Oscillator representations and systems of ordinary differential equations

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Using representation-theoretic methods, we determine the spectrum of the 2 × 2 system

$$Q(x, D_x) = A\left(-\frac{\partial_x^2}{2} + \frac{x^2}{2}\right) + B\left(x\partial_x + \frac{1}{2}\right),\ x \in \mathbb{R},$$

with $A, B \in \mathrm{Mat}_2(\mathbb{R})$ constant matrices such that $A = {}^tA > 0$ (or <0), $B = -{}^tB \neq 0$, and the Hermitian matrix $A + iB$ positive (or negative) definite. We also give results that generalize (in a possible direction) the main construction.

1. Introduction

It is a natural and important problem to find an efficient and invariant way of studying the spectrum of systems of differential equations. We deal first with the spectral problem of particular systems that are the *Weyl* quantization of noncommutative quadratic form (that we shall call *noncommutative harmonic oscillators*) of the kind

$$Q(x, \xi) = \frac{1}{2}A(x^2 + \xi^2) + iBx\xi \quad (i = \sqrt{-1}),$$

where $(x, \xi) \in \mathbb{R}^2 \simeq T^*\mathbb{R}$, $A, B \in \mathrm{Mat}_2(\mathbb{R})$ are constant 2 × 2-matrices with $A = {}^tA$ *definite*, either *positive* or *negative*, $B = -{}^tB \neq 0$, and $A + iB > 0$ (or < 0). The Weyl quantization of the above noncommutative quadratic form is

$$Q(x, D_x) := Q^w(x, D_x) = A\left(-\frac{\partial_x^2}{2} + \frac{x^2}{2}\right) + B\left(x\partial_x + \frac{1}{2}\right),$$

$Q(x, D_x)$: $\mathcal{S}(\mathbb{R}; \mathbb{C}^2) \to \mathcal{S}(\mathbb{R}; \mathbb{C}^2)$, $Q(x, D_x)$: $\mathcal{S}'(\mathbb{R}; \mathbb{C}^2) \to \mathcal{S}'(\mathbb{R}; \mathbb{C}^2)$ *continuously*, and it is important to remark that in the above hypotheses the system $Q(x, D_x)$, as an unbounded operator in $L^2(\mathbb{R}; \mathbb{C}^2)$ (with maximal domain $B^2(\mathbb{R}; \mathbb{C}^2) := \{u \in L^2(\mathbb{R}; \mathbb{C}^2); x^\alpha\partial_x^\beta u \in L^2(\mathbb{R}; \mathbb{C}^2), \forall\alpha, \beta, 0 \le \alpha + \beta \le 2\}$) is *self-adjoint* with *discrete spectrum* made of eigenvalues $\mu_k \in \mathbb{R}$ (with finite multiplicities) such that $|\mu_k| \to +\infty$ as $k \to +\infty$. The main problem in analyzing the spectrum of such systems comes from two sources: the noncommutativity of matrices and the noncommutativity of the *quantized* variables x and ξ.

Our aim is in the first place to find a systematic way of solving *algebraically in* $L^2(\mathbb{R}; \mathbb{C}^2)$ (that is, taking into account L^2-convergence or, at most, $\mathcal{S}'$-convergence) the above kind of systems. The operator $Q(x, D_x)$ naturally possesses an $\mathfrak{sl}_2(\mathbb{R})$-action (and more generally a metaplectic group action), due to the Weyl quantization. Hence, the *algebraic* viewpoint consists of using this $\mathfrak{sl}_2(\mathbb{R})$-symmetry and the symmetries carried by A and B to develop a method that determines, as a first step, a candidate for an eigenvalue along with the formal sequence of coefficients of a potential eigenfunction belonging to that eigenvalue (in the sense that the eigenfunction has that sequence as coefficients with respect to some Schwartz L^2-basis obtained by means of the oscillator representation of $\mathfrak{sl}_2(\mathbb{R})$; this may be thought of as a necessary condition). Then the second step consists of determining which candidate eigenvalues can be actual ones, by studying the convergence properties (in L^2 or $\mathcal{S}'$) of the related formal sequences of coefficients (see ref. 1). In *Section 2*, we shall give a survey of the results of refs. 2 and 3 described above. As it will be seen below, the key step is to obtain a *three-term* recurrence system that can be "diagonalized." In *Section 3*, we shall next consider a different system of ordinary differential operators and treat the relative spectral problem by employing the tensor-product of the oscillator representation and the vector (standard) representation of $\mathfrak{sl}_2(\mathbb{R})$. That system (which is also interesting, for the most general noncommutative harmonic oscillator is given by $A(-\partial_x^2/2) + B(x\partial_x + 1/2) + C(x^2/2) + D$, with $A = {}^tA$, $B = -{}^tB$, $C = {}^tC$ and $D = {}^tD$) might look more difficult at first glance than the ones considered in Section 2, because of the presence of a zeroth-order nonconstant matrix-valued term. However, the difficulty is only "virtual," for the use of the aforementioned tensor product representation of $\mathfrak{sl}_2(\mathbb{R})$ allows one to treat the eigenvalue problem in a completely straightforward way: the above system is just a pair of "harmonic oscillators" in disguise. We shall also exhibit a family of systems parametrized by $\varepsilon \in \mathbb{R}$, which "interpolates" the pair of "harmonic oscillators" and a system of the above type. This family possesses the remarkable property that all the eigenvalues have multiplicity 1, provided $\varepsilon \notin (1/2)\mathbb{Z}$. This approach will be generalized in ref. 4 to treat more general 2 × 2-systems and higher-rank cases in a unified way.

