THE INPUT IMPEDANCE OF AN ASSEMBLY OF RANDOMLY BRANCHING ELASTIC TUBES

M. G. TAYLOR

From the Department of Physiology, University of Sydney, Sydney, Australia

ABSTRACT Computations are presented of the input impedance of assemblies of randomly bifurcating elastic tubes, as a generalized model of the arterial system. Account is taken of the viscosity of the fluid, the viscoelastic properties of the walls, the variation of elasticity in the different orders of branches, and the variation in cross-sectional area at the bifurcations. The results show that the distributed and scattered nature of the terminations of such an assembly greatly reduces the influence of reflections upon the behavior of the input impedance. The variation of impedance with frequency is very similar in form to that found in animal experiments for the input impedance of the aorta. The architecture of the arterial system may thus be considered to play an important part in determining the favorably low impedance presented to the heart by the aorta.

INTRODUCTION

In most discussions of models of the arterial system it has been customary to consider a single line, with either constant or variable properties along its length, but in any case with one defined termination. The results of calculations based on such simple models have been found to agree reasonably well with findings in animal experiments. This agreement is generally satisfactory for low frequency disturbances (e.g. the first four or five harmonic components of the normal pulsatile events), but when the analysis is carried to higher frequencies, discrepancies from the expected behavior are found. In particular, the influence of reflections appears to become smaller with increasing frequency.

There are a number of factors which may be responsible for this and they will be discussed in detail below, but briefly they are: (*a*) the fact that the arterial system is *not* a single tube, but consists of an assembly of branching tubes; (*b*) the attenuation of traveling waves by the viscous properties of blood and the arterial wall. These factors have not all been included in simple models of the arterial system, and it is the aim of this paper to examine them and their interactions.

In 1955, the late J. R. Womersley (McDonald, 1960) pointed out that the arterial system, with its many branches of differing length, and its scattered terminations, ought to behave like a sound absorber, and on this ground he was doubtful whether reflection effects could be expected to play any important role in its behavior. In subsequent work, however, McDonald and Taylor (1959) found that for oscillations of low frequency (up to the fifth harmonic) the terminations of an artery, or an arterial bed, constituted at least an "effective" reflection site. In the dog the effective site of reflections for the thoracic aorta was placed at the level of the pelvis, while the corresponding region for the femoral artery was at the level of the knee. More recent work in this laboratory, on the input impedance of the ascending aorta, femoral artery, and other vascular beds, agrees with results published by Patel, de Freitas, and Fry (1963). It is found that with increasing frequency, after a steep initial fall, the impedance reaches a minimum, after which it exhibits succeeding maxima and minima, but these are of relatively small size. The phase angle of the input impedance is at first large and negative, approaching $-\frac{1}{2}\pi$ during the steep fall of the modulus, passing through zero again as the modulus reaches its minimum, and thereafter showing small fluctuations. Whatever may be the reasons for the initial fall in the modulus, the fact that a minimum value is attained certainly indicates that reflection effects are present. The smallness of the subsequent variations in impedance after the first minimum indicates that the influence of reflections is relatively small at these higher frequencies. At least some of the diminution of the effects of reflection can be explained in the way that Womersley suggested; with increasing frequency the wavelengths become shorter, so that there will be an increase in the phase differences between reflected components arriving at the origin from terminations lying at different distances from it. The reflections will therefore tend to cancel out, and, viewed from the origin, the system will thus appear to have absorbed the incident energy. If, in addition, allowance is made for the attenuation of traveling waves by the viscous losses in the system, a still greater depression of the effects of reflection is to be expected.

This paper presents the results of a series of computations of the input impedance of assemblies of branching, nonuniformly elastic tubes, to illustrate the way in which the presence of many scattered terminations influence the input impedance, and to examine the influence of attenuation due to viscosity of both the contained fluid and the wall material, as well as the effects of variation in total cross-section of the system.

Most of the calculations have been made in relation to an assembly with properties chosen so that it may represent a reasonable model of a generalized arterial system, followed out to the seventh order of branching. A few calculations have also been made for an assembly with properties appropriate to the arteries of a single regional vascular bed.

The only comparable simulation of the properties of a branching assembly is that which has been carried out by Noordergraaf and his colleagues in Utrecht (Noordergraaf et al., 1964), who have constructed a multisegment delayline model of the entire human arterial tree. Such an analogue, which has produced waveforms in fair agreement with those actually encountered, is, however, hardly a suitable instrument with which to explore the detailed effects of variations in vascular properties or dimensions; to do this would require rebuilding it for every example.

METHODS OF COMPUTATION

The particular assumptions will be given for the individual examples presented below, but since they are at different levels of complexity, it is convenient to deal with all their possible features together. The relationships are taken to be linear in all instances.

The Basic Unit. The basic unit for all calculations is an elastic tube, filled with an inviscid fluid of mass, M, per unit length of tube, and having a perfectly elastic wall, giving it a capacitance, C, per unit length. In terms of pressure-volume relations

$$V \cdot dP/dV = 1/C$$

If we consider the relation between pressure and linear velocity of flow (not volume flow), we find that the characteristic impedance of such a tube is given by;

$$Z_o = (M/C)^{1/2}, \tag{1}$$

and the propagation constant by;

$$\lambda = i\omega(M \cdot C)^{1/2}. \tag{2}$$

A pressure oscillation of circular frequency $\omega = 2\pi f$ (where f is the frequency), traveling in the direction of x increasing, can be expressed by;

$$P(x, t) = P(t) \cdot e^{-\lambda x} = P_o e^{i\omega[t-(x/c)]}, \tag{3}$$

where c is the wave velocity, which may be complex in what follows.

If we take a length, l, of this tube, terminated in an impedance Z_T at $x = l$, then the input impedance at the origin $x = 0$, is defined by

$$(\mathcal{Z}_o) = Z_o \cdot \left\{ \frac{1 + R \cdot e^{-2\lambda l}}{1 - R \cdot e^{-2\lambda l}} \right\} \tag{4}$$

where the reflection coefficient is given by

$$R = \frac{Z_T - Z_o}{Z_T + Z_o} \tag{5}$$

In what follows, frequent use is made of this relationship, or its reciprocal, the input admittance,

$$\mathcal{Y}_o = (\mathcal{Z}_o)^{-1} \tag{6}$$

The Viscoelastic Wall. The viscosity of the wall material may be taken into account by making the capacitance, C, complex. It has been found experimentally that the dynamic elastic constant, E, for arterial walls (Hardung, 1953; Bergel, 1961) may be expressed as a complex quantity

$$E = |E| \cdot e^{i\phi} \tag{7}$$

where the phase angle ϕ has been found to increase rapidly from zero frequency, and then to remain almost constant, of the order of 5° to 10°, depending upon the artery in question. The modulus $|E|$ has also been found to increase slightly with increasing frequency. In the present computations, this property has been approximated by neglecting the increase in $|E|$, but replacing the capacitance C by $C \cdot e^{-i\phi}$. It can be seen that Z_o now becomes

$$Z_o = (M/C)^{1/2}(\cos \tfrac{1}{2}\phi + i \sin \tfrac{1}{2}\phi) \tag{8}$$

while λ becomes

$$\lambda = i\omega \cdot (M \cdot C)^{1/2} \cdot (\cos \tfrac{1}{2}\phi - i \sin \tfrac{1}{2}\phi) \tag{9}$$

The variation in phase angle ϕ with frequency ω, has been approximated by taking

$$\phi = \phi_o(1 - e^{-\gamma\omega}) \tag{10}$$

and in the scale of these computations γ has been taken to be 1 or 2.

Fluid Viscosity. The treatment of Womersley (1957) has been followed. If the tube is considered to be in a condition of limiting longitudinal constraint (the so called "tethered tube"), and if the fluid within it is viscous, then we find, in the present notation

$$Z_o = (M/C)^{1/2} \cdot (1 - F_{10})^{-1/2} \tag{11}$$

and

$$\lambda = i\omega \cdot (M \cdot C)^{1/2} \cdot (1 - F_{10})^{-1/2} \tag{12}$$

Here the term $(1 - F_{10})$ is Womersley's notation for

$$\left\{1 - \frac{2 J_1(\alpha i^{3/2})}{\alpha i^{3/2} \cdot J_0(\alpha i^{3/2})}\right\}$$

in which $\alpha = R\ (\omega/\mu)^{1/2}$ where R is the radius of the vessel and μ is the kinematic viscosity of the fluid, while J_o and J_1 are Bessel's functions of the first kind, zero and first order respectively.

Womersley's treatment also included a term involving σ, the Poisson ratio of the wall material, but this has been dropped from the present consideration, as it influences the results only by a constant, in fixing the limiting value of the wave velocity for large values of α.

The α-dependent terms in the computations were found, for various frequencies and vessel sizes, by choosing an appropriate value of α for a given vessel of cross-section A_o, and at a particular value of ω_o, so that for any other values of A_n and ω, we have

$$\alpha_n = \alpha_o\{(A_n/A_o) \cdot (\omega/\omega_o)\}^{1/2} \tag{13}$$

The influence of a viscoelastic wall together with that of a viscous fluid, has been taken into account by replacing C by $C \cdot e^{-i\phi}$ as in equations (8) and (9); this is permissible provided that the model is of the tethered tube.

