Effective dimensional reduction algorithm for eigenvalue problems for thin elastic structures: A paradigm in three dimensions

Evgueni E. Ovtchinnikov and Leonidas S. Xanthis

Centre for Techno-Mathematics and Scientific Computing Laboratory, University of Westminster, London HA1 3TP, United Kingdom

Communicated by Peter D. Lax, New York University, New York, NY, November 16, 1999 (received for review July 3, 1999)

We present a methodology for the efficient numerical solution of eigenvalue problems of full three-dimensional elasticity for thin elastic structures, such as shells, plates and rods of arbitrary geometry, discretized by the finite element method. Such problems are solved by iterative methods, which, however, are known to suffer from slow convergence or even convergence failure, when the thickness is small. In this paper we show an effective way of resolving this difficulty by invoking a special preconditioning technique associated with the effective dimensional reduction algorithm (EDRA). As an example, we present an algorithm for computing the minimal eigenvalue of a thin elastic plate and we show both theoretically and numerically that it is robust with respect to both the thickness and discretization parameters, i.e. the convergence does not deteriorate with diminishing thickness or mesh refinement. This robustness is sine qua non for the efficient computation of large-scale eigenvalue problems for thin elastic structures.

robust preconditioning in thickness and discretization parameters | vibrations | shells | plates | rods

Prolegomena

> Mnemosyne, Archimedean Muse: Ivo Babuška and his legacy to computational mathematics, mechanics, and finite element culture.*

It may be useful to begin with some reflections through which the present contribution can be contextualized. The first aspect that will impress the newcomer to the field of thin elastic structures, such as shells, plates, and rods, is a myriad of papers and books that have been written on a plethora of theoretical, numerical, and computational themes. The second aspect that will—upon reflection—appear is that the field is almost exclusively occupied by (a hierarchy of) lower-dimensional models (e.g., two-dimensional plates and shells) bearing the names of their most prominent proponents: Kirchhoff, Love, Reissner, Mindlin, von Kármán, Koiter, Naghdi, etc. (see, e.g., refs. 1–8). The *raison d' être* for these models has been to provide tractable, "valid" approximations to the governing partial differential equations of thin three-dimensional elastic bodies. However, questions relating to the validity of these models have never ceased to be raised since, say, 1888, when Love published his seminal paper on "The small vibrations and deformation of a thin elastic shell" (ref. 9; for a historical excursion, see ref. 10).

Even today, despite substantial progress in the numerical solution of three-dimensional large-scale problems, the study of lower-dimensional models continues to be a crowded monodrome. This (lower-dimensional) paradigm is eloquently exemplified by some prominent specialists in the field, who, in a recent state-of-the-art statement—"sign-posted" in cyberspace (ref. 11)—explain: ". . . An accurate, fully three-dimensional, simulation of a very thin body is beyond the power of even the most powerful computers and computational techniques . . . Thus the need for two-dimensional shell models."

In this paper we face up to the three-dimensional challenge—with respect to the efficient computation of eigenvalue problems for thin elastic structures. In fact, thorough scrutiny of the literature reveals that this is the first paper to deal with the numerical solution of *three-dimensional* eigenvalue problems for *thin* elastic structures—including theoretical convergence estimates for the iterative algorithm. The (new) paradigm in three dimensions, rooted in the epistemological approach introduced earlier (refs. 12–16) bears the hallmark of a *paradigm shift*—to adopt Thomas Kuhn's terminology (ref. 17)—from lower- to higher-dimensional thinking in this field (see also ref. 2).

Interestingly, the genesis of the new paradigm was heralded in Stuart Antman's inspiring and visionary "dreams for a final theory"†: "The progress in the numerical analysis of problems of solid mechanics suggests that a day will come when rod and shell theories will lose their distinctive identities within computational mechanics and be subsumed under a general theory for the numerical treatment of three-dimensional problems, endowed with useful error estimates" (ref. 1, page 600; see also ref. 6).

