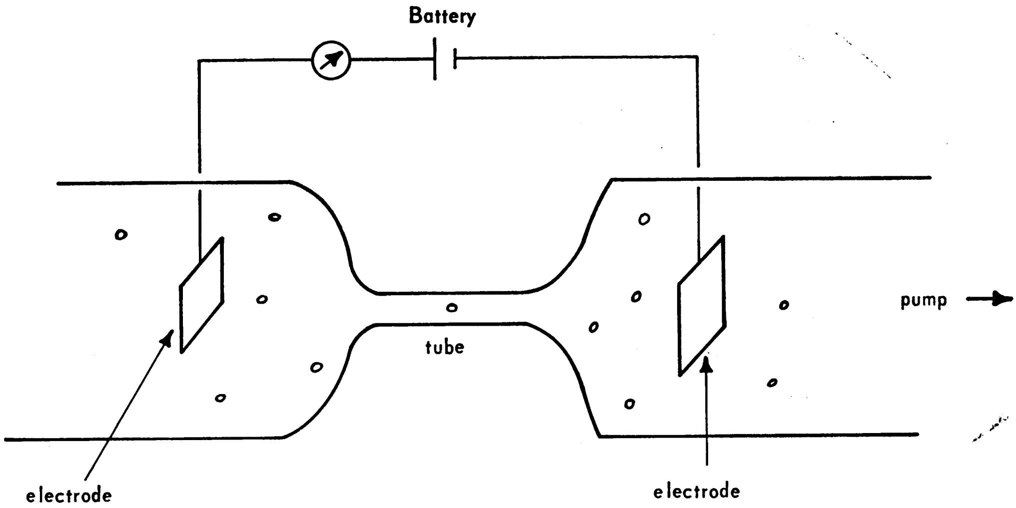

FIGURE 1 Idealization of the Coulter counter.

And

$$\phi = \tfrac{1}{2}V \quad \text{on the surface} \quad z = -l/2$$

$$\phi = -\tfrac{1}{2}V \quad \text{on the surface} \quad z = l/2.$$

V represents the potential drop across the tube, l the length of the tube, and $\partial\phi/\partial n$ the normal derivative of ϕ. The geometry is illustrated in Fig. 2.

FIGURE 2

We now employ a device used by Rayleigh (2) to solve an equivalent problem. For any functions ϕ and ψ we have from Green's theorem that

$$\int_\tau (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d\tau = \oint \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) dS,$$

where the surface integral is over the closed surface bounding the volume τ. The normal is the outward normal to this surface. We choose ϕ to be our electrostatic potential function and ψ to be the z coordinate. Both are solutions of Laplace's equation so that

$$\oint \left(\phi\frac{\partial z}{\partial n} - z\frac{\partial\phi}{\partial n}\right) dS = 0. \tag{1}$$

Now $\partial z/\partial n = 0$ on the sides of the tube, -1 on the left end, and $+1$ on the right end of the tube. $\partial\phi/\partial n$ is zero on the sides of the tube, on the particle surface, $-\mathbf{E}$ on the left end of the tube, and $+\mathbf{E}$ on the right end. Observing that the current

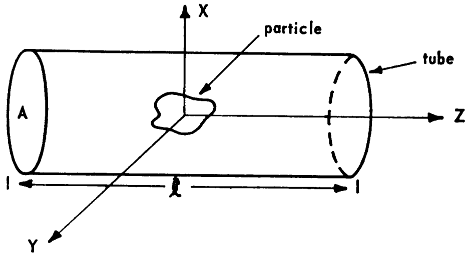

FIGURE 2 Geometry of the boundary value problem.

density $\mathbf{J} = \sigma \mathbf{E}$ (σ is the conductivity of the fluid) and that the total current $I = \int \mathbf{J} \cdot \mathbf{d}S = \sigma \int \mathbf{E} \cdot d\mathbf{S}$ where the integral is performed over either end of the tube, we have from equation 1

$$-VA + \frac{lI}{\sigma} + \oint \phi \frac{\partial z}{\partial n} dS = 0, \tag{2}$$

where the integral is performed over the particle surface only. The normal is directed toward the interior of the particle. If we set $R_0 = l/\sigma A$ (the resistance of the fluid in the tube in the absence of the particle), we have

$$(R - R_0)IA = \oint \phi \frac{\partial z}{\partial n} dS, \tag{3}$$

where we have used Ohm's law $V = IR$. We now assume that the particle is small so that $R - R_0$ is small (say of order ϵ). The difference between I and I_0 ($I_0 = V/R_0$) will also be small so that we have

$$(R - R_0)I = (R - R_0)I_0 + O(\epsilon^2)$$

where $O(\epsilon^2)$ means: of order ϵ^2.