Nonuniform Elasticity. In order to include in the model the known elastic nonuniformity of the arterial system, the capacitance of the tube was prescribed to vary with the order of branching in such a way that the wave velocity varied according to the arbitary but reasonable expression

$$(M \cdot C)^{1/2} = 3 - 2(2/3)^n \tag{14}$$

Thus, the wave velocity in the original tube is 1 unit, and with increasing order of branching, approaches 3 units. This range of values makes the present computations comparable with the results obtained previously for a single nonuniform line (Taylor, 1965).

Properties of the Bifurcating Assembly. Most of the computations were carried out for a model consisting of a set of bifurcating tubes, with the general properties given below. It should be noted that no account has been taken of the hydrodynamic peculiarities of a bifurcation as such.

If a single tube of cross-sectional area A_o divides into two branches, of equal cross-sectional area A_1, such that

$$A_1 = A_o(\tfrac{1}{2}\cdot d) \tag{15}$$

then if $d = 1$, the total cross-sectional area of the system is unchanged after branching, but for values of $d > 1$, the total cross-section is increased. Values of d in the arterial system (McDonald, 1960) lie between 1.2 and 1.3. After n such divisions the area of the n^{th} order branch is thus $A_o(\frac{1}{2}d)^n$, and the total cross-section becomes $A_o \cdot d^n$. (In the computations $A_o = 1$, for convenience.)

The system chosen for computations had branching up to the seventh order; that is, there were 128 final branches, terminating in identical pure resistances. The value of the terminal impedance was chosen to give various values of the nominal reflection coefficient at the terminations.

The length of a branch in the assembly was chosen, as will be described below, to belong to a population of lengths randomly distributed about a mean value depending on the order of the branch; that is, if the mean length of the original tube was $\bar{L}_o$, then the mean length of an n^{th} order branch $\bar{L}_n$ was set to be

$$\bar{L}_n = \frac{1}{n+1}\cdot \bar{L}_o \tag{16}$$

In the computations, it was generally taken that $\bar{L}_o = 1$.

The kind of assembly to which this choice of lengths could give rise is illustrated in Fig. 1.

Generation of the Random Lengths. The lengths of the branches of given order were chosen to be distributed about the mean length for that order, according to the normalized form of the gamma distribution function. If the mean length of a group is $\bar{L}$, then the

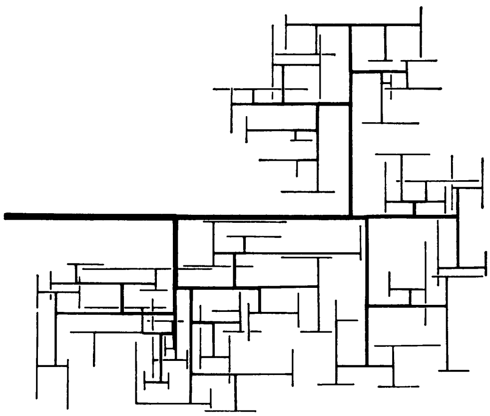


FIGURE 1 Diagram of an assembly with bifurcation to seventh order; the lengths of the branches are to scale, but the diameters are only indicated qualitatively. This assembly was the basis of all calculations except those shown in Figs. 7, 8, and 12.

frequency of occurrence of branches having a length between L and $L + dL$, is given by

$$F(L) = \left(\frac{k+1}{\bar{L}}\right)^{k+1} \cdot \frac{L^k}{k!} \cdot e^{-\{L \cdot (k+1)/\bar{L}\}} \cdot dL \tag{17}$$

where k is an integer, the order of the function.

To generate a set of lengths obeying this distribution function, the method of inverse interpolation of the cumulative function $G(L)$ was employed (Abramowitz and Stegun, 1964). If $F(L) = dG(L)$ with $G(0) = 0$, the desired values of L can be found by solving

$$G(L) - N = 0 \tag{18}$$

where N is a random number, uniformly distributed in the range 0 to 1.

In the calculations for the bifurcating assembly, the value $k = 2$ was chosen, with mean value $\bar{L} = 1$; and since $G(L)$ was a known function, equation (18) was solved for values of N provided by a random number routine in the computer, by simple iteration of the Newton-Raphson method. The lengths of individual branches of order n were obtained by scaling by the appropriate mean length for that order. Fig. 2 shows the form of $F(L)$ for various values of k, and a set of 32 random lengths, generated as described above, with $\bar{L} = 1$ and $k = 2$.

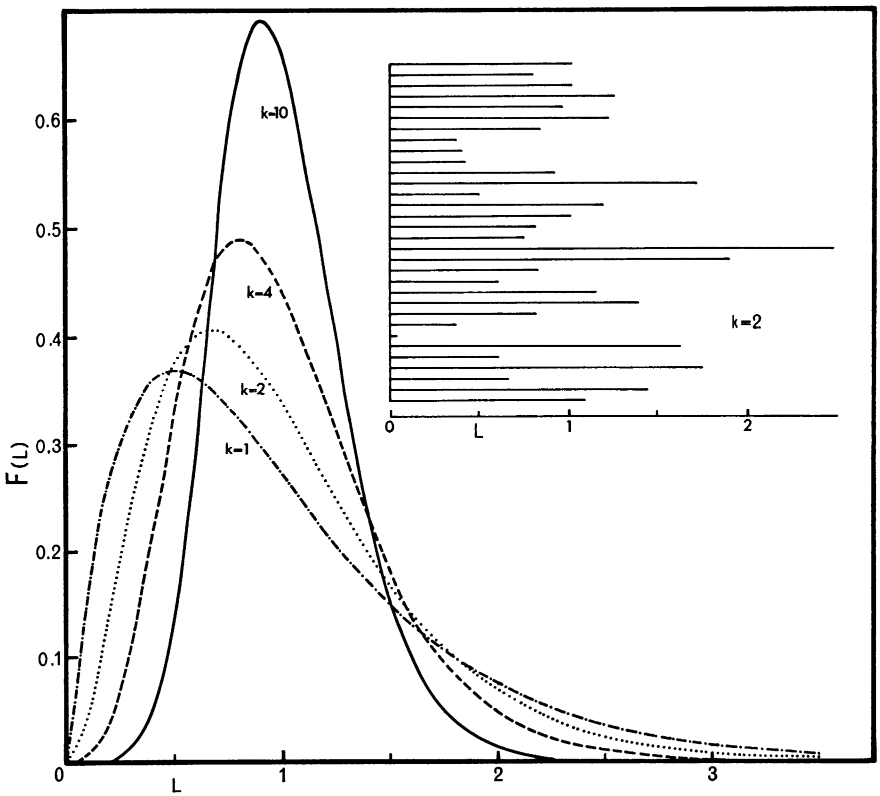


FIGURE 2 Gamma distribution function (equal 17) for four values of the order k. Inset, 32 random lengths generated for $k = 2$, mean value at $L = 1$.

Computation of the Input Impedance of the Assembly. The general scheme of the computation is shown in Fig. 3. Given a particular set of values of the parameters involved (α_o, ϕ_o, γ, d, R), and having generated the appropriate sets of random numbers giving the lengths of the branches, the calculations for a particular value of ω were begun by finding the input admittance of each of the final branches. These were then taken, two by two, to determine the terminal impedances of the next-to-last branches, of which the input admittances could then be calculated; and so on, working backwards, to arrive at the input admittance of the original tube, and thus finally at the input impedance of the whole assembly. The process was repeated for each value of ω, in the range 0 to 4π in steps of π40.

It is perhaps necessary to remark that the impedances were scaled according to the area of the individual branch concerned, since this definition of the impedance is in relation to average flow velocity, in particular the flow velocity at the origin of the assembly.

The computations were carried out on the English Electric K.D.F. 9 computer at the Basser Computing Laboratory of the School of Physics of this University. The programs were written in the mnemonic machine language USERCODE, and the running time for each set of 160 impedance values was approximately 3 min. In all, over 100 different sets were calculated, of which a selection is given in the examples of this paper.