1. Introduction

Vibrations and associated eigenvalue problems are all-pervasive in the natural and man-made world and as such are investigated by many branches of mathematical physics, computational mathematics, and engineering. For example, in structural dynamics, to which this paper relates, the eigenvalues of the elasticity system of equations represent the frequencies of free vibrations of an elastic structure, and the eigenfunctions describe vibration modes. In this paper we are primarily concerned with the question of how to solve efficiently eigenvalue problems for three-dimensional large-scale *thin* elastic structures, such as shells, plates and rods of arbitrary geometry. We address in particular the hitherto unresolved difficulties encountered in the numerical solution of such problems resulting from the presence of a small thickness parameter. The term "dimensional reduction" in the title and throughout this paper must be distinguished from that commonly associated with the familiar *lower-dimensional* shell, plate, and rod models (see, e.g., refs. 1 and 4). Here the term means reduction of the complexity in the numerical solution of the *three-dimensional* problem. This will become apparent from the ensuing discussion (see Section 7).

The discretization of an eigenvalue problem for a differential equation using, e.g., the finite element method yields an algebraic eigenvalue problem. We note that in this paper we do not address questions of accuracy of discretization; our main concern is the efficient numerical solution of the discretized problem (assumed to be an adequate approximation of the eigenvalue problem for the equations of *three-dimensional* elasticity). The

APPLIED MATHEMATICS

*Mnemosyne (archetypal image of cultural and intellectual memory, mother of the nine Muses) has been called upon to record our homage to the life and pre-eminent scientific work of Professor Ivo Babuška. His insightful and challenging questions occasioned our exploration of thin elastic structures.

†Title of a book by Steven Weinberg, Nobel Laureate.

numerical algorithms for solving algebraic eigenvalue problems are among the oldest known in the literature; they are "150 years old and still alive" as the authors of ref. 18 wittingly put it to emphasize that eigenproblems continue to be a major focus of modern numerical analysis—especially in connection with large-scale computations. Unlike the case of linear systems of equations—which can be solved by *direct* methods, i.e., at a finite number of steps—methods for solving algebraic eigenvalue problems are intrinsically *iterative*. When developing an iterative method it is important to secure that it is *robust*, i.e., its convergence rate is not adversely affected by the various parameters involved. In large-scale finite element computations the convergence may deteriorate as the size of the discretized problem increases. In the case of *thin* elastic structures, such as shells, plates, and rods, one encounters, additionally, the deterioration of the convergence owing to the small thickness of the structure. This poses a hitherto insurmountable difficulty even for modern high performance numerical methods such as multilevel methods, as witnessed, e.g., in the numerical experiments of refs. 19 and 20. Interestingly, as ref. 19 states, this convergence deterioration is especially pronounced in the case of thin *plates* (the case treated here) rather than thin shells; moreover, very small plate thickness may result not only in slow convergence but sometimes in convergence failure.

Recent progress in robust iterative algorithms is associated with a technique called *preconditioning* (see, e.g., refs. 21 and 22 and the references therein). To describe this technique, let us first consider the solution of a linear system $A\mathbf{u} = \mathbf{f}$ with a symmetric positive definite matrix A. The solution $\mathbf{u}$ of this system yields the minimum to the functional $J(\mathbf{u}) = (A\mathbf{u}, \mathbf{u}) - 2(\mathbf{f}, \mathbf{u})$, where $(\cdot,\cdot)$ is the Euclidean scalar product; and it can be found using, e.g., the steepest descent algorithm. The convergence of this algorithm is determined by the spectral condition number (i.e., the ratio of the maximal and minimal eigenvalues) of the matrix A. Hence, if A is, e.g., the stiffness matrix of the finite element method, the convergence deteriorates when the finite element mesh is refined. In the case of a thin elastic structure, e.g., a plate or a shell, the spectral condition number of A depends, additionally, on the thickness t as $O(t^{-2})$. Thus, the convergence deteriorates also when t is small (cf. ref. 21). However, if one calculates the gradient of J using the scalar product $(B\cdot,\cdot)$, where B is a symmetric positive definite matrix, then the convergence of the steepest descent algorithm is determined by the spectral condition number of $B^{-1}A$ rather than by the spectral condition number of A. Therefore, the deterioration of the convergence caused by a parameter can be avoided provided that the spectral condition number of $B^{-1}A$ is uniformly bounded with respect to that parameter. The matrix B is called a *preconditioner* for A, and the technique just described is the preconditioning we mentioned earlier.

