A pseudo zeta function and the distribution of primes

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The Riemann zeta function is given by:

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s = \prod_{p\ \text{prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \text{ for Re } s > 1.$$

$\zeta(s)$ may be analytically continued to the entire s-plane, except for a simple pole at $s = 0$. Of great interest are the complex zeros of $\zeta(s)$. The Riemann hypothesis states that the complex zeros all have real part 1/2. According to the prime number theorem, $p_n \approx n \log n$, where p_n is the nth prime. Suppose that p_n were exactly $n \log n$. In other words, in the Euler product above, replace the nth prime by $n \log n$. In this way, we define a pseudo zeta function $C(s)$ for Re $s > 1$. One can show that $C(s)$ may be analytically continued at least into the half-plane Re $s > 0$ except for an isolated singularity (presumably a simple pole) at $s = 0$. It may be shown that the pseudo zeta function $C(s)$ has no complex zeros whatsoever. This means that the complex zeros of the zeta function are associated with the irregularity of the distribution of the primes.

For Re $s > 1$, the Riemann zeta function is defined by the series $\zeta(s) = \Sigma_{n=1}^{\infty} 1/n^s$. Euler (1,2) observed that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ \text{prime}} \left(1 - \frac{1}{p^s}\right)^{-1}. \qquad [1]$$

This is a simple consequence of unique factorization.

Riemann (1,2) showed that $\zeta(s)$ continues analytically to the entire s-plane, except for a simple pole at $s = 1$. The ζ function has real zeros at $s = -2, -4, -6, \ldots$. Of great interest are the *complex* zeros of $\zeta(s)$. The celebrated Riemann hypothesis states that the complex zeros all have real part 1/2.

From Eq. **1** above,

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right),$$

so the Riemann hypothesis says that $1/\zeta(s)$ has *no poles* with Re $s \neq 1/2$ in the critical strip $0 \leq$ Re $s \leq 1$. Because of the symmetry implied by Riemann's functional equation (see *An Unsolved Problem* below), it is equivalent to show that $1/\zeta(s)$ has no poles with Re $s > 1/2$.

Now, for Re $s > 1$, we have (2)

$$\begin{aligned}\log\zeta(s) &= -\sum_p \log\left(1 - \frac{1}{p_s}\right)\\ &= \sum_p \sum_{k=1}^{\infty} \frac{p^{-sk}}{k}\\ &= \sum_{k=1}^{\infty} \frac{1}{k} \sum_p p^{-sk}.\end{aligned}$$

If we write

$$\varphi(s) = \sum_p \frac{1}{p^s} = \sum_p p^{-s}, \qquad [2]$$

then we have

$$\log\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k}\varphi(ks). \qquad [3]$$

Note that the series $\Sigma_{k=2}^{\infty} 1/k\varphi(ks)$ is uniformly convergent, hence analytic, in any half-plane Re $s \geq (1/2) + \delta$, $\delta > 0$, for then Re$(ks) \geq 1 + 2\delta$, and $|\varphi(ks)| \leq \Sigma_p p^{-k(1/2+\delta)}$. This series is uniformly convergent, because

$$\begin{aligned}\text{Re } s \geq \frac{1}{2} + \delta \text{ implies} |\varphi(ks)| &\leq \sum_p p^{-k(1/2+\delta)}\\ &\leq \sum_{n=2}^{\infty} n^{-k(1/2+\delta)}\\ &\leq \int_1^{\infty} x^{-k(1/2+\delta)} dx\\ &= \frac{1}{k\left(\frac{1}{2} + \delta\right) - 1} = O\left(\frac{1}{k}\right), \ k \geq 2.\end{aligned}$$

Accordingly, $\Sigma_{k=2}^{\infty} 1/k|\varphi(ks)|$ is dominated by the constant series $M\ \Sigma_{k=2}^{\infty} 1/k^2 < \infty$.

So only the first term $\varphi(s)$ in Eq. **2** can cause trouble; hence the Riemann hypothesis is *equivalent* to the claim that $\varphi(s)$ can be analytically continued into the strip $1/2 <$ Re $s < 1$. (NB φ will have a logarithmic singularity at $s = 1$.)

Motivated by the prime number theorem, which states that the nth prime $p_n \approx n \log n$, we consider the "comparison series"

$$\psi(s) = \sum_{n=2}^{\infty} (n \log n)^{-s}. \qquad [4]$$

Can ψ be continued analytically into the critical strip? This question is at least vaguely related to the Riemann hypothesis, because presumably the properties of ψ are related to those of φ.