RESULTS

The Influence of Scattered Terminations on the Input Impedance. As was

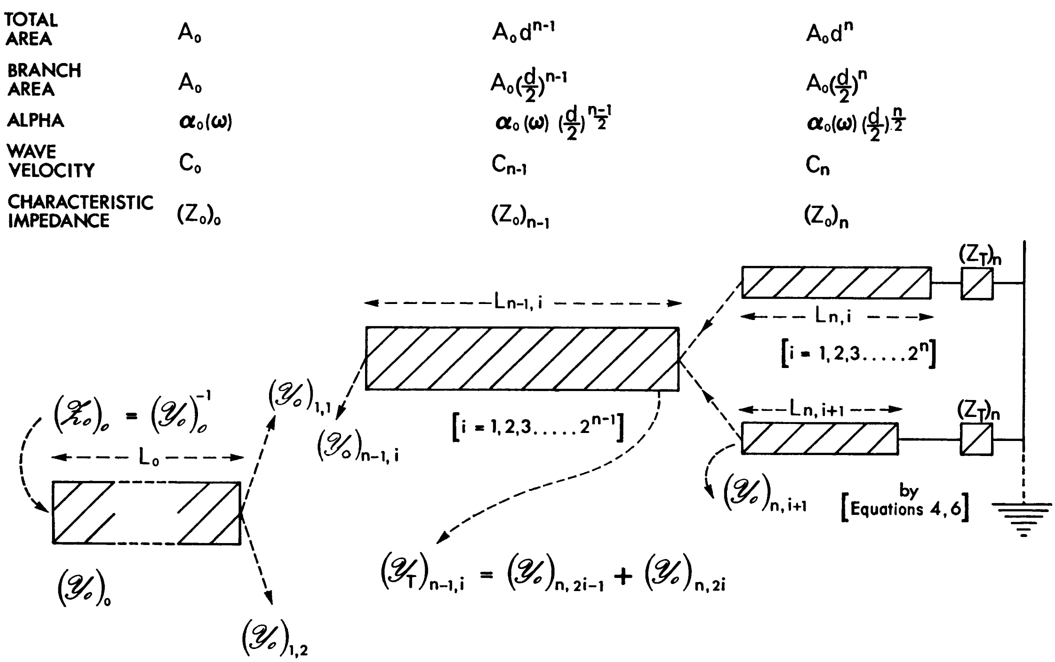


FIGURE 3 The scheme of computation. Values of the various properties of the branches are tabulated above for the original tube, and the penultimate and final branches. As described in the text, the computation begins by finding all 2^n values of $(\mathcal{Y}_o)_{n,i}$ and then proceeds back to $(\mathcal{Y}_o)_o$.

indicated in the Introduction, most models of the arterial system have specified only a single termination; the first example to be presented now is admittedly not very close to the facts of the situation, but is a useful illustration of an extreme case. The second example is more realistic, but at the same time involves also other features than scattering.

Random Lines in Parallel. The assembly here consisted of a number of tubes, connected in parallel at their origins, as indicated in the upper part of Fig. 2. If we take these elementary divisions to have all the same wave velocity and the same characteristic impedance, then the input impedance of the assembly is given by

$$\mathcal{Z}_o = \left\{ \int_0^\infty \mathcal{Y}_o(L) \cdot F(L) \cdot dL \right\}^{-1} \tag{19}$$

where $\mathcal{Y}_o(L)$ denotes the input admittance of an elementary division of length L, given by equations (4 through 6). The examples in Fig. 4 were computed by taking 100 values of L, from 0(0.1)10, integrating the real and imaginary parts of the expression separately by Simpson's rule, and finally obtaining $\mathcal{Z}_o$ in modulus and phase. Here $c = 1$, $Z_o = 1$, $\bar{L} = 2$, $R = 0.3$, with $k = 1, 2, 4$, and 10. The input impedance of a single line, of length 2 and with $R = 0.3$ is also given, for comparison.

It can be seen that, as might be expected, the more scattered are the terminations (small k) the greater is the mutual interference of the reflections, and the less their influence on the input impedance; for $k = 1$, the modulus of the input impedance falls rapidly to a very shallow minimum, and then slowly approaches $Z_o = 1$; in other words the system comes to behave very nearly as though it were without terminal reflections at all. The variance of L about its mean value $\bar{L}$, in populations obeying equation (17), is given by $(\bar{L})^2/(k + 1)$; it can thus be seen that for larger values of k the distribution of the terminations is less scattered about the mean length. For large k the input impedance shows greater variations with increasing ω, and behaves more as if the system had a definite end, but it is still a long way from showing the regular maxima and minima found for the single tube. This rather extreme example illustrates very well the manner in which the existence of a random scattering of the terminations can lead to the "absorbing" properties which Womersley saw in the architecture of the arteries.

The Randomly Branching Assembly. For this example, the input impedance has been calculated for a random assembly of elastic tubes, with branching to the seventh order. The parameters of this assembly are: inviscid fluid (i.e. no α-dependence); perfectly elastic wall (i.e. $\phi_o = \gamma = 0$); unchanging total cross-sectional area ($d = 1$); lengths of branches given by equation (15), together with equation (17), with $k = 2$; wave velocity according to equation (16); reflection coefficient $R = 0.6$ at the terminations. The input impedance has also been calculated for a single "equivalent" tube of 8 segments, each having a length corresponding to the mean of the lengths of the branches of that order. The wave velocity in each segment was that

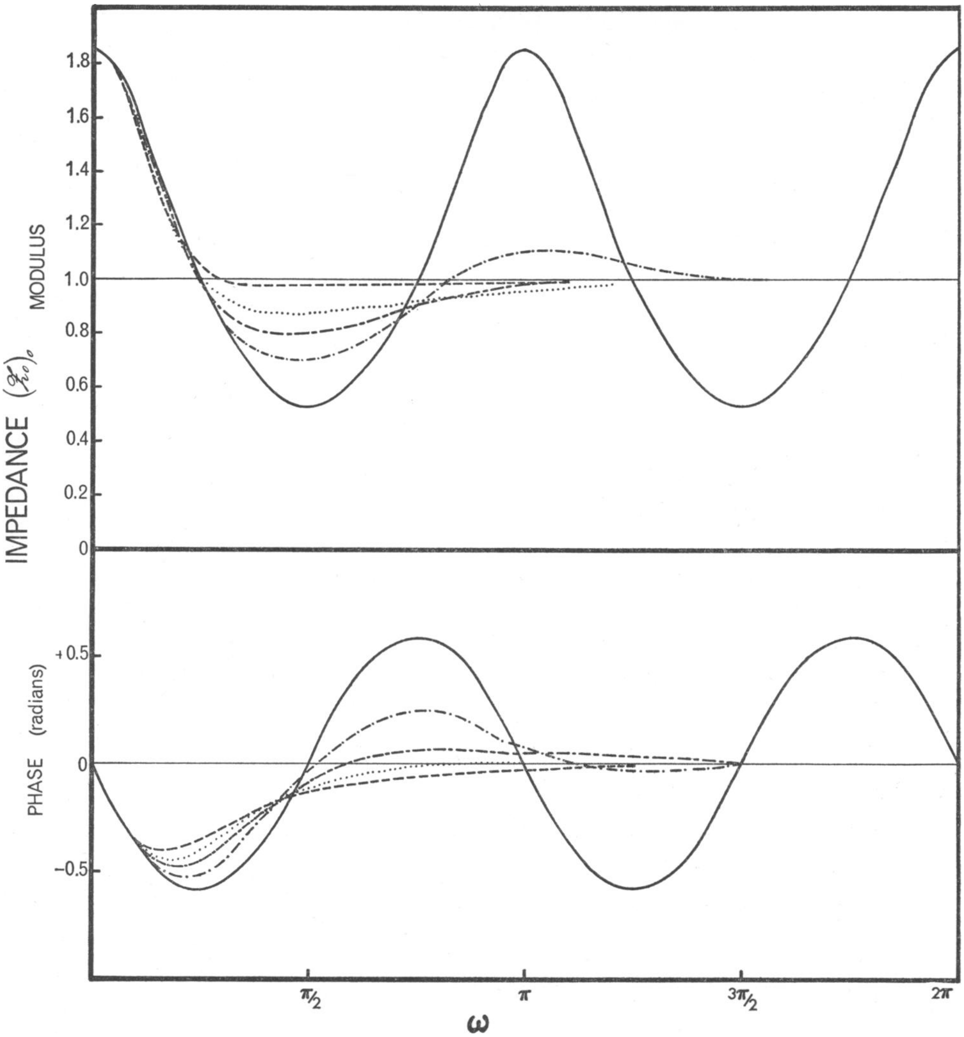


FIGURE 4 Input impedance of assemblies of 100 tubes of random length, connected in parallel at their origins. The paramters are given in the text. Solid line, equivalent single tube. $k = 1$ — — — — ; $k = 2$ . . . . ; $k = 4$ — — — — — ; $k = 10$ — . — . — . .

for the appropriate order of branching; the terminal reflection coefficient was the same, namely $R = 0.6$.

The results of these calculations are presented in Fig. 5. It can be seen that the input impedance of the single equivalent tube varies considerably with frequency, but it should be noted that it nowhere returns to the value at zero frequency. This is to be ascribed to the influence of elastic nonuniformity, which has previously been shown