The above preconditioning technique can be applied to eigenvalue problems as well. Suppose that we are interested in the minimal eigenvalue of the matrix A. This eigenvalue is the minimum of the *Rayleigh quotient* functional $\lambda(\varphi) = (A\varphi, \varphi)/(\varphi, \varphi)$, which can be found using the preconditioned steepest descent algorithm described above. In Section 5 below, we present an estimate for the convergence of this algorithm obtained from that of ref. 23. This estimate shows that the convergence is determined by the spectral condition number of $B^{-1}A$ and the ratio $(\lambda_1 - \lambda_0)/\lambda_1$, where λ_0 and λ_1 are the two smallest eigenvalues of A. In the case of a thin elastic plate the asymptotic analysis of ref. 24 (see Section 6) demonstrates that $(\lambda_1 - \lambda_0)/\lambda_1$ is uniformly positive in the vicinity of $t = 0$. Thus, if we use a preconditioner that is robust with respect to both the thickness and the discretization parameters then the corresponding preconditioned steepest descent method for finding the minimal eigenvalue of A is robust also with respect to these parameters.

Today, we find a plethora of preconditioned iterative methods that are robust with respect to the discretization parameters (see, e.g., ref. 25). By contrast, there is a scarcity of iterative methods for thin elastic structures that are robust with respect to the thickness—and these invariably address one- and two-dimensional models (see, e.g., refs. 26–28). The question of thickness-robustness for full three-dimensional elasticity was first addressed—and radically resolved—in refs. 13–16. In these papers, a methodology was introduced for the development of robust numerical methods for *thin* elastic structures based on the synergy of the so-called *Effective Dimensional Reduction Algorithm* (EDRA; ref. 12) and the concept of the *Korn's type inequality in subspaces* (see, e.g., refs. 14 and 16). In EDRA special basis functions are used for the discretization of the problem in the transverse direction of the thin structure, whereas in the lateral direction any suitable discretization can be employed—or, in fact, no discretization at all, yielding a *semi-discrete* system of equations (see, e.g., refs. 12 and 16). The (semi-) discretized problem is then solved by a suitable preconditioned iterative algorithm with a block-diagonal preconditioner in which each block corresponds to basis functions with a fixed transversal component (see Section 7). In the present paper, we show that this methodology can also be applied to eigenvalue problems of full three-dimensional elasticity for thin structures. As an example, we consider the solution of the minimal eigenvalue problem for a thin elastic plate using the steepest descent algorithm and we show both theoretically and numerically that employing the above block-diagonal preconditioner makes the convergence of this algorithm robust with respect to both the thickness and the discretization parameters. This robustness is *sine qua non* for the efficient computation of *large-scale* eigenvalue problems for thin elastic structures.

2. Notation

In the paper, we use the notation $(\cdot,\cdot)_\Omega$ for the scalar product in the Lebesgue space $L^2(\Omega)$. The same notation is used for vector functions, i.e., for $\mathbf{u}, \mathbf{v} \in (L^2(\Omega))^3$, $(\mathbf{u}, \mathbf{v})_\Omega$ denotes the scalar product of $\mathbf{u}$ and $\mathbf{v}$ in $(L^2(\Omega))^3$. Throughout the paper, we use boldface for vector functions and their spaces, i.e., $\mathbf{u} = (u_1, u_2, u_3)$, etc. The standard notation is used for Sobolev spaces and their subspaces, e.g., $H^1_0(\Omega)$ is the Sobolev space of functions vanishing on the boundary $\partial\Omega$ of the domain Ω with weak derivatives in $L^2(\Omega)$.