On the right-hand side of equation 3, we approximate ϕ by ϕ_0 where ϕ_0 is the solution of the boundary value problem

$$\nabla^2 \phi_0 = 0,$$

$$\partial \phi_0 / \partial n = 0 \quad \text{on the surface of the particle,}$$

and

$$\phi_0 \rightarrow -E_0 z \quad \text{as} \quad \mathbf{r} \rightarrow \infty.$$

E_0 is the uniform field that would exist inside the tube in the absence of the particle. E_0 and I_0 are related by the following equation:

$$I_0 = \int \mathbf{J}_0 \cdot d\mathbf{S} = J_0 A = \sigma E_0 A.$$

We may write equation 3, correct to first order in ϵ,

$$\frac{R - R_0}{R_0} = \frac{\oint \phi_0 \frac{\partial z}{\partial n} dS}{\tau E_0}, \tag{4}$$

where $\tau = Al$ is the volume of the tube. From equation 4 we can determine the change in resistance for a particle of arbitrary shape in the tube, provided we can calculate ϕ_0, the potential field about the particle in a *uniform* field, E_0.

APPLICATIONS

We shall illustrate equation 4 with two examples. Consider first a sphere of radius a. Then

$$\phi_0 = -E_0\left(r \cos \theta + \frac{a^3 \cos \theta}{2r^2}\right)$$

and

$$\frac{\partial z}{\partial n} = -\frac{\partial r \cos \theta}{\partial r} = -\cos \theta$$

so that

$$\frac{R - R_0}{R_0} = \frac{\oint \phi_0 \frac{\partial z}{\partial n} dS}{E_0 \tau} = \frac{\int_0^\pi \frac{3}{2} E_0 a \cos^2 \theta \, 2\pi a^2 \sin \theta \, d\theta}{E_0 \tau}$$

$$= \frac{2\pi a^3}{\tau} = \frac{3}{2}\left(\frac{4/3 \pi a^3}{\tau}\right).$$

That is, the fractional change in resistance is 3/2 the ratio of the volumes. This result was first derived by H. A. Lorentz (3) and later, more rigorously, by Rayleigh (2).

We next consider a prolate spheroid. We may construct ϕ_0 as follows. Two linearly independent solutions of Laplace's equation are

$$\frac{1}{2} a\eta\xi$$

and

$$\frac{1}{2} a\eta\left[\xi \ln \frac{\xi + 1}{\xi - 1} - 2\right],$$

where a, η, and ξ are defined in Morse and Feshbach's book (4). The surface of the spheroid is defined by $\xi = \xi_0$. Now since $z = 1/2(a\eta\xi)$, the solution of Laplace's equation for which $\phi_0 \rightarrow -E_0 z$ and $r \rightarrow \infty$ and $\partial\phi_0/\partial n = (1/h_\xi)(\partial\phi_0/\partial\xi) = 0$ on $\xi = \xi_0$ is

$$\phi_0 = -E_0 \frac{1}{2} a\eta\left[\xi - A(\xi_0)\xi \ln \frac{\xi + 1}{\xi - 1} + 2A(\xi_0)\right],$$

where $A(\xi)$ is defined by the equation

$$1 - A(\xi)\frac{d}{d\xi}\left[\xi \ln \frac{\xi + 1}{\xi - 1}\right] = 0.$$

We have then

$$\frac{R - R_0}{R_0} = \oint \phi(\xi_0, \eta)\frac{\partial}{h_\xi \partial_\xi}\left(\frac{1}{2} a\eta\xi_0\right) h_\phi \, d\phi \, h_\eta \, d\eta$$

$$= \frac{a^3\pi}{3\tau} \frac{1}{\dfrac{2\xi_0}{\xi_0^2 - 1} - \ln \dfrac{\xi_0 + 1}{\xi_0 - 1}}. \tag{5}$$