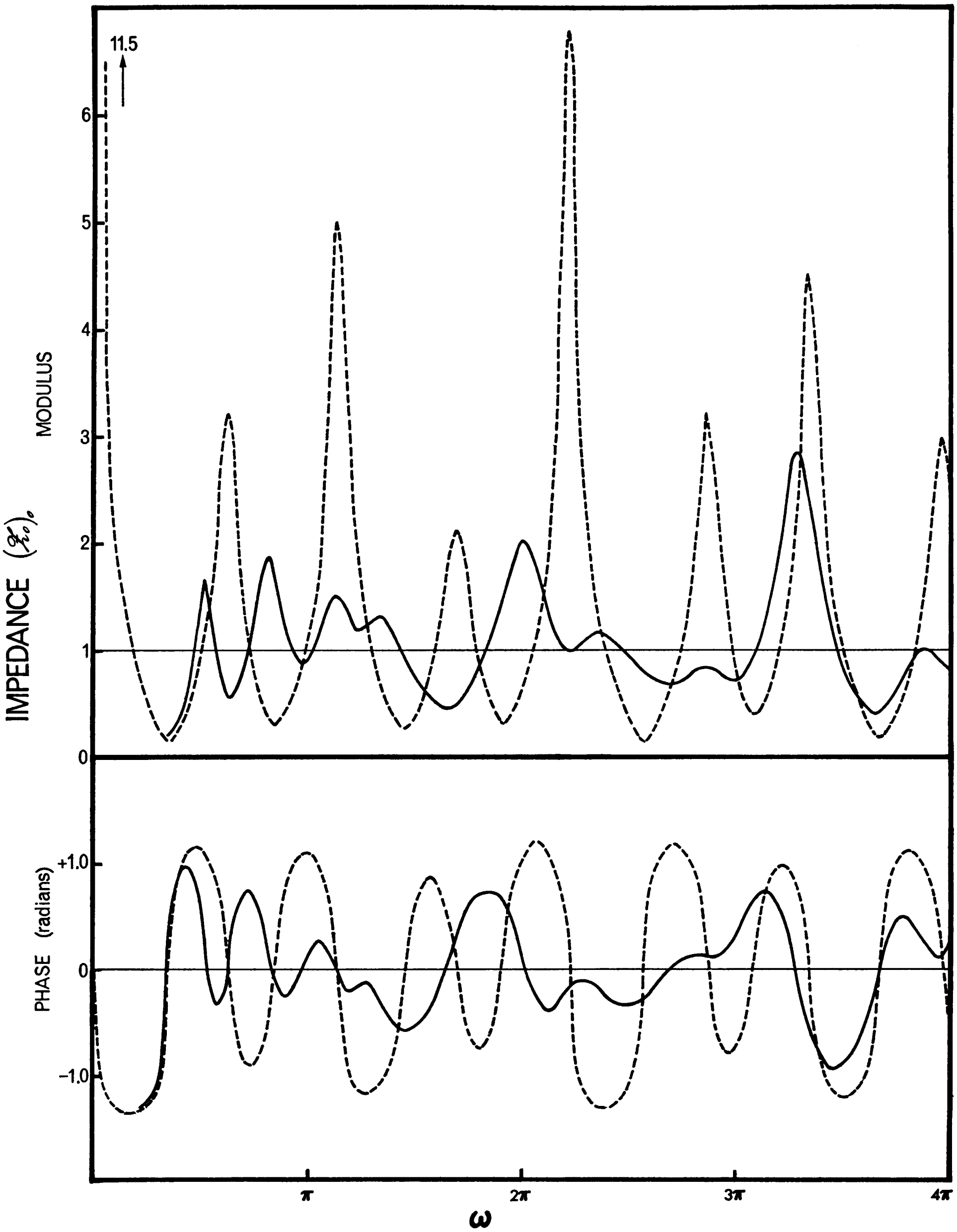


FIGURE 5 Input impedance of the branching assembly (solid line) compared with that of the "equivalent" single tube (dashed line). Reflection coefficient 0.6 in both cases so that $|\mathcal{Z}_o|_o = 11.5$ at $\omega = 0$. Note the much greater stability of the impedance for the assembly.

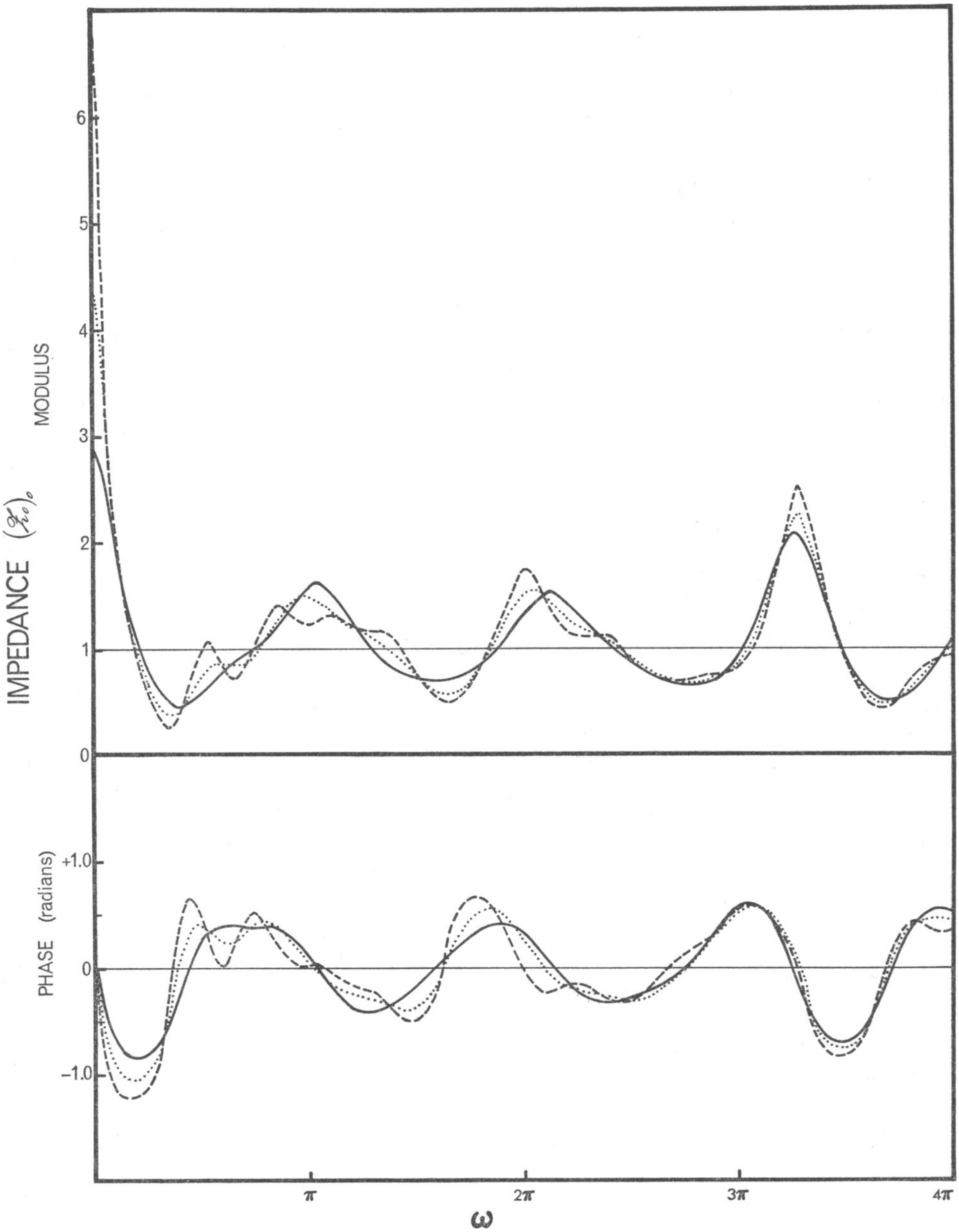


FIGURE 6 Effect of variation of reflection coefficient upon input impedance of the branching assembly. $R = 0.0$ ———— ; $R = 0.2$ . . . . ; $R = 0.4$ — — — — . Note that the effect is mainly on the values at low frequency.

(Taylor, 1965) to have a marked effect on the input impedance. Its presence leads to "isolation" of the termination, and as the frequency increases, the input impedance approaches the nominal characteristic impedance of the system at its origin.

By contrast with the single tube, the branching assembly shows much less fluctuation of its input impedance; because of the asymmetry of the system, $|\mathfrak{Z}_o|_o$ shows two small peaks after the initial minimum, but it can be seen that at higher frequencies the maxima and minima become considerably broadened, indicating the progressively greater interaction and interference of the reflections from the terminations. It is thus clear that the presence of many terminations lying at various distances from the origin, also acts to isolate the termination, and to leave the input impedance close to the nominal characteristic impedance. In the notation of this paper $(\mathfrak{Z}_o)_o \rightarrow (Z_o)_o$, and the effect of elastic nonuniformity is thus considerably enhanced.

These two examples illustrate an important common feature of these systems, namely, the steep fall of $|\mathfrak{Z}_o|_o$ from a value equal to the total terminal impedance at zero frequency (that is, the "peripheral resistance" of the circulatory system), to a minimum value at some low frequency. Both the elastic nonuniformity and the scattering of the terminations have the effect of reducing the subsequent maxima and minima, so that if the system is operating at any frequency above some minimum value, the input impedance is relatively insensitive to change in frequency, and remains stable and low. In what follows, it will be seen that the effects of damping due to fluid viscosity and the viscosity of the wall material are slight in comparison with these two others, but act to increase still further the stability of $(\mathfrak{Z}_o)_o$. Furthermore, the influence of all these factors is to reduce the sensitivity of $(\mathfrak{Z}_o)_o$ to changes in the terminal reflection coefficient, so that even a tenfold increase in the total terminal impedance causes relatively little change in $(\mathfrak{Z}_o)_o$, provided that the operating frequency is sufficiently high. The effect of changing the terminal impedance from 2.9 units (nominal reflection coefficient $R = 0.0$) to 4.3 units ($R = 0.2$) and to 6.7 units ($R = 0.4$), is shown in Fig. 6, where it can be seen that the changes in $|\mathfrak{Z}_o|_o$ after the initial minimum are relatively slight, even though the terminal impedance has been more than doubled. More extreme cases of variation in R are presented below.