3. Eigenvalue Problem for a Thin Elastic Structure

Let Ω be a domain of the form $\omega \times (-t, t)$, where ω is a two-dimensional bounded domain with piecewise smooth Lipschitz boundary $\partial\omega$. Thus, t is the (half-) thickness of Ω.

In this paper, we are interested in the computation of the minimal eigenvalue of the following eigenvalue problem:

$$E(\mathbf{u}, \mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v})_\Omega \ \forall \mathbf{v} \in \mathbf{V}. \quad [1]$$

Here $\mathbf{V}$ is the space of admissible displacements given by

$$\mathbf{V} = \{\mathbf{u} \in (H^1(\Omega))^3 : \mathbf{u} = \mathbf{0} \text{ on } \Gamma = \partial\omega \times (-t, t)\}, \quad [2]$$

and E is the elastic energy form given by

$$E(\mathbf{u}, \mathbf{v}) = \int_\Omega \sum_{i,j=1}^{3} \sigma_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})d\Omega,$$

where ε_{ij} are the components of the *strain* tensor given by

$$\varepsilon_{ij}(\mathbf{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

and σ_{ij} are the components of the *stress* tensor given by

$$\sigma_{ij}(\mathbf{u}) = \frac{E}{1+\nu}\,\varepsilon_{ij}(\mathbf{u}) + \frac{E\nu}{(1+\nu)(1-2\nu)}\,\delta_i^j \sum_{k=1}^{3} \varepsilon_{kk}(\mathbf{u}),$$

where E is the Young's modulus and ν is the Poisson's ratio.

4. Discretization

To discretize **1** we introduce the finite-dimensional approximation $\mathbf{V}^{pq}$ of the space of admissible displacements V given by

$$\mathbf{V}^{pq} = \{(u_1, u_2, u_3) \in \mathbf{V}: u_k = \sum_{i=0}^{p} u_{ki}(x, y) w_i(z)\}, \quad [3]$$

where $u_{ki} \in V^q$,

$$V^q = \{u \in H_0^1(\omega): u = \sum_{j=1}^{N_q} c_j v_j(x, y)\}, \quad [4]$$

and $v_j(x, y)$ and $w_i(z)$ are some given *basis* functions. The Galerkin projection onto $\mathbf{V}^{pq}$ yields the following discretized counterpart of **1**:

$$E(\mathbf{u}^{pq}, \mathbf{v}) = \lambda(\mathbf{u}^{pq}, \mathbf{v})_\Omega \ \forall \mathbf{v} \in \mathbf{V}^{pq}. \quad [5]$$

Let us enumerate the elements of the set J_{pq} of triple indices (i, j, k), where i ranges between 0 and p, j ranges between 1 and N_q, and k ranges between 1 and 3:

$$J_{pq} = \{0, \ldots, p\} \times \{1, \ldots, N_q\} \times \{1, 2, 3\} = \{(i_n, j_n, k_n)\}_{n=1,N_{pq}}.$$

The vector functions $\mathbf{w}_n^{pq} = w_{i_n}(z) v_{j_n}(x, y) \mathbf{e}_{k_n}$ where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$, form the basis of $\mathbf{V}^{pq}$, and the eigenvalue problem **5** can be expressed in matrix form as

$$A\varphi = \lambda B\varphi, \quad [6]$$

where the *stiffness matrix* A and the *mass matrix* B are $N_{pq} \times N_{pq}$ matrices with the elements a_{mn} and b_{mn} given by

$$a_{mn} = E(\mathbf{w}_m^{pq}, \mathbf{w}_n^{pq}),\ b_{mn} = (\mathbf{w}_m^{pq}, \mathbf{w}_n^{pq})_\Omega \quad [7]$$

and φ is the vector of the coordinates of the eigenfunction $\mathbf{u}^{pq}$ in the basis $\{\mathbf{w}_n^{pq}\}$, i.e.,

$$\mathbf{u}^{pq} = \sum_{i=1}^{N_{pq}} \varphi_i \mathbf{w}_i^{pq}.$$

We observe from **7** that the matrices A and B are symmetric positive definite.