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An Integral Formula

From Eq. **4**, we have the Stieltjes integral formula

$$\begin{aligned}\psi(s) &= \int_{2-}^{\infty} (x \log x)^{-s} d[x] \\ &= (x \log x)^{-s}[x]\big|_{2-}^{\infty} + s\int_{2}^{\infty} (x \log x)^{-(s+1)} \cdot \\ &\quad \cdot (\log x + 1)[x]dx \\ &= -(2\log 2)^{-s} + s\int_2^{\infty} [x](x\log x)^{-(s+1)}(\log x + 1)dx.\end{aligned}$$

Write $\{x\} = x - [x]$, the fractional part of x. Then the integral above is

$$\int_2^{\infty} (x \log x)^{-s}\left(1 + \frac{1}{\log x}\right)dx - \int_2^{\infty} \frac{\{x\}(\log x + 1)}{(x \log x)^{s+1}}\, dx. \quad \textbf{[5]}$$

Note that the second integral in Eq. **5** defines an analytic function for Re $s > 0$.

Denote the first integral in Eq. **5** by $I(s)$. Then $I(s) = J(s) + K(s)$, where

$$J(s) = \int_2^{\infty} (x \log x)^{-s}dx = \int_2^{\infty} x^{-s}(\log x)^{-s}dx$$

and

$$K(s) = \int_2^{\infty} x^{-s}(\log x)^{-(s+1)}dx.$$

If we integrate K by parts, we get

$$\begin{aligned} u &= x^{-(s-1)} & dv &= (\log x)^{-(s+1)}\tfrac{dx}{x} \\ du &= -(s-1)x^{-s}dx & v &= -\frac{1}{s}(\log x)^{-s}, \end{aligned}$$

whence

$$\begin{aligned} K(s) &= \frac{1}{2}\cdot 2^{-(s-1)}(\log 2)^{-s} - \left(\frac{s-1}{s}\right)\int_2^{\infty} x^{-s}(\log x)^{-s}dx \\ &= \frac{2\cdot(2\log 2)^{-s}}{s} + \left(\frac{1}{s} - 1\right)J(s). \end{aligned}$$

Therefore,

$$I(s) = J(s) + K(s) = \frac{2\cdot(2\log 2)^{-s}}{s} + \frac{1}{s}J(s). \quad \textbf{[6]}$$

Next, we integrate $J(s)$ by parts:

$$\begin{aligned} u &= x^{-(s-1)} & dv &= (\log x)^{-s}\tfrac{dx}{x} \\ du &= -(s-1)x^{-s}dx & v &= -\left(\frac{1}{s-1}\right)(\log x)^{-(s-1)}. \end{aligned}$$

We obtain

$$J(s) = \frac{(2\log 2)^{-(s-1)}}{s-1} - \int_2^{\infty} x^{-s}(\log x)^{-(s-1)}dx.$$

In the latter integral, make the substitution $x = e^t$; the integral becomes

$$\int_{\log 2}^{\infty} t^{-(s-1)}e^{-(s-1)t}dt,$$

which equals

$$\int_0^{\infty} t^{-(s-1)}e^{-(s-1)t}dt - \int_0^{\log 2} t^{-(s-1)}e^{-(s-1)t}dt = W(s) - Y(s). \quad \textbf{[7]}$$

The second integral $Y(s)$ in Eq. **7** is an analytic function if Re$(s - 1) < 1$, i.e., Re $s < 2$. As for the first integral $W(s)$, it is convergent for $1 <$ Re $s < 2$, and we have the explicit formula

$$W(s) = (s-1)^{-s}\Gamma(2-s) = -(s-1)^{-(s-1)}\Gamma(1-s).$$

Now $(s-1)^{-(s-1)} = \exp[-(s-1)\log(s-1)]$ is a single-valued analytic function on the (simply connected) slit plane $\mathbb{C}\backslash[1, \infty)$. Then for $s \notin [1, \infty)$, we have the relation

$$J(s) = \frac{(2\log 2)^{-(s-1)}}{s-1} + (s-1)^{-(s-1)}\Gamma(1-s) + Y(s), \quad \textbf{[8]}$$

whence $J(s)$ can be analytically continued into the entire s plane minus the point $\{1\}$.