In order to illustrate the fact that this kind of behavior is a general property of branching systems, the input impedance was calculated for two further assemblies, generated by two different sets of random numbers, again with $k = 2$. They are shown in the upper parts of Figs. 7 and 8 respectively. In each case the parameters of the assemblies were the same: inviscid fluid; viscoelastic wall, $\phi_o = 10°$, $\gamma = 1$; lengths by equations (15 and 17); wave velocity by equation (16); reflection coefficient $R = 0.4$. It is clear that apart from some minor differences in the behavior of $|\mathfrak{Z}_o|_o$, the main and important feature of a steep descent from the value of the terminal impedance at $\omega = 0$, is common to all three realizations of the branching assembly.

The Influence of a Viscoelastic Wall. Calculations were made of the input impedance of an assembly with the following parameters: inviscid fluid; $d = 1$; lengths of branches by equations (15) and (17); wave velocity by equation (16); reflection coefficient $R = 0.6$; and viscoelastic wall, $\gamma = 1, 2, 4$; $\phi_o = 0°, 10°, 20°$. That is, we are to examine the effect of variation in the type of viscoelasticity.

The effect of introducing viscosity into the wall material is shown in Fig. 9. It may be seen that the fluctuations in $(\mathcal{Z}_o)_o$ are considerably reduced by even the attenuation resulting from a phase angle of $\phi_o = 10°$, while for $\phi_o = 20°$ the effects are still more marked. The effect of values of γ greater than 1 was almost indistinguishable from the others, so these results have not been included. The reason for the insensitivity of the system to changes in γ is that for lower frequencies the attenuation due to wall viscosity is in any case small, so that bringing about a more rapid increase of ϕ towards its asymptotic value ϕ_o in the low frequency range has relatively little effect on the impedance.

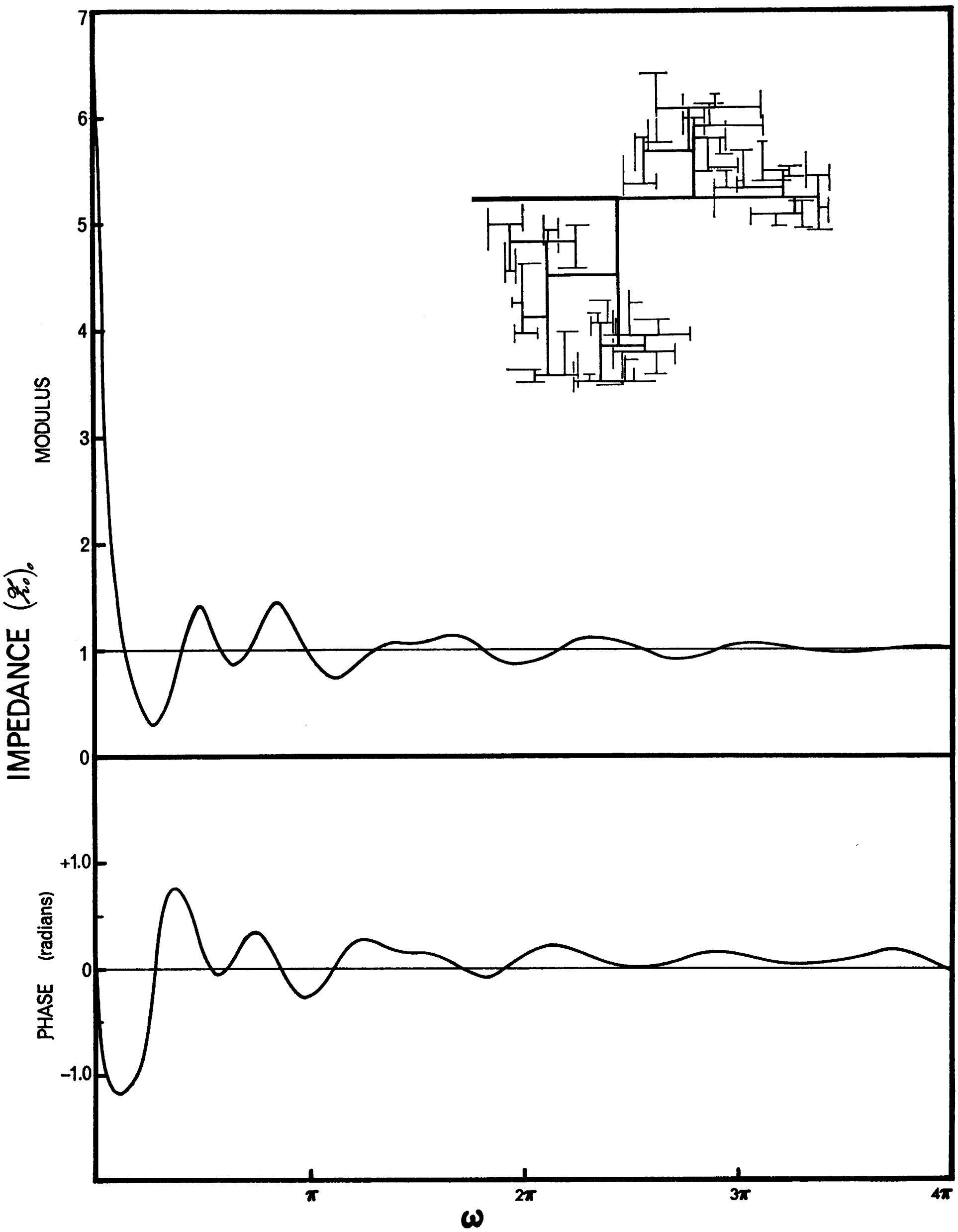
For an oscillation traveling in a tube with a viscoelastic wall, the attenuation per wavelength is given by the term $\exp(-2\pi \tan \frac{1}{2}\phi)$, so that if $\phi = 10°$ we find that after traveling one wavelength an oscillation would be reduced to 58% of its original amplitude; for $\phi = 20°$ the reduction would be to 33%.

The Influence of Fluid Viscosity. Fig. 10 shows the results of calculations of $(\mathcal{Z}_o)_o$ for the same basic assembly as in the previous section, for the three cases: (*a*) no viscous elements; (*b*) wall viscosity only, $\gamma = 1$, $\phi_o = 10°$; (*c*) fluid viscosity only, $\alpha_o = 10$.

It can be seen that the effects of attenuation, due either to the viscosity of the wall or of the fluid, are much the same, although with these particular values the wall viscosity appears to have the greater effect. One minor feature which is of interest, though of slight practical importance, is the reduction of the frequency at which the first minimum occurs, in the presence of a viscous fluid. As Womersley has shown, the wave velocity of oscillations traveling in tubes with a low value of α, is reduced particularly for $\alpha < 1$. In this present example, the value of α at the frequency of the minimum of $|\mathcal{Z}_o|_o$, is almost 10 in the original branch, but only 1.25 in the 6$^{\text{th}}$ order branches, and less than 1 in the terminal branches. With these values of α in the more distant branches, the wave velocity in them will be about one half the nominal value, so that some reduction in the frequency of the "resonance" responsible for the development of the first minimum is quite to be expected.

The Influence of Changing Cross-Sectional Area. As has been stated in previous sections, the parameter d denotes the ratio of the sum of the areas of the two equal branches to the area of the parent vessel. To show the effect of variation in this, calculations, presented in Fig. 11, were carried out for an assembly with the following characteristics: viscous fluid $\alpha = 10$; viscoelastic wall, $\gamma = 1$, $\phi_o = 10°$; lengths by equations (15 and 17); wave velocity by equation (16); $R = 0.8$; $d =$ 1.0, 1.1, 1.2, 1.3, 1.4.

The effect on the total cross-section of the bed, after n branchings, is shown in the upper part of Fig. 11. It will be seen that for $d = 1.4$, the total cross-section increases rapidly with the order of branching, and for the terminal branches is more than 10 times the original area.



IMPEDANCE (Z/Z_0)
MODULUS
PHASE (radians)
7
6
5
4
3
2
1
0
+1.0
0
-1.0
π
2π
3π
4π
ω