5. Preconditioned Iterative Solution of an Algebraic Eigenvalue Problem

The problem of finding the smallest eigenvalue λ_0 (and corresponding eigenvector φ_0) of **6** can be formulated equivalently as the following variational problem:

$$\lambda_0 = \lambda(\varphi_0) = \min_{\varphi} \lambda(\varphi), \quad [8]$$

where $\lambda(\varphi)$ is the *Rayleigh quotient* functional given by

$$\lambda(\varphi) = \frac{(A\varphi, \varphi)}{(B\varphi, \varphi)},$$

and $(\cdot, \cdot)$ stands for the usual scalar product in N_{pq}-dimensional Euclidean space. The problem **8** can be solved iteratively by the steepest descent algorithm: given an approximation φ_0^n to the eigenvector φ_0, we compute the new approximation φ_0^{n+1} and the corresponding minimal eigenvalue approximation λ_0^{n+1} by solving the following one-dimensional variational problem:

$$\lambda_0^{n+1} = \lambda(\varphi_0^{n+1}) = \min_{\tau} \lambda(\varphi_0^n + \tau\nabla\lambda(\varphi_0^n)). \quad [9]$$

From **9** we see that $\lambda_0 \le \lambda_0^{n+1} \le \lambda_0^n$ and, therefore, the iterations converge. Furthermore, it is easy to verify that if $\lambda_0^n \to \lambda$ then λ is an eigenvalue of **6**. Thus, if $\lambda_0^m < \lambda_1$ for some m then $\lambda_0^n \to \lambda_0$. However, the convergence rate of the iterations **9** generally depends on the thickness t and discretization parameters p and q. When the thickness is small or the finite element mesh is refined, i.e., p and N_q are large, the convergence may be very slow. To avoid the deterioration of the convergence we naturally resort to preconditioning.

Let C be a symmetric positive definite $N_{pq} \times N_{pq}$ matrix. If we use the scalar product $(C\cdot, \cdot)$ instead of $(\cdot, \cdot)$, then the gradient of the functional $\lambda(\varphi)$ becomes

$$\nabla\lambda(\varphi) = \frac{2}{(B\varphi, \varphi)} C^{-1}(A\varphi - \lambda(\varphi)B\varphi). \quad [10]$$

The matrix C is called the *preconditioner* and the iterative algorithm **9** with the gradient computed according to **10** is called the *preconditioned* steepest descent algorithm. For the convergence rate of this algorithm the following estimate can be derived from that given in ref. 23 (see also ref. 29). Assuming that λ_0 is simple, and $\lambda_0^n < \lambda_1$, where λ_1 is the second smallest eigenvalue of **6**, we have

$$r(\lambda_0^{n+1}) \le \left(1 - \frac{\delta_0}{\delta^0}\frac{\lambda_1 - \lambda_0}{\lambda_1}\right) r(\lambda_0^n), \quad [11]$$

where

$$r(\lambda) = \frac{\lambda - \lambda_0}{\lambda_1 - \lambda},\ \delta_0 = \min_{\varphi} \frac{(A\varphi, \varphi)}{(C\varphi, \varphi)},\ \delta^0 = \max_{\varphi} \frac{(A\varphi, \varphi)}{(C\varphi, \varphi)}.$$

From **11** we observe that the convergence of the algorithm **9** is affected by two factors: (*i*) the spectral condition number $\delta = \delta^0/\delta_0$ of the matrix $C^{-1}A$ and (*ii*) the ratio $\delta_\lambda = (\lambda_1 - \lambda_0)/\lambda_1$. In Section 6 below, we describe the asymptotic behavior of δ_λ with respect to the thickness and discretization parameters, and in Section 7, we introduce the preconditioner C for which δ is bounded by a constant independent of these parameters.