Applications of the study of the spectrum of $N \times N$ noncommutative harmonic oscillators are in the field of lower bounds and hypoellipticity of systems of pseudodifferential operators (see refs. 5–10).

We finally remark that the spectral problem for $Q(x, D_x)$ can be translated into a family of Fuchsian type third-order equations with four regular singularities, in the complex unit disk (see ref. 11). We believe that the results on the multiplicity of the spectrum of $Q(x, D_x)$ may provide a crucial information to determine the monodromy of the ordinary differential equation discussed in ref. 11.

2. Study of the Spectrum of *Q*

The condition that the Hermitian matrix $A + iB$ be *definite*, either positive or negative, is equivalent to requiring that det $A > (\mathrm{pf}(B))^2$, and also equivalent to requiring that the operator $Q(x, D_x)$ be elliptic, i.e., det $Q(x, \xi) \neq 0$ for $(x, \xi) \neq (0, 0)$, det $Q(x, \xi)$ being positively homogeneous of degree 4 in (x, ξ). Hence, $Q(x, D_x)$, as an *unbounded* operator with domain (maximal domain)

$$\mathcal{D}(Q) := \{u \in L^2(\mathbb{R}; \mathbb{C}^2); Qu \in L^2(\mathbb{R}; \mathbb{C}^2)\}$$

$$= B^2(\mathbb{R}; \mathbb{C}^2),\ \langle Qu|\bar{\varphi}\rangle_{\mathcal{S}',\mathcal{S}} = (u, Q\varphi)_{L^2},$$

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for any given $\varphi \in \mathcal{S}(\mathbb{R}; \mathbb{C}^2)$, has a discrete spectrum made of real eigenvalues with finite multiplicities, diverging (in absolute value) to $+\infty$.

Define

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, I(\alpha, \beta) = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}.$$

Since $B = \pm \mathrm{pf}(B)J$, and by the commutativity of J with $SO(2)$, one can easily reduce the study of Q to that of

$$Q_{(\alpha,\beta)}(x; D_x) = I(\alpha, \beta)\left(-\frac{\partial^2}{2} + \frac{x^2}{2}\right) + J\left(x\partial + \frac{1}{2}\right),$$

where, after possibly conjugating by K, we may assume α, $\beta > 0$. We may thus suppose $A = I(\alpha, \beta)$, $B = J$. Set $\ell = \sqrt{\alpha\beta - 1} > 0$, and define

$$\Psi^{\dagger}(\ell) = \frac{1}{\sqrt{2\ell}}(xI + \partial J - \ell xJ), \text{ and } \Psi(\ell) = \frac{1}{\sqrt{2\ell}}(xI + \partial J + \ell xJ). \quad [1]$$

Since $[\Psi(\ell), \Psi^{\dagger}(\ell)] = I$, putting

$$X^+ := \frac{\Psi^{\dagger}(\ell)^2}{2}, X^- := -\frac{\Psi(\ell)^2}{2}, H := \Psi(\ell)\Psi^{\dagger}(\ell) - \frac{1}{2}I,$$

yields that H, X^+, X^- satisfy the commutation-relations of $\boldsymbol{sl}_2(\mathbb{R})$

$$[X^+, X^-] = H, [H, X^{\pm}] = \pm 2X^{\pm}. \quad [2]$$

This gives in fact the tensor product of the oscillator representation and the two-dimensional trivial representation of $\boldsymbol{sl}_2(\mathbb{R})$. For $v \in \mathbb{C}^2 \backslash \{0\}$, set

$$\xi_0(v) := \exp(x^2J/2)\exp(-\ell x^2 I/2)v, \quad \text{and} \quad \xi_N(v) := \Psi^{\dagger}(\ell)^N \xi_0(v), N \in \mathbb{Z}_+.$$