FIGURE 7

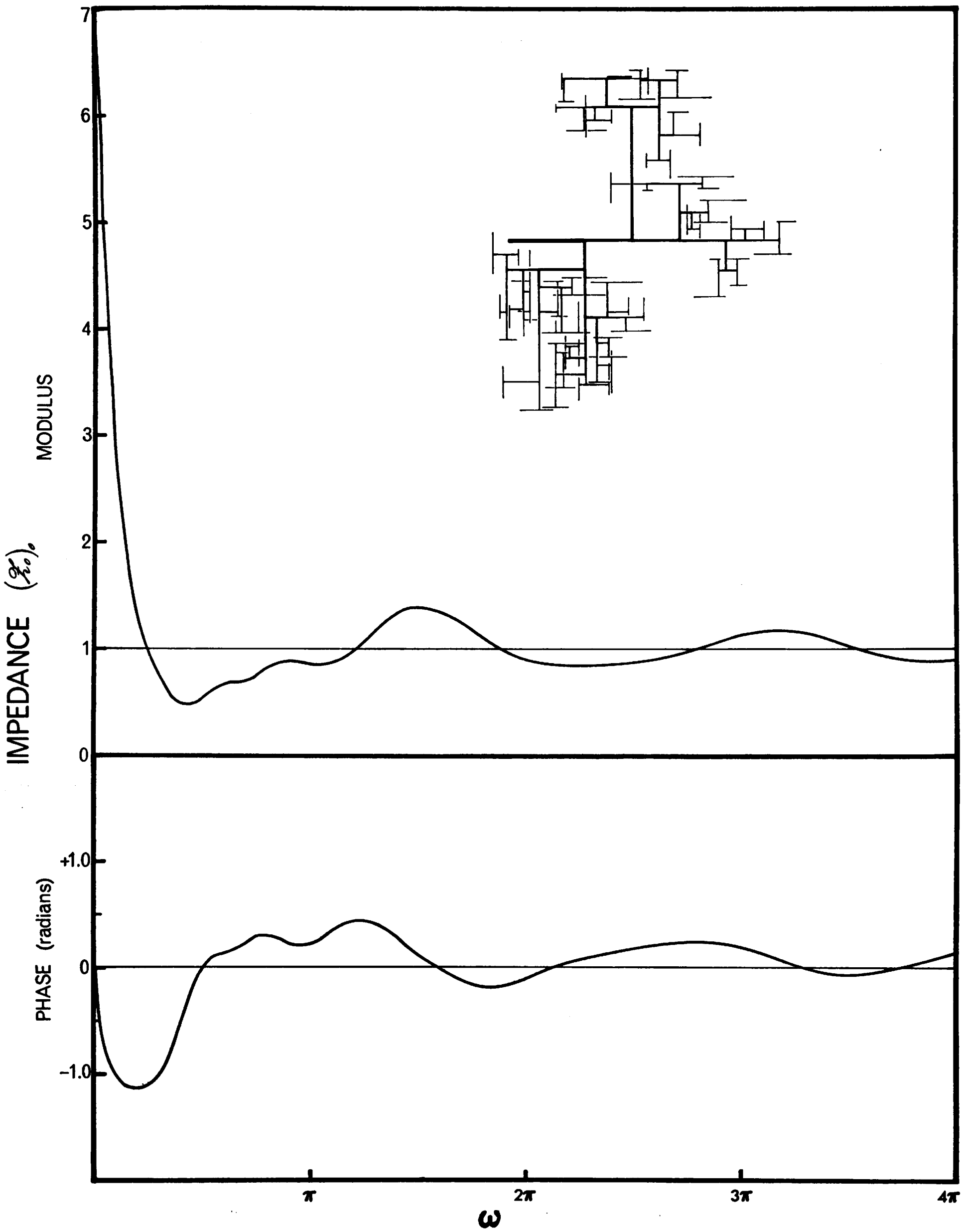


FIGURE 8

FIGURES 7 AND 8 Two examples of input impedance of assemblies other than Fig. 1. In each case the particular assembly is inset. Parameters in text, viscoelastic wall, inviscid fluid, $R = 0.6$. Note the general similarity of these results, particularly the steep fall of modulus from the dc values, and the initial large negative values of phase.

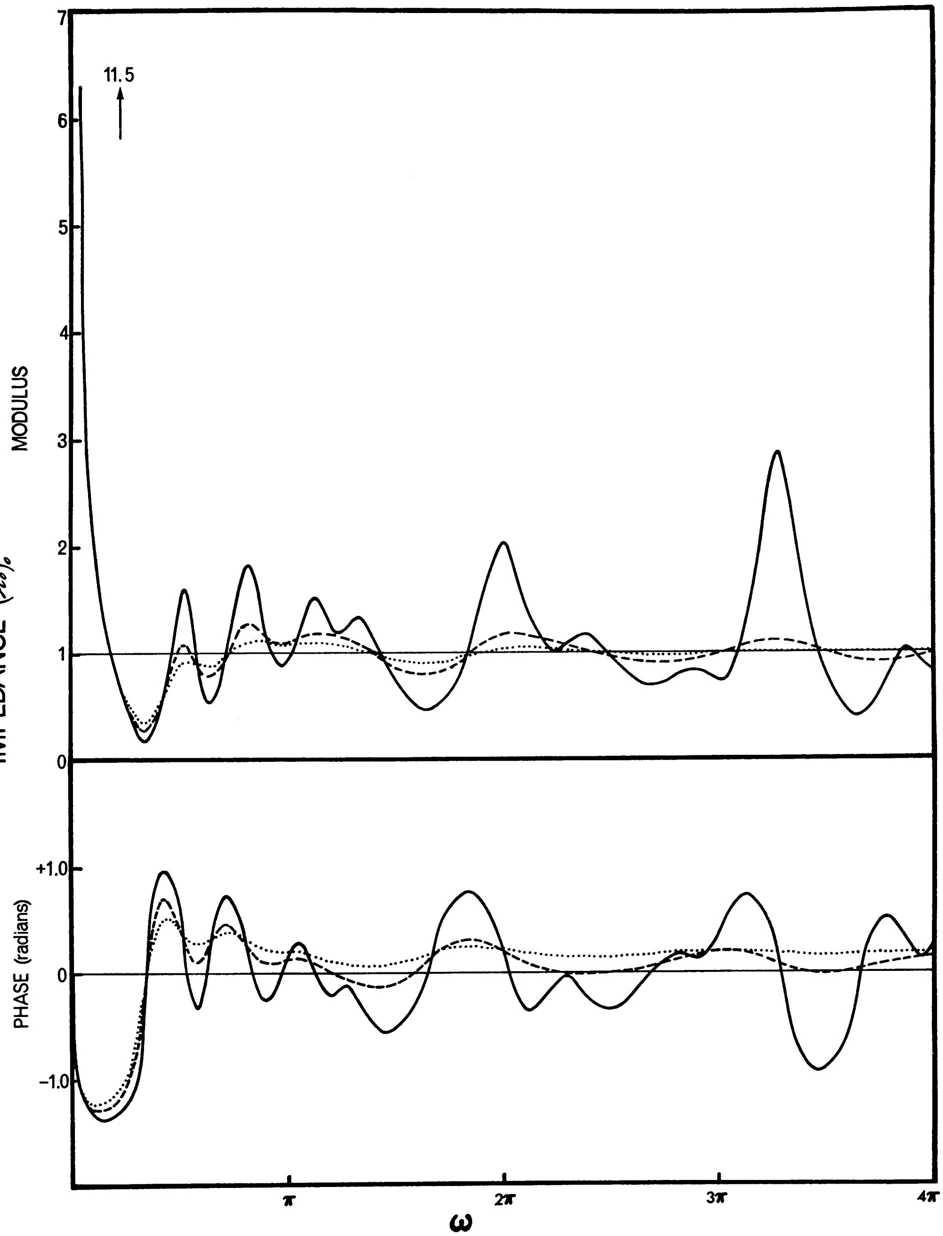


FIGURE 9 Comparison of effects of different degrees of wall viscosity upon input impedance of the assembly. Parameters in text; No viscosity ———— ; $\phi_o = 10°$ – – – – – – ; $\phi_o = 20°$ .... Note that the effect on impedance modulus is greatest for large ω; note also that the introduction of wall viscosity shifts the phase angle of $(\mathcal{Z}_o)_o$ (Equation 8).

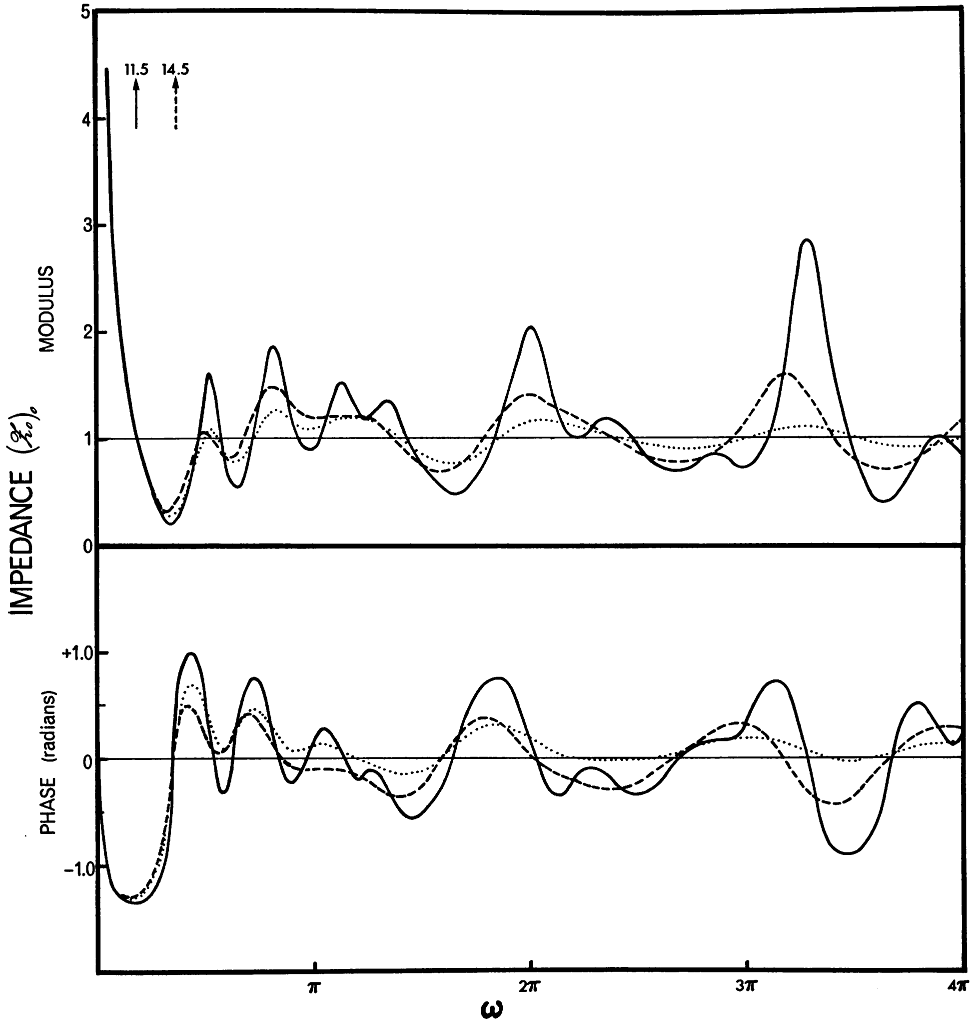


FIGURE 10 Comparison of the effects of fluid viscosity and wall viscosity upon input impedance of the assembly. Parameters in text. No viscosity ————————; wall viscosity only . . . . ; fluid viscosity only — — — — — — . Note increase in dc resistance with fluid viscosity.