The Main Theorem

From the results of the preceding section,

$$I(s) = \frac{2\cdot(2\log 2)^{-s}}{s} + \frac{1}{s}J(s)$$

also has an analytic continuation into the critical strip. That is, in particular, it has no singularities there.

Hence

$$\psi(s) = I(s) - \int_2^{\infty} \frac{\{x\}(\log x + 1)}{(x\log x)^{s+1}}\,dx$$

has an analytic continuation into the critical strip $0 <$ Re $s \leq 1$ in which it has no singularities. [This is in contrast with $\varphi(s)$, which must be singular at the complex zeros of $J(s)$.]

Conclusion. THEOREM. *Let* $C(s) = \Pi_{n=2}^{\infty}(1 - (n\log n)^{-s})^{-1}$. *Then $C(s)$ continues analytically into the critical strip and has no zeros there.*

Significance of the theorem: If the primes were distributed more regularly (i.e., if $p_n \equiv n \log n$), then the Riemann hypothesis would be trivially true. In reality, the zeros of $J(s)$ are related to the *irregularities* in the distribution of the primes. [Of course, the latter fact was known to Riemann; see his "explicit formula" (1) for $\pi(x)$, the number of primes less than x.]

Some Related Pseudo ζ Functions

It is, of course, impossible for the nth prime p_n to equal $n \log n$, for the latter is not even an integer. But what if p_n is replaced by $[n \log n]$, the greatest integer in $n \log n$? More generally, what if p_n is replaced by $n \log n + \varepsilon_n$, where $(\varepsilon_n)_2^{\infty}$ is a bounded sequence? It turns out that the corresponding pseudo zeta function also has no complex zeros.

To see this, consider the difference

$$\sum_{n=2}^{\infty} \frac{1}{(n\log n)^s} - \sum_{n=2}^{\infty}\frac{1}{(n\log n + \varepsilon_n)^s}$$

$$= \sum_{n=2}^{\infty} \left\{ \frac{1}{(n \log n)^s} - \frac{1}{(n \log n + \varepsilon_n)^s} \right\}$$

$$= \sum_{n=2}^{\infty} \int_0^{\varepsilon_n} \frac{s}{(n \log n + t)^{s+1}} \, dt.$$

This series converges uniformly for $\mathrm{Re}\, s > 0$. Indeed, writing $\sigma = \mathrm{Re}\, s$, we have

$$\| |n \log n + t|^{s+1}| = |n \log n + t|^{\sigma+1} \geq (n \log n - t)^{\sigma+1},$$

so that our series is dominated by

$$|s| \sum_{n=2}^{\infty} \int_0^{|\varepsilon_n|} \frac{dt}{(n \log n - t)^{\sigma+1}} \leq |s| \sum_{n=2}^{\infty} \int_0^{\varepsilon} \frac{dt}{(n \log n - t)^{\sigma+1}},$$

where $\varepsilon = \sup_n |\varepsilon_n|$. Interchanging the order of summation and integration, we have

$$|s| \int_0^{\varepsilon} \sum_{n=2}^{\infty} \frac{1}{(n \log n - t)^{\sigma+1}} \, dt,$$

which converges uniformly for $\sigma > 0$.

Accordingly, the two series

$$\sum_{n=2}^{\infty} \frac{1}{(n \log n + \varepsilon_n)^s} \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{1}{(n \log n)^s}$$

have the same analytic behavior in the critical strip. Hence $\Sigma_{n=2}^{\infty} 1/(n \log n + \varepsilon_n)^s$ has no complex singularities in the critical strip. But this means that the associated pseudo zeta function

$$\tilde{C}(s) = \prod_{n=2}^{\infty} (1 - (n \log n + \varepsilon_n)^{-s})^{-1}$$

has no zeros in the critical strip.

An Unsolved Problem

The Riemann zeta function satisfies the functional equation (1, 2)

$$f(s) = f(1 - s),$$

where $f(s) = \Gamma(s/2)\zeta(s)\pi^{-s/2}$.

Question: Does the pseudo zeta function (say with p_n replaced by $n \log n$) also satisfy some sort of functional equation? This would be quite interesting if true.

1. Edwards, H. M. (1974) *Riemann's Zeta Function* (Academic, New York) 1–38, 299–307.
2. Widder, D. V. (1971) *An Introduction to Transform Theory* (Academic, New York) 51–55, 60–63, 85–90.

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