Since X^- annihilates $\xi_0(v)$ [$\xi_0(v)$ is a lowest-weight vector], $\xi_N(v)$ is a weight vector of H with weight $N + 1/2$, and it also follows that whenever v and w are nonzero vectors such that $\langle v, w\rangle_{\mathbb{C}^2} = 0$, $\{\xi_N(v), \xi_N(w)\}_{N\in\mathbb{Z}_+}$ is an orthogonal basis of $L^2(\mathbb{R}; \mathbb{C}^2)$. The operator $Q_{(\alpha,\beta)}$ is unitarily equivalent to $(\ell/\sqrt{\alpha\beta})\tilde{Q}$, where $\tilde{Q} := A^{1/2}HA^{1/2}$. The problem is therefore to understand the structure of the spectrum of $\tilde{Q}$. There are now (at least) two ways of studying the spectral problem for $\tilde{Q}$: (1) studying the equivalent problem of finding λ and η such that $(H - \lambda A^{-1})\eta = 0$ (the *twisted eigenvalue problem*); and (2) studying directly the eigenvalue problem $(\tilde{Q} - \lambda I)\eta = 0$. In both cases, the key point is to get an appropriate recurrence formula that allows one to control the coefficients of the eigenfunctions. (One can easily see that an expansion in terms of the usual Hermite functions is not "convenient" for this purpose.) As we shall explain below, we get a system of recurrence equations that, by suitable rotations, can be diagonalized into a *scalar three-term* recurrence equation (this is highly nontrivial, for we do not know *a priori* that that is possible). We remark that the crucial point is the choice of the basis to be used in the discussion. Since $A^{-1} = \sigma_+ I + \sigma_- KJ$, $A^{1/2} = \mu_+ I + \mu_- KJ$, in both problems 1 and 2 the action of the operator K appears. Since K *does not* commute with H (whereas J does), what is missing here is the explicit formula for the action of K on the $\xi_N(v)$ in terms of the ξ. Set $\xi_N^j = \xi_N(e_j)$, $j = 1, 2$, where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{C}^2$, and, for $v_+ := [\begin{smallmatrix} 1 \\ -i \end{smallmatrix}]$, define $\xi_N^+ := \xi_N(v_+)$.

LEMMA 2.1. *The set* $B_{\mathrm{mix}} := \{\xi_N^+, K\xi_N^+\}_{N\in\mathbb{Z}_+}$ *is an orthogonal basis of* $L^2(\mathbb{R}; \mathbb{C}^2)$.

As the system preserves parity, we shall consider hereafter only the *even* case.

Dealing with Problem 1. Since the action of H on the $K\xi_N$ involves both the ξ and $K\xi$, we use both bases $B_{\mathrm{can}}^+ := \{\xi_{2N}^1, \xi_{2N}^2\}_{N\in\mathbb{Z}_+}$ and $B_{\mathrm{can}}^{+,K} := \{K\xi_{2N}^1, K\xi_{2N}^2\}_{N\in\mathbb{Z}_+}$ at the same time, and the fact that they are related by K. The latter means that if $(\{a_N\}_{N\in\mathbb{Z}_+}, \{b_N\}_{N\in\mathbb{Z}_+})$ are the coordinates of a solution η to the twisted eigenvalue problem with respect to the basis B_{can}^+, and $(\{c_N\}_{N\in\mathbb{Z}_+}, \{d_N\}_{N\in\mathbb{Z}_+})$ those of η with respect to the basis $B_{\mathrm{can}}^{+,K}$, then there exists an involutive linear function

$$\tilde{K}: (\{a_N\}_{N\in\mathbb{Z}_+}, \{b_N\}_{N\in\mathbb{Z}_+}) \mapsto (\{c_N\}_{N\in\mathbb{Z}_+}, \{d_N\}_{N\in\mathbb{Z}_+}).$$

Using $\tilde{K}$, we get a nonconstant matrix-coefficient system of recurrence equations in $\mathbb{C}^4$ for the (a_N, b_N, c_N, d_N) that can be reduced to two recurrence systems for (a_N, b_N) and (c_N, d_N) plus a linear relation between (a_N, b_N) and (c_N, d_N). At this point there exists an explicitly known constant matrix $M_0(\ell)$ such that $(1/\sqrt{c_\ell})\, M_0(\ell) \in SO(2)$, where

$$c_\ell := \frac{\ell^2 + 1}{\ell^4}, \text{ and, upon setting } v_N := M_0(\ell)^N \begin{bmatrix} a_N \\ b_N \end{bmatrix}$$

$$\text{and } w_N := M_0(\ell)^{*N} \begin{bmatrix} c_N \\ d_N \end{bmatrix},$$

we arrive at the recurrence equations

$$(P1_N^\lambda) \quad d_{2N}(\lambda)v_N + c_\ell \Lambda_{2(N-1)}(\lambda)v_{N-1} + 2(N+1)(2N+1)\Lambda_{2(N+1)}(\lambda)v_{N+1} = 0,$$