6. Asymptotic Behavior of Eigenvalues of Thin Plates

From the theory of spectral approximation (see, e.g., ref. 30) we know that the two smallest eigenvalues of **5** converge to the two smallest eigenvalues of **1** provided that the space $\mathbf{V}^{pq}$ approximates the corresponding eigenfunctions. For the case of a thin isotropic and homogeneous plate, it was first shown in ref. 24 that, as $t \to 0$, the eigenvalues of **1** divided by t^2 converge to the eigenvalues θ of the biharmonic equation

$$\frac{E}{3(1-\nu^2)} \int_\omega \Delta u \Delta v \, d\omega = \theta \int_\omega u v \, d\omega \ \forall v \in H_0^2(\omega). \quad [12]$$

APPLIED MATHEMATICS

Thus, we conclude that

$$\lim_{t\to 0}\lim_{p,q\to\infty}\frac{\lambda_1-\lambda_0}{\lambda_1}=\frac{\theta_1-\theta_0}{\theta_1}, \qquad [13]$$

where θ_0 and θ_1 are the two smallest eigenvalues of **12**. Assuming that the first eigenvalue of **12** is simple, the constant in the right-hand side of **13** is positive.

7. Effective Dimensional Reduction Preconditioner for Thin Elastic Plates

Let us set $w_0(z) = 1$, $w_1(z) = z$ and assume that the subsequent basis functions are the eigenfunctions corresponding to all but the two smallest eigenvalues of the eigenvalue problem

$$w_i \in \Pi^p\colon \int_{-t}^{t}\frac{dw_i}{dz}\frac{dw}{dz}\,dz=\lambda\int_{-t}^{t} w_i w\,dz \;\; \forall w\in\Pi^p, \qquad [14]$$

where Π^p is the space of polynomials of degree $\le p$. Further, let us assume that the triple indices (i_n, j_n, k_n) are enumerated in such a way that i_n is the slowest index and j_n is the fastest one, and let us take the following block-diagonal matrix for the preconditioner C:

$$C = \mathrm{diag}\{C_0, C_{21}, C_{22}, C_{23}, \ldots, C_{p1}, C_{p2}, C_{p3}\}, \qquad [15]$$

where C_0 coincides with the square block of A containing all elements a_{mn} for which $i_n = i_m \le 1$ and C_{ik} coincides with the square block of A containing all a_{mn} for which $i_n = i_m = i$ and $k_n = k_m = k$.

According to **10** and **15** the computation of the gradient in the preconditioned steepest descent algorithm involves the solution of the systems

$$C_0\mathbf{g}_0 = \mathbf{f}_0 \qquad [16]$$

and

$$C_{ij}\mathbf{g}_{ij} = \mathbf{f}_{ij}. \qquad [17]$$

From the definition of the matrices C_0 and C_{ij}, we observe that they can be viewed as stiffness matrices of some auxiliary two-dimensional problems. Thus, the solution of the discretized *three-dimensional* problem is reduced to the solution of a sequence of discretized *two-dimensional* problems. This key feature explains the use of the term "dimensional reduction" for algorithms based on the above preconditioner. One such algorithm is the so-called EDRA which the authors have previously introduced to solve multidimensional boundary value problems for elliptic systems of partial differential equations (see, e.g., refs. 12 and 16; ‡). For this algorithm a convergence rate estimate is given in ref. 16 which is independent of the number of lower-dimensional blocks C_{ij} in the preconditioner C. It should be noted that if we introduce the block partitioning of the matrix A corresponding to the partitioning of C given above, then A itself can be formally viewed as the matrix of a system of discretized two-dimensional equations. However, unlike **16–17**, this system is *coupled*. The above result on the convergence rate of the EDRA implies that the use of the preconditioner C effectively *decouples* this system, and, for this reason, we have called this preconditioning technique *effective* dimensional reduction. Furthermore, from the Korn's type inequality in subspaces it follows that the constants δ_0 and δ^0 are uniformly bounded from below and above not only with respect to the discretization parameters but also with respect to the thickness (see refs. 15 and 16). Thus, the convergence of the EDRA is not adversely affected by either the small thickness or large number of elements (i.e., basis functions) in a finite element discretization.