Employing the more customary parameterization of the ellipsoid in terms of its semiminor axis b and semimajor axis c we have

$$a = 2\sqrt{c^2 - b^2} \tag{6}$$

and

$$\xi_0 = c/\sqrt{c^2 - b^2},$$

so that equations 5 and 6 determine the fractional change in resistance in terms of the minor and major axes. Since the volume of a prolate spheroid ($\xi = \xi_0$) is

$$\tau_p = \frac{\pi a^3}{6} \xi_0(\xi_0^2 - 1),$$

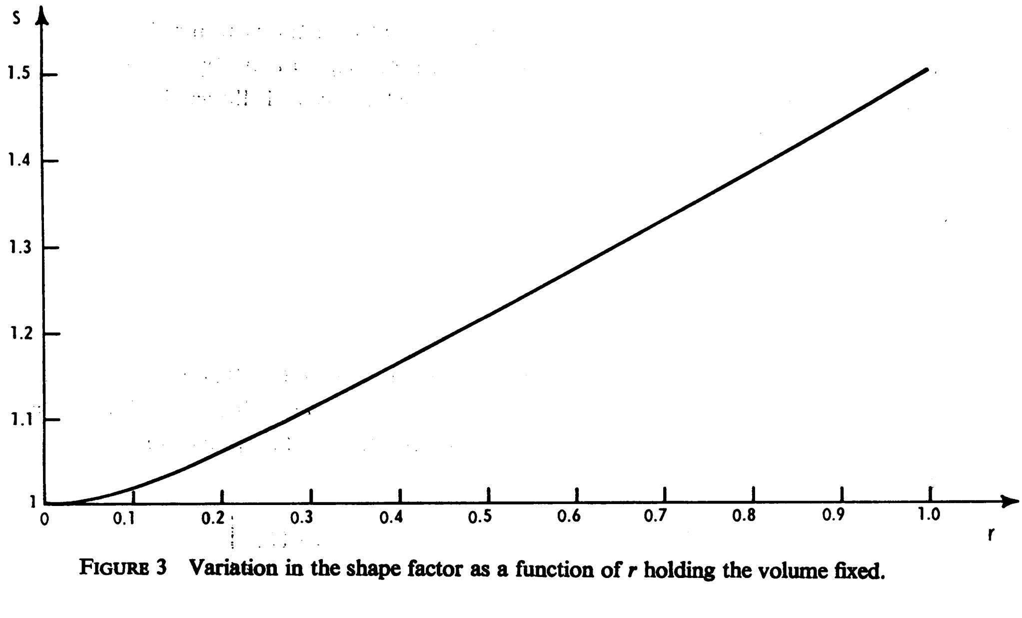

FIGURE 3 Variation in the shape factor as a function of r holding the volume fixed.

we see that the change in resistance is *not* proportional to the ratio of the volumes of the particle and tube as was the case for the sphere but depends on the shape of the particle.

The magnitude of this effect is best illustrated by expressing the fractional change in resistance in terms of τ_p and ξ_0 thus:

$$\frac{R - R_0}{R_0} = \frac{\tau_p}{\tau} \frac{2}{2\xi_0^2 - \xi_0(\xi_0^2 - 1)\ln\frac{\xi_0 + 1}{\xi_0 - 1}},$$

$$= \frac{\tau_p}{\tau} S$$

where S is the "shape factor" and defined by the factor multiplying τ_p/τ. If we set $r = b/c$, the ratio of the semiminor to the semimajor axis, we have from equation 6

$$\xi_0 = \frac{1}{\sqrt{1 - r^2}}.$$

In Fig. 3 we have plotted S as a function of r holding τ_p constant.

FIGURE III

The curve is nearly linear near $r = 1$ (a sphere) with a slope of 0.6. To let $r \to 0$ holding τ_p constant, c must approach infinity (and $b \to 0$). When c becomes comparable to the size of the tube the approximation employed breaks down.

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REFERENCES

1. GREGG, E. C., and K. DAVID STEIDLEY. 1965. *Biophys. J.* **5**:393.
2. LORD RAYLEIGH. 1892. *Phil. Mag.* **34**:481.
3. LORENTZ, H. A. 1880. *Ann. Phys. Chem. Ser. II.* **9**:641.
4. MORSE, PHILIP M., and HERMAN FESHBACH. 1953. *In* Methods of Theoretical Physics. McGraw-Hill Book Company, New York. 1284, 1327.