$$(P2_N^\lambda) \quad d_{2N}(\lambda)w_N + c_\ell \Lambda_{2N}(\lambda)w_{N-1} + 2(N+1)(2N+1)\Lambda_{2N}(\lambda)w_{N+1} = 0,$$

$$(P3_N^\lambda) \quad \Lambda_{2N}(\lambda)v_N = \lambda\sigma_- M_0(\ell)^N JK(M_0(\ell)^*)^{-N} w_N,$$

where $N \in \mathbb{Z}_+$, $v_{-1} = w_{-1} = 0$ and

$$d_{2N}(\lambda) := \left(1 + \frac{2}{\ell^2}\right)\Lambda_{2N}(\lambda)^2 + \frac{2}{\ell^2}\lambda\sigma_+\Lambda_{2N}(\lambda) - \lambda^2\sigma_-^2,$$

$$\Lambda_{2N}(\lambda) = 2N + \frac{1}{2} - \lambda\sigma_+.$$

The key point now is the *compatibility condition*: *Suppose* λ *gives rise to a solution* $(\{a_N\}_{N\in\mathbb{Z}_+}, \{b_N\}_{N\in\mathbb{Z}_+})$ *to* $(P1_N^\lambda)$ *that defines a function* $u_1 := \Sigma_{N=0}^{+\infty} (a_N\xi_{2N}^1 + b_N\xi_{2N}^2) \in L^2(\mathbb{R}; \mathbb{C}^2)$, *then with* $(\{a_N\}_{N\in\mathbb{Z}_+}, \{b_N\}_{N\in\mathbb{Z}_+})$ *we associate, in an algebraic fashion through* $(P3_N^\lambda)$, *a solution* $(\{c_N\}_{N\in\mathbb{Z}_+}, \{d_N\}_{N\in\mathbb{Z}_+})$ *to* $(P2_N^\lambda)$ *that corresponds to a function* $u_2 := \Sigma_{N=0}^{+\infty} (c_N K\xi_{2N}^1 + d_N K\xi_{2N}^2) \in L^2(\mathbb{R}; \mathbb{C}^2)$, *for which* $u_1 = u_2$ *iff* $\tilde{K}((\{a_N\}_{N\in\mathbb{Z}_+}, \{b_N\}_{N\in\mathbb{Z}_+})) = (\{c_N\}_{N\in\mathbb{Z}_+}, \{d_N\}_{N\in\mathbb{Z}_+})$. *In this and only in this case,* $\eta = u_1 = u_2$ *is a solution to the twisted eigenvalue problem.*

To study the recurrence $(P1_N^\lambda)$, we first observe that by virtue of the initial condition $v_{-1} = 0$, it is actually a *scalar* recurrence, depending only on the initial condition $v_0 \in \mathbb{C}^2$. At this point, all the solutions to $(P1_N^\lambda)$ [and hence to $(P2_N^\lambda)$] are of the form $\mathbf{h}(\lambda) \otimes v_0$, for some sequence $\mathbf{h}(\lambda) = \{h_N(\lambda)\}_{N\in\mathbb{Z}_+} \in \mathbb{C}^{\mathbb{Z}_+}$ depending only on λ. One next sees that it is possible to

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construct, for $\lambda \in \mathbb{R}$, only two kinds of solutions: the ones for which there exists N_0 such that $h_N(\lambda) = 0$ for all $N \geq N_0 + 1$ [such solutions are called *finite-type* (λ, N_0)*-solutions*] that are hence obviously associated with Schwartz candidates as solutions to the twisted eigenvalue equation; the ones that are *not* of finite type, for which there exists $N_0 \geq 0$ such that $h_N(\lambda) = 0$ for every $N \leq N_0 - 1$ *and* $h_N(\lambda) \neq 0$ for *infinitely many* $N \geq N_0$ [one sets $h_{-1}(\lambda) \equiv 0$]. The latter solutions are not in general associated with any reasonable distribution in $\mathcal{S}'(\mathbb{R}; \mathbb{C}^2)$ *unless* λ satisfies an equation of the kind $q_{N_0}(\lambda) = f_{N_0}(\lambda)$, where q_{N_0} is a suitably defined rational function and f_{N_0} is a suitably defined *continued fraction* [both depending on $\Lambda_{2N}(\lambda)$ and $d_{2N}(\lambda)$], case in which the associated function is actually a Schwartz function [such solutions are called *infinite-type* (λ, N_0)*-solutions*]. Notice that the *algebraic-companion* solutions $\mathbf{k}(\lambda) \otimes w_0$ to $(P2_N^\lambda)$, algebraically obtained from a solution $\mathbf{h}(\lambda) \otimes v_0$ of $(P1_N^\lambda)$ (of finite or infinite type) through $(P3_N^\lambda)$, are *automatically* associated with Schwartz functions. Hence, for a fixed λ that gives rise to finite-type or infinite-type solutions, we get at least two solutions, for we have the freedom of the choice of $v_0 \in \mathbb{C}^2$. On the other hand, the aforementioned *compatibility condition* rules out that freedom: if one defines for $\mathbf{h}(\lambda)(\neq 0)$ of finite or infinite type