Returning to the eigenvalue problem, we observe that, in view of the discussion in the previous section, one can easily derive from **11** the following asymptotic estimate for the convergence of the iterations **9**:

$$\lim_{t\to 0}\lim_{p,q\to\infty}\lim_{n\to\infty}\frac{|\lambda_0^{n+1}-\lambda_0|}{|\lambda_0^{n}-\lambda_0|}=1-\frac{\delta_0}{\delta^0}\frac{\theta_1-\theta_0}{\theta_1}=q_0<1.$$

This estimate shows that the convergence rate does not deteriorate for large values of the discretization parameters p and N_q (cf. **3** and **4**) and small values of the thickness parameter t. Thus, the above algorithm for solving the minimal eigenvalue problem for thin elastic structures is *robust* with respect to these parameters.

Table 1. The smallest eigenvalue (λ_0) and the number of iterations (*N*) vs. thickness (*t*) and polynomial degrees in the transverse (*p*) and lateral (*q*) directions

t	*p*	*q*	λ_0	*N*
0.04	2	6	1062890.0	6
0.02	2	6	266137.0	6
0.01	2	6	66568.1	6
0.005	2	6	16644.8	6
0.01	1	6	72600.5	2
0.01	2	6	66568.1	6
0.01	3	6	66566.3	6
0.01	4	6	66566.3	6
0.01	2	2	78726.6	6
0.01	2	4	66789.3	6
0.01	2	6	66568.1	6
0.01	2	8	66497.1	6

8. Numerical Example

In this section, we present numerical results for a three-dimensional cantilevered rectangular plate $\Omega = (0, 1) \times (0, 2) \times (0, t)$ clamped at $y = 0$ and free on all other edges. The elastic moduli are: $E = 10^{10}$ and $\nu = 0.25$. We discretize problem **1** using the p-version of the finite element method (see, e.g., ref. 8). We have carried out numerical experiments using one and more rectangular brick elements to subdivide the given domain. Here we present typical convergence results for the case when the domain is subdivided into 4 elements by the planes $x = 0.5$ and $y = 0.5$. We use the basis functions described in Section 7 in the (transverse) variable z and (piecewise) polynomials of degree $\le q$ in the (lateral) variables x and y. The discretized problem **6** is solved using iterations **9** which are repeated until the relative energy norm $\sqrt{(A\mathbf{r}^n, \mathbf{r}^n)/\lambda_0^n}$ of the (preconditioned) residual $\mathbf{r}^n = C^{-1}(A\varphi_0^n - \lambda_0^n B\varphi_0^n)$ becomes less than 0.001. Table 1 displays the smallest eigenvalue λ_0 and the number of iterations for various values of the thickness parameter t and the discretization parameters p and q. We note that these results fully confirm the theoretical predictions made above.

Epilegomena

It may be useful to end with a reflection on the versatility of the preconditioner presented above. In this paper we have presented a methodology for computing the minimal eigenvalue for thin isotropic and homogeneous plates and we have demonstrated

‡In the present paper, for simplicity and consistency of notation, we interpret the results on EDRA in the algebraic form.

robustness of the preconditioner with respect to the thickness and discretization parameters. Our current research indicates that this preconditioner exhibits the same robustness also for anisotropic and heterogeneous (e.g., multilayered) thin elastic structures, such as shells, plates and rods of arbitrary geometry. But these and further results including the computation of several eigenvalues for three-dimensional thin elastic structures will be the subject of another communication.