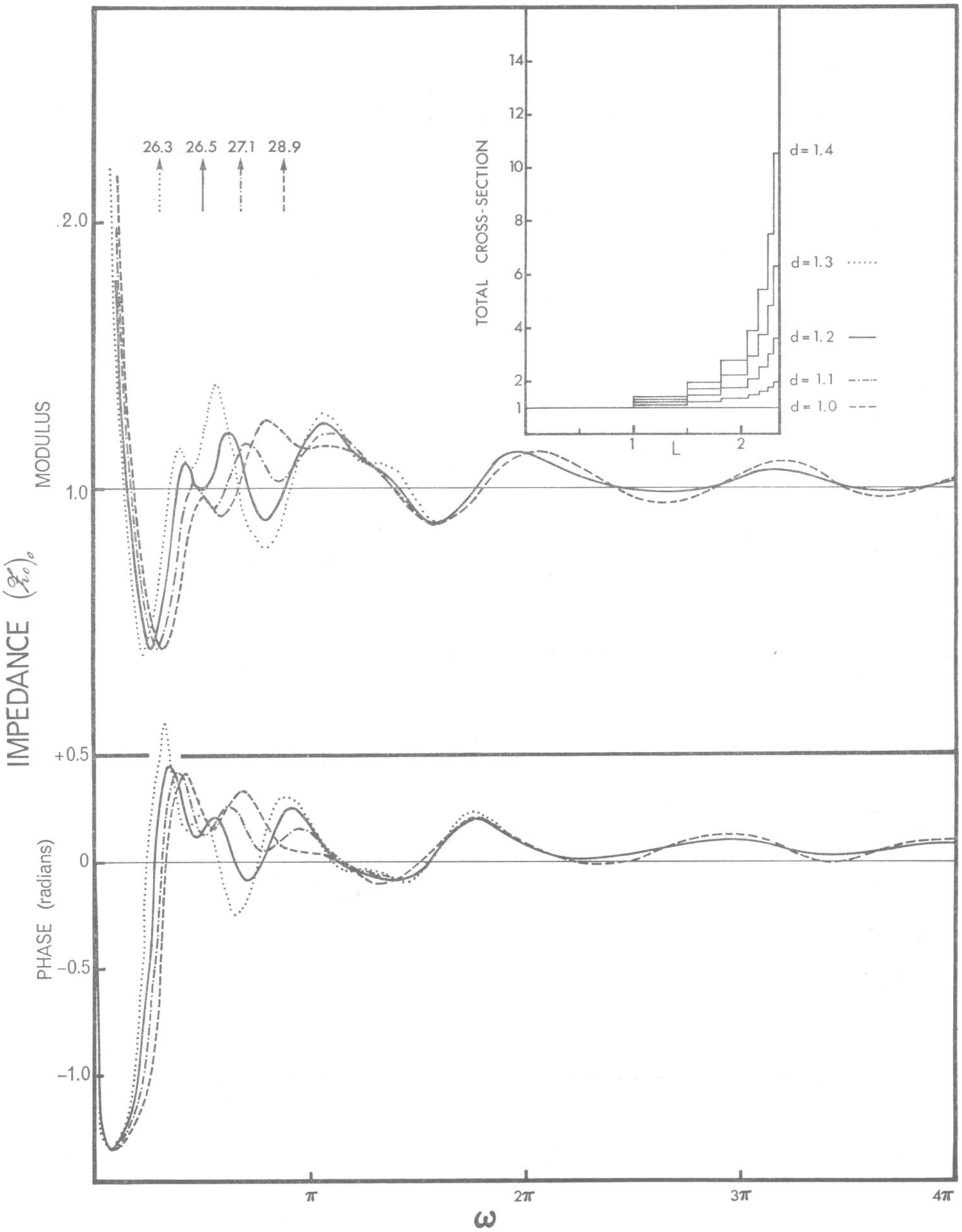


FIGURE 11 Comparison of effects of changing cross-sectional area of branching. Inset shows effect on total cross-section. Note decreasing dc resistance with increasing values of *d;* note also displacement of first minimum towards lower frequencies. In other respects the variations in cross-section have little influence on the impedance.

Although in these examples, the same value was used for the terminal impedance (25.9 units), corresponding to a nominal reflection coefficient of 0.8 with $d = 1$, it will be observed that the value of $|\mathcal{Z}_o|_o$ at zero frequency differs with different values of d. This is because the presence of a "viscous fluid" in the system now introduces a viscous resistance in the vessels of the assembly itself, which must be added to that of the terminations. With increasing values of d, and hence increasing vessel diameter, this extra resistance becomes smaller, being 2.95 units for $d = 1.0$, and falling to 0.22 units for $d = 1.4$.

The general behavior of $(\mathcal{Z}_o)_o$ for different values of d is remarkably similar. The only important difference is that with increasing values of d there is a shift of the first minimum towards lower frequencies. It might be expected that, with higher values of d and hence larger values of α in the more distant branches, the phase velocity in these would be increased, leading to a reverse of the changes discussed in the previous section, and to an increase in the "resonant frequency." The explanation of the actual outcome is to be found in a consideration of the reflection conditions at the successive branchings. Womersley (1957) has shown that division of a tube into two branches, without change in total cross-sectional area, leads to the production of reflections at the junction, by reason of the different values of α on either side of the junction. The amount of this mismatch at a bifurcation decreases with increasing d, reaching a minimum at a value of d generally between 1.2 and 1.4, depending upon the difference in elastic properties on either side of the junction. In the calculations presented here, we see, therefore, that as d is increased and the reflections from the intermediate junctions become less, the reflections originate more exclusively from the terminations. The "effective" site of reflections in the assembly is thus moved further from the origin, and it is this change which is probably responsible for the reduction in resonant frequency.

A Model "Regional" Vascular Bed. The computations in the previous sections have been in relation to a model of the whole arterial system, with variation in wave velocity in successive divisions, and with, where applicable, a large value of α_o. It is interesting, therefore, to consider the behavior of a system more analogous to a regional vascular bed, for example, that of the femoral artery.

The model for this is much the same, except that the lengths have been scaled down to one quarter, and the wave velocity made constant in all branches. The parameters of the system can thus be listed as: viscous fluid, $\alpha_o = 3$; viscoelastic wall, $\gamma = 2$, $\phi_o = 10°$; lengths as by equations (15 and 17); wave velocity 3 units, constant; reflection coefficient 0.0 to 0.95.

The behavior of $(\mathcal{Z}_o)_o$ for various values of R is shown in Fig. 12. The resonant frequency is now some four or five times higher than for the model of the whole arterial system, by reason of the shorter total length and the higher wave velocity, even taking account of the decreased phase velocity in the distant branches due to the low value of α there.

It is interesting to note that even though there is no elastic nonuniformity in the