$$W_\lambda(\mathbf{h}) := \{\mathbf{h}(\lambda) \otimes v_0;\ \mathbf{h}(\lambda) \otimes v_0 \text{ and } \mathbf{k}(\lambda) \otimes w_0 \text{ are } \textit{compatible},\ v_0 \in \mathbb{C}^2\},$$

then $\dim_{\mathbb{C}} W_\lambda(\mathbf{h}) \leq 1$. The reason why $W_\lambda(\mathbf{h})$ might be zero-dimensional (corresponding to $v_0 = w_0 = 0$) comes from the fact that the operator $\tilde{K}$ acts in a highly unknown fashion, and we cannot *a priori* conclude that a solution $\mathbf{h}(\lambda) \otimes v_0$ to $(P1_N^\lambda)$ and the *algebraic companion* $\mathbf{k}(\lambda) \otimes w_0$ relative to $(P2_N^\lambda)$ are related (for the *same* λ) through the compatibility condition.

Let us define

$$\textstyle\sum_0^+ := \{\lambda \in \mathbb{R}; \text{there exists a finite-type } \lambda\text{-solution } \mathbf{h}(\lambda) \otimes v_0\},$$

$$\textstyle\sum_\infty^+ := \{\lambda \in \mathbb{R}; \text{there exists an infinite-type } \lambda\text{-solution } \mathbf{h}(\lambda) \otimes v_0\}$$

(and analogously the sets Σ_0^- and Σ_∞^- relative to the odd case). One has also the following description of Σ_0^+: upon defining polynomials $j_{2N} \in \mathbb{R}[\lambda]$ inductively by the recurrence formula

$$j_{2N}(\lambda) = d_{2N}(\lambda) j_{2N-2}(\lambda) - c_\ell 2N(2N-1) \Lambda_{2N-2}(\lambda)\Lambda_{2N}(\lambda) j_{2N-4}(\lambda),\ N \geq 1,$$

$j_{-2}(\lambda) = 1$, $j_0(\lambda) = d_0(\lambda)$ [they appear as determinants of particular Jacobi matrices (see ref. 2)], then

$$\textstyle\sum_0^+ = \{\lambda \in \mathbb{R};\ \exists\, N_0 \in \mathbb{Z}_+,\ \Lambda_{2N_0}(\lambda) = j_{2N_0}(\lambda) = 0\}$$

(and analogously in the odd case). Put $\Sigma_0 := \Sigma_0^+ \cup \Sigma_0^-$, $\Sigma_\infty := \Sigma_\infty^+ \cup \Sigma_\infty^-$, and for $\lambda \in \Sigma_0 \cup \Sigma_\infty$ define $V_\lambda^+ := \{u \in L^2(\mathbb{R}; \mathbb{C}^2);\ \tilde{Q}(x, D_x)u = \lambda u$ and u is *even*$\}$ and $V_\lambda^- := \{u \in L^2(\mathbb{R}; \mathbb{C}^2);\ \tilde{Q}(x, D_x)u = \lambda u$ and u is *odd*$\}$. It is important to notice that at this point we do not know as yet that the $V_\lambda^\pm$ are eigenspaces belonging to λ (because we do not know as yet that any given λ in $\Sigma_0 \cup \Sigma_\infty$ is an eigenvalue). Thus, as proved in ref. 2, we have the following theorem (that might be thought of as the first half of *Theorem 2.4* below).

THEOREM 2.2. *One has*

$$\mathrm{Spec}(\tilde{Q}(x, D_x)) \subset \textstyle\sum_0 \cup \sum_\infty,$$

and

$$\dim V_\lambda^+ \leq \begin{cases} 2 \text{ if } \lambda \in \sum_0^+ \\ 1 \text{ if } \lambda \in \sum_\infty^+ \backslash \sum_0^+ \end{cases} \quad \dim V_\lambda^- \leq \begin{cases} 2 \text{ if } \lambda \in \sum_0^- \\ 1 \text{ if } \lambda \in \sum_\infty^- \backslash \sum_0^- \end{cases}.$$