1. Antman, S. S. (1995) *Nonlinear Problems of Elasticity* (Springer, New York).
2. Babuška, I. & Li, L. (1991) *Comput. Struct.* **40,** 419–430.
3. Bernadou, M. (1996) *Finite Element Methods for Thin Shell Problems* (Wiley, London).
4. Ciarlet, P. G. (1997) *Mathematical Elasticity, Volume II: Theory of Plates* (North-Holland, Amsterdam).
5. Libai, A. & Simmonds, J. G. (1998) *The Nonlinear Theory of Elastic Shells* (Cambridge Univ. Press, Cambridge, U.K.).
6. Noor, A. K., Belytschko, T. & Simo, J. C. (1989) *Analytical and Computational Models of Shells*, CED-Vol. 3, ASME.
7. Sanchez-Hubert, J. & Sanchez-Palencia, E. (1997) *Coques Elastiques Minces: Propriétés Asymptotiques* (Masson, Paris).
8. Szabó, B. & Babuška, I. (1991) *Finite Element Analysis* (Wiley, New York).
9. Love, A. E. H. (1888) *Philos. Trans. R. Soc. London A* **17,** 491–546.
10. Calladine, C. R. (1988) *Proc. Inst. Mech. Eng.* **202,** A3, 141–149.
11. Arnold, D. N., Babuška, I., Brezzi, F., Ciarlet, P. & Pitkäranta, J. (1998) Quotation by the organizers of the workshop on *Elastic Shells: Modeling, Analysis and Numerics*, April 2000, Math. Sci. Res. Inst., Berkeley, CA. http://msri.org/activities/programs/9900/fem/eshells
12. Ovtchinnikov, E. E. & Xanthis, L. S. (1995) *C. R. Acad. Sci. Paris Sér. I.* **320,** 879–884.
13. Ovtchinnikov, E. E. & Xanthis, L. S. (1996) *Comput. Methods Appl. Mech. Engrng.* **138,** 299–315.
14. Ovtchinnikov, E. E. & Xanthis, L. S. (1997) *Proc. R. Soc. London A* **453,** 1793–2016.
15. Ovtchinnikov, E. E. & Xanthis, L. S. (1998) *Comput. Methods Appl. Mech. Engrng.* **160,** 23–37.
16. Ovtchinnikov, E. E. & Xanthis, L. S. (1998) *Proc. R. Soc. London A* **454,** 2023–2278.
17. Kuhn, T. S. (1996) *The Structure of Scientific Revolutions* (Chicago Univ. Press, Chicago), 3rd Ed.
18. van der Vorst, H. A. & Golub, G. H. (1997) in *The State of the Art in Numerical Analysis*, eds. Duff, I. S. & Watson, G. A. (Clarendon, Oxford), pp. 93–119.
19. Cho, Y. & Yong, Y. K. (1996) *Comput. Struct.* **58,** 575–583.
20. Hwang, T. & Parsons, I. D. (1992) *Int. J. Numer. Methods Eng.* **35,** 1677–1696.
21. D'yakonov, E. G. (1996) *Optimization in Solving Elliptic Problems* (CRC, Boca Raton, FL).
22. Knyazev, A. V. (1998) *Electron. Trans. Numer. Anal.* **7,** 104–123.
23. Knyazev, A. V. (1990) *Int. Ser. Numer. Math.* Oberwolfach **96,** 143–154.
24. Ciarlet, P. G. & Kesavan, S. (1981) *Comput. Methods Appl. Mech. Engrng.* **26,** 145–172.
25. Mandel, J., Farhat, C. & Cai, X.-C. (1998) *Domain Decomposition Methods 10* (Am. Math. Soc., Providence, RI).
26. Arnold, D. N., Falk, R. S. & Winther, R. (1997) *Math. Model. Numer. Anal.* **31,** 517–557.
27. Bramble, J. H. & Sun, T. (1998) *Math. Comp.* **67,** 901–916.
28. Brenner, S. (1996) *Math. Model. Numer. Anal.* **30,** 265–297.
29. D'yakonov, E. G. & Orekhov, M. Yu. (1980) *Math. Notes* **27,** 382–391.
30. Babuška, I. & Osborn, J. (1991) in *Handbook of Numerical Analysis*, eds. Ciarlet, P. G. & Lions, J.-L. (North-Holland, Amsterdam), Vol. II, Part 1, pp. 641–787.