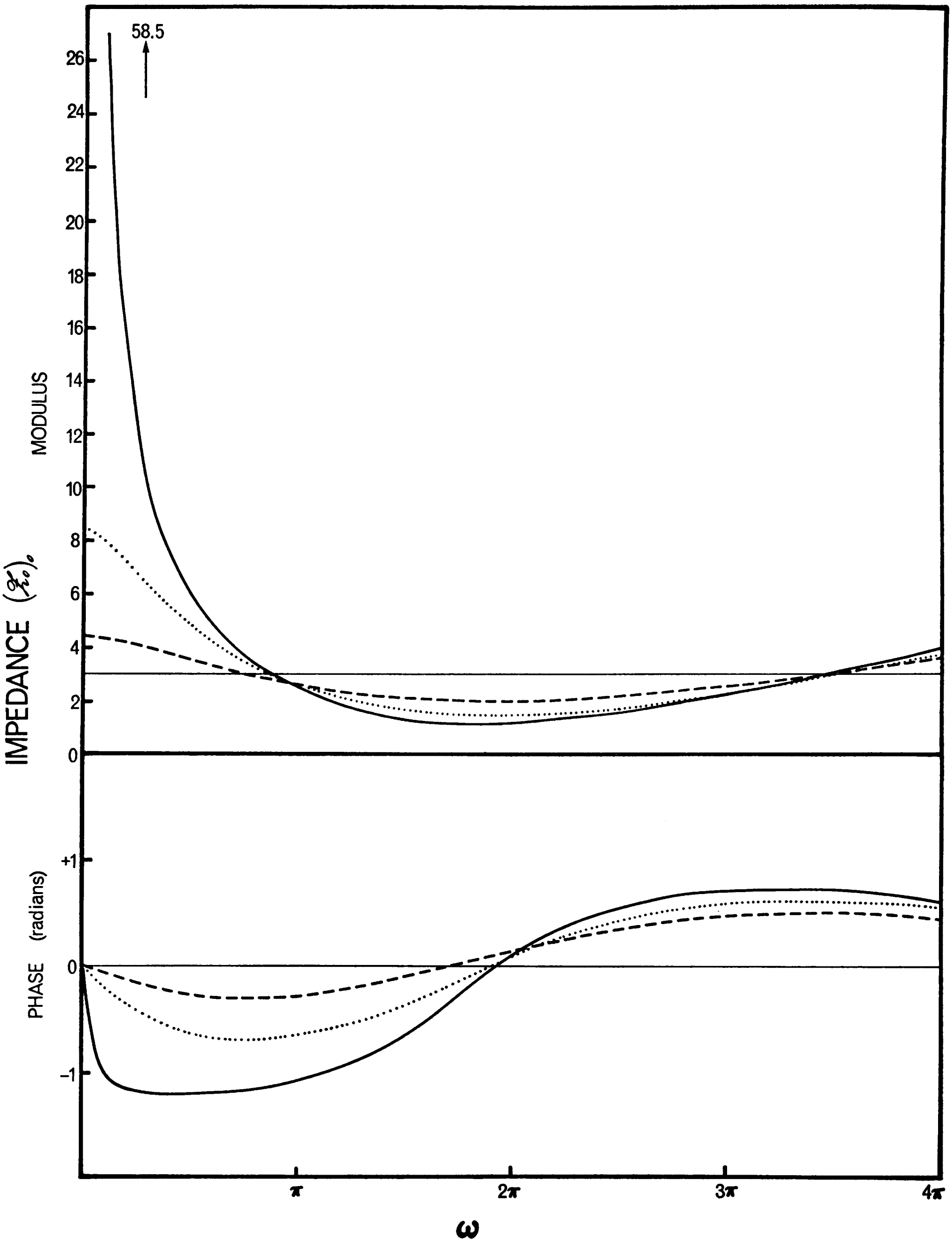


FIGURE 12 Input impedance of an assembly appropriate to a "regional" vascular bed, for three degrees of reflection. $R = 0.2$ — — — — — ; $R = 0.4$ . . . . $R = 0.9$ ——————— .

system, the combination of effects of attenuation due to viscosity, and the interference of reflections due to the scattering of the terminations, is sufficient to prevent the development of large values of $|\mathbf{Z}_o|_o$ after the first minimum. It is possible therefore to have relatively enormous values of the total terminal impedance without producing any large variations in impedance after the first minimum. This kind of behavior is very similar to that encountered in animal experiments.

DISCUSSION

These computations represent a further attempt to include in a fairly general linear model as many of the significant attributes of the arterial system as possible, in order to provide a basis for the interpretation of the results of animal experiments. From the calculated examples it appears that the main features of the plots of input impedance of the whole arterial system, viewed from the ascending aorta, can be ascribed to the combined effects of elastic nonuniformity and the presence of many scattered terminations. The dependence of input impedance upon the viscosity of the wall material, the viscosity of the blood, or the variations in total cross-sectional area is, in comparison with these other two features, relatively very slight.

The input impedance of this kind of system possesses a frequency dependence which can conveniently be divided into two ranges: (*a*) a low frequency range, in which the system behaves in much the same way as a single elastic tube with a terminal impedance at one point; (*b*) a high frequency range, in which it behaves like a single elastic tube with the properties of the very first part of the assembly, but with apparently a matched termination, providing little or no reflection.

The low frequency range of behavior consists of a steep fall of the impedance modulus from its value at zero frequency, that is, from the value of the dc resistance of the system, down to a minimum value, after which it rises again to a low maximum. The phase angle of the impedance, in this range, becomes large and negative, approaching 90°, passes through zero when the modulus is at or near its minimum, and then becomes positive.

The high frequency range of behavior can be taken as beginning at or after the first maximum following the first minimum of the impedance modulus. The modulus shows a number of fluctuations in this range, varying about the value of the nominal characteristic impedance of the first part of segment of the assembly; the phase angle also shows fluctuations, being alternately positive and negative. All these variations, however, are very slight compared with the changes seen in the low frequency range, and in the presence of attenuation due to viscous losses become still smaller with increasing frequency.

It appeared from previous studies (Taylor, 1964, 1965) that the presence of elastic nonuniformity in the arterial system, giving higher wave velocities in the peripheral vessels, was an important determinant of the input impedance. The present computations have shown, however, that in a more realistic model which incorporates the branching architecture of the arterial tree, the fact that the terminations of the arteries lie distributed at different distances from its origin, can also be expected to have a considerable influence on the input impedance. Womersley's original misgivings about the role of reflections in the arterial system thus appear to be justified, but not entirely so.

The consideration of the effects of elastic nonuniformity or of scattering of the terminations depends very greatly on the wavelength of the oscillation concerned, and thus, in a given system, on the frequency. For low frequencies and long wavelengths the effects of elastic nonuniformity are slight, and the system behaves very much as though it were uniform. In the same way we find that provided the path lengths are small in comparison with a wavelength, the phase differences between reflected components returning from scattered terminations are small, and they do not cancel each other out; again, the system behaves very much as though it had a single termination. Reflection effects can thus be appreciable at low frequencies, and at least partial "resonances" can be expected. The behavior of shorter wavelength oscillations in the high frequency range is considerably different. In the first place, the effect of elastic nonuniformity becomes appreciable, and the nature of the termination of the system comes to have less and less effect upon the impedance at the origin. In the second place, the differences in pathlength come to represent larger differences in phase angle between the reflected waves returning to the origin, and these now tend to interfere so that their effect upon the input impedance is reduced, and the apparent reflection from the "periphery" becomes small.

The same kind of variation in apparent reflection effects can be expected in different parts of the arterial system. In the smaller peripheral branches, the division between the low and high frequency ranges will occur at a higher frequency than in the main parent branches. This is to be explained in terms of the shorter path lengths to the terminations, and also in the lesser degree of elastic nonuniformity along these paths. Thus, while an oscillation of a given frequency may appear to be strongly reflected, when examined in a peripheral artery, it may be apparently far less reflected when examined in a central vessel such as the aorta. The behavior of apparent phase velocities in different vessels, discussed by McDonald and Taylor (1959), indicated very strong reflection effects in the femoral artery, but very little in the thoracic aorta. In the abdominal aorta of the dog, they noted that "the fact that the velocity of the higher frequencies is changed very little by a major occlusion is puzzling, but the relative stability of the phase velocity of oscillatory terms of 10 c/s and above has frequently been found in our experiments." This stability is considerably less puzzling if considered in relation to the properties of the arterial system discussed above.

With regard to the physiological significance of the input impedance of the arterial system, it has been shown elsewhere (Taylor, 1964, 1965) that the provision of a low input impedance to pulsatile flow acts to reduce the amount of external work done by the heart in providing a (necessarily) pulsatile flow of blood to the body. The design of the arterial system is, in fact, so favorable in this regard, that the amount of pulsatile external work of the heart is normally only about 5 to 15% of the total. Attention was previously directed to the influence of the nonuniform elastic properties of the arterial tree in determining these favorable input conditions; it is necessary now to add that the distributed nature of the terminations of the arteries must also be acknowledged to play an important contributory role.

This work was supported by a Grant-in-Aid from the National Heart Foundation of Australia which is gratefully acknowledged. The author is also indebted to Miss Sandra Williamson for the preparation of the illustrations, and to Miss Anne Williamson for assistance with the manuscript.

Received for publication 4 June 1965.

REFERENCES

ABRAMOWITZ, M., and STEGUN, I. A., (editors), 1964, Handbook of Mathematical Functions (Section 26.8, 949), National Bureau of Standards, Applied Mathematics Series, **55.**

BERGEL, D. H., 1961, *J. Physiol.,* **156,** 458.

HARDUNG, V., 1953, *Helv. Physiol. et Pharmacol. Acta,* **10,** 482.

McDONALD, D. A., 1960, Blood Flow in Arteries, London, Arnold.

McDONALD, D. A., and TAYLOR, M. G., 1959, *Progr. Biophysics and Biophysic. Chem.* **9,** 107.

NOORDERGRAAF, A., VERDOUW, P. D., VAN BRUMMELEN, A. G. W., and WIEGEL, F. W., 1964, *in* Pulsatile Blood Flow, (E. O. Attinger, editor), New York, McGraw-Hill Book Company, Inc.

PATEL, D., DE FREITAS, F. M., and FRY, D. L., 1963, *J. Appl. Physiol.,* **18,** 134.

TAYLOR, M. G., 1964, *in* Pulsatile Blood Flow, (E. O. Attinger, editor), New York, McGraw-Hill Book Company, Inc.

TAYLOR, M. G., 1965, *Physics Med. and Biol.,* **10,** 539.

WOMERSLEY, J. R., 1957, Wright Air Development Centre, technical report, WADC-TR, 56.