Dealing with Problem 2. In this case, the key ingredients for dealing directly with problem 2 are the formula

$$\tilde{Q} = \mu_+^2 H + \mu_-^2 KHK + \mu_+\mu_-(HK + KH)J$$

and the use of the basis B_{mix} (see *Lemma 2.1*). Again, we restrict to the even-eigenfunction case. Hence, consider the equation $(\tilde{Q} - \lambda I)\eta = 0$, with η of the form $\eta = \Sigma_{N=0}^{+\infty} (a_N \xi_{2N}^+ + b_N K \xi_{2N}^+)$. Upon setting $a_{-1} = b_{-1} = 0$, $z_N := \ell^{-2N}(1 - i\ell)^N (\mu_- a_N - i\mu_+ b_N)$, we get the recurrence equations

$$(P2_N^\lambda)' \quad d_{2N}(\lambda) z_N + c_\ell \Lambda_{2N}(\lambda) z_{N-1} + 2(N+1)(2N+1)\Lambda_{2N}(\lambda) z_{N+1} = 0,$$

i.e., *exactly* the recurrence equations $(P2_N^\lambda)$. It is a key observation now, and the reason why the basis B_{mix} is of fundamental importance, that the *compatibility condition* is automatically satisfied. Hence, since $\lambda \in \Sigma_0 \cup \Sigma_\infty$ gives rise to Schwartz solutions to the eigenvalue equation, it follows from $(P2_N^\lambda) = (P2_N^\lambda)'$ that $\lambda \in \mathrm{Spec}(\tilde{Q})$. Thus $\mathrm{Spec}(\tilde{Q}(x, D_x)) = \Sigma_0 \cup \Sigma_\infty$. Now, since *any finite-type* $(\lambda, N_0 + 1)$*-solution of* $(P1_N^\lambda)$ *corresponds to a finite-type* (λ, N_0)*-solution of* $(P2_N^\lambda)$, *and hence of* $(P2_N^\lambda)'$, we have the following crucial fact.

LEMMA 2.3. *One has* $\Sigma_0^+ \subset \Sigma_\infty^+$ *and* $\Sigma_0^- \subset \Sigma_\infty^-$.

As a consequence of this approach and of *Theorem 2.2* above, we get, as proved in ref. 3, the following rather complete description.

THEOREM 2.4. *One has*

$$\mathrm{Spec}(\tilde{Q}(x, D_x)) = \textstyle\sum_0 \cup \sum_\infty,$$

and

$$\dim V_\lambda^+ = \begin{cases} 2 \text{ if } \lambda \in \sum_0^+ \\ 1 \text{ if } \lambda \in \sum_\infty^+ \setminus \sum_0^+ \end{cases} \quad \dim V_\lambda^- = \begin{cases} 2 \text{ if } \lambda \in \sum_0^- \\ 1 \text{ if } \lambda \in \sum_\infty^- \setminus \sum_0^- \end{cases}.$$

Remark 2.5: In general $\Sigma_0^\pm \neq \emptyset$ (see ref. 2).

3. A New System

As an example of some possible directions of generalization of the system defined by the operator

$$Q_{\mathrm{h}}(x, D_x) = \frac{-\partial_x^2 + 2x^2}{2} I + \left(x\partial_x + \frac{1}{2}\right) J = \Psi(1)\Psi^\dagger(1) - \frac{1}{2} I$$

(see Eq. **1**), we propose here the spectral problem relative to the operator

$$Q_{\mathrm{vect}}(x, D_x) = \frac{-\partial_x^2 + 2x^2}{2} I + \left(x\partial_x + \frac{1}{2}\right) J + \begin{bmatrix} \cos(x^2) & \sin(x^2) \\ \sin(x^2) & -\cos(x^2) \end{bmatrix},$$

i.e., the study of the equation

$$\begin{cases} \left(\dfrac{-\partial_x^2 + 2x^2}{2} + \cos(x^2)\right)u_1 - \left(x\partial_x + \dfrac{1}{2} - \sin(x^2)\right)u_2 = \lambda u_1 \\ \left(\dfrac{-\partial_x^2 + 2x^2}{2} - \cos(x^2)\right)u_2 + \left(x\partial_x + \dfrac{1}{2} + \sin(x^2)\right)u_1 = \lambda u_2. \end{cases}$$

Notice that, as an unbounded operator in $L^2(\mathbb{R}; \mathbb{C}^2)$ with domain $B^2(\mathbb{R}; \mathbb{C}^2)$, $Q_{\mathrm{vect}}(x, D_x)$ is self-adjoint with compact resolvent (this again by virtue of the global ellipticity of its principal part). It seems quite nontrivial to obtain the eigenvalues of $Q_{\mathrm{vect}}(x, D_x)$. We want to show that exploiting the oscillator representation allows one to solve the spectral problem. Notice that $Q_{\mathrm{h}}(x, D_x)$ is unitarily equivalent (through a symplectic scaling) to $Q_{(\sqrt{2},\sqrt{2})}(x, D_x)$ (in the notations of *Section 2*). The main problem here is to treat the zeroth-order part of $Q_{\mathrm{vect}}(x, D_x)$. That seems difficult, but that is not the case, by virtue of the tensor product representation of $\boldsymbol{sl}_2(\mathbb{R})$ to be constructed below. Hence, let $\{H, X^+, X^-\}$ be the basis of $\boldsymbol{sl}_2(\mathbb{R})$, which satisfies Eq. **2**. Put

$$\psi := \frac{x + \partial_x}{\sqrt{2}}, \quad \psi^\dagger := \frac{x - \partial_x}{\sqrt{2}}.$$

Then it is clear that $[\psi, \psi^\dagger] = 1$, and hence the map ω: $\boldsymbol{sl}_2(\mathbb{R}) \to \mathrm{End}_{\mathbb{C}}\,(\mathcal{S}(\mathbb{R}))$

$$\omega(H) = \psi\psi^\dagger - \frac{1}{2}, \quad \omega(X^+) = \frac{(\psi^\dagger)^2}{2}, \quad \omega(X^-) = -\frac{\psi^2}{2},$$

gives the oscillator representation of $\boldsymbol{sl}_2(\mathbb{R})$ on $\mathcal{S}(\mathbb{R})$. Because the action of $\boldsymbol{sl}_2(\mathbb{R})$ leaves the parity invariant, we have

$$\mathcal{S}(\mathbb{R}) = \mathcal{S}_{\mathrm{even}}(\mathbb{R}) \oplus \mathcal{S}_{\mathrm{odd}}(\mathbb{R}) =: \mathcal{S}_+(\mathbb{R}) \oplus \mathcal{S}_-(\mathbb{R}),$$

the irreducible decomposition of ω. Put $\omega^\pm := \omega|_{\mathcal{S}_\pm(\mathbb{R})}$. Then $v_0 = e^{-x^2/2}$ (resp. $\psi^\dagger v_0$) gives the lowest weight vector of the irreducible representation of $(\omega^+, \mathcal{S}_+(\mathbb{R}))$ (resp. of $(\omega^-, \mathcal{S}_-(\mathbb{R}))$) (see ref. 12). Let (π, V) $(V \simeq \mathbb{C}^2)$ be the vector representation of $\boldsymbol{sl}_2(\mathbb{R})$:

$$\pi(H) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \pi(X^+) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \pi(X^-) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We have the following proposition.

PROPOSITION 3.1. *Put* $\rho(H) := Q_{\mathrm{vect}}(x, D_x)$,

$$\rho(X^+) := \frac{1}{2}\left\{\Psi^\dagger(1)^2 + \begin{bmatrix} -\sin(x^2) & 1 + \cos(x^2) \\ -1 + \cos(x^2) & \sin(x^2) \end{bmatrix}\right\},$$

$$\rho(X^-) := \frac{1}{2}\left\{-\Psi(1)^2 + \begin{bmatrix} -\sin(x^2) & -1 + \cos(x^2) \\ 1 + \cos(x^2) & \sin(x^2) \end{bmatrix}\right\}.$$

Then $(\rho, \mathcal{S}(\mathbb{R}; \mathbb{C}^2))$ *defines a representation of* $\boldsymbol{sl}_2(\mathbb{R})$. *Furthermore,* ρ *is equivalent to the tensor product representation* $(\omega \otimes \pi, \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^2)$. *In fact, the operator* $\exp(x^2 J/2)$ *defines the intertwining operator between these representations:*

$$\rho(X)e^{x^2J/2} = e^{x^2J/2}\omega \otimes \pi(X),$$

where, recall, $\omega \otimes \pi(Y) = \omega(Y) \otimes 1 + 1 \otimes \pi(Y)$ *[we have here identified* $\mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^2$ *with* $\mathcal{S}(\mathbb{R}; \mathbb{C}^2)$*]. In particular, the system defined by the operator* $Q_{\mathrm{vect}}(x, D_x)$ *is unitarily equivalent to the system defined by the operator* $\omega \otimes \pi(H)$.

Proof: All the statements follow from the fact

$$\rho(H) = e^{x^2J/2}\omega \otimes \pi(H)e^{-x^2J/2},$$

$$\rho(X^\pm) = e^{x^2J/2}\omega \otimes \pi(X^\pm)e^{-x^2J/2}.$$

Details are left to the reader.

Using the irreducible decomposition of the tensor product representation $\omega \otimes \pi$ we have the following theorem.

THEOREM 3.2. *Put* $\xi_0^\pm := e^{x^2J/2}\,\varphi_0^\pm$ *and* $\xi_1^\pm := e^{x^2J/2}\,\varphi_1^\pm$, *where*

$$\varphi_0^+ := v_0 \otimes \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], \varphi_0^- := \psi^\dagger v_0 \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\varphi_1^+ := (\psi^\dagger)^2 v_0 \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} + v_0 \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\varphi_1^- := (\psi^\dagger)^3 v_0 \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \psi^\dagger v_0 \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Define, for $N \geq 0$, $\xi_{0,N}^\pm := \rho(X^+)^N\,\xi_0^\pm$, $\xi_{1,N}^\pm := \rho(X^+)^N\,\xi_1^\pm$. *Then*

$$Q_{\mathrm{vect}}(x, D_x)\xi_{0,N}^\pm = \left(2N \mp \frac{1}{2}\right)\xi_{0,N}^\pm,$$

$$Q_{\mathrm{vect}}(x, D_x)\xi_{1,N}^\pm = \left(2N + 2 \mp \frac{1}{2}\right)\xi_{1,N}^\pm.$$

Remark 3.3: As an $\boldsymbol{sl}_2(\mathbb{R})$-module, the above theorem implies the irreducible decomposition of $(\rho, \mathcal{S}(\mathbb{R}; \mathbb{C}^2))$:

$$\mathcal{S}(\mathbb{R}; \mathbb{C}^2) \simeq \underbrace{\overline{\mathrm{Span}\{\xi_{0,N}^+\}_{N\geq 0}} \oplus \overline{\mathrm{Span}\{\xi_{1,N}^+\}_{N\geq 0}}}_{\text{even}} \bigoplus \underbrace{\overline{\mathrm{Span}\{\xi_{0,N}^-\}_{N\geq 0}} \oplus \overline{\mathrm{Span}\{\xi_{1,N}^-\}_{N\geq 0}}}_{\text{odd}}$$

where the closure refers to the $\mathcal{S}$-topology (the same decomposition holds for $L^2(\mathbb{R}; \mathbb{C}^2)$ with closure in the L^2-topology). In particular, ξ_0^+, ξ_1^+ and ξ_0^-, ξ_1^- give the lowest-weight vectors of the irreducible summands, respectively. The L^2-structure of Spec-$(Q_{\mathrm{vect}}(x, D_x))$ is given by

eigenvalue	$-1/2$	$2N - 1/2\ (N \geq 1)$	$1/2$	$2N + 1/2\ (N \geq 1)$
eigenvector	ξ_0^+	$\xi_{0,N}^+, \xi_{1,N-1}^+$	ξ_0^-	$\xi_{0,N}^-, \xi_{1,N-1}^-$
multiplicity	1	2	1	2

Furthermore, the proof of the above theorem gives the following result (see ref. 4).

THEOREM 3.4. *Let*

$$Q_\varepsilon(x, D_x) := \frac{-\partial_x^2 + 2x^2}{2}I + \left(x\partial_x + \frac{1}{2}\right)J$$
$$+ \varepsilon\begin{bmatrix} \cos(x^2) & \sin(x^2) \\ \sin(x^2) & -\cos(x^2) \end{bmatrix}, \quad \varepsilon \in \mathbb{R}.$$

The system $Q_\varepsilon(x, D_x)$ *interpolates systems* $Q_{\mathrm{h}}(x, D_x)$ *and* $Q_{\mathrm{vect}}(x, D_x)$, *it has spectrum given by the numbers* $2N + 1/2 \pm \varepsilon$ *(with* even *relative eigenfunctions) and* $2N + 3/2 \pm \varepsilon$ *(with* odd *relative eigenfunctions), where* $N \in \mathbb{Z}_+$, *with* multiplicity one *for any* $N \geq 0$ *when* $\varepsilon \notin (1/2)\mathbb{Z}$.

Remark 3.5: We remark that the eigenfunctions of $Q_\varepsilon(x, D_x)$ do not depend on ε (see ref. 4). This is the main reason why we think the eigenfunction basis of Q_ε may serve as a useful tool for studying more general systems.

We shall give further generalizations in ref. 4. One of our main motivations is to provide a class of examples whose spectral problems can be explicitly solved in a unified way. Indeed, if we take any unitary transformation $U(x)$ in place of $e^{x^2J/2}$ in *Proposition 3.1*, we may write down a number of examples that *look* more difficult, but that actually are obvi-

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ously all unitarily equivalent to system $Q_{\text{vect}}(x, D_x)$. Moreover, it is also quite interesting to consider the eigenvalue problem for

$$Q(x, D_x) = A\left(-\frac{\partial_x^2}{2} + \frac{x^2}{2}\right) + B\left(x\partial_x + \frac{1}{2}\right) + C\begin{bmatrix} \cos(x^2) & \sin(x^2) \\ \sin(x^2) & -\cos(x^2) \end{bmatrix},$$

even for the special case $C = A$, for one may write

$$Q(x, D_x) = A\left(-\frac{\partial_x^2}{2}\right) + B\left(x\partial_x + \frac{1}{2}\right) + (A + 2CK)\left(\frac{x^2}{2}\right) + CKJ + O(x^4)$$

(A and C are real self-adjoint matrices, B real skew-adjoint).

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