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Mathematics

General covariance and harmonic maps

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ABSTRACT **The principle of general covariance is used to derive the conditions for a map to be harmonic. Modification to the harmonic map equations due to the presence of a Yang–Mills field is described.**

1. Introduction

Einstein, Infeld, and Hoffmann derived the "geodesic principle" saying that a particle moves along a geodesic in response to the semi-Riemann geometry determined by the mass distribution of the universe from general symmetry considerations—the "principle of general covariance" in ref. 1. Their result not only stated that the trajectory of a particle is a geodesic, γ, but also that the associated energy momentum tensor is proportional to the tensorial distribution $\gamma' \otimes \gamma'$ supported on γ. In the 1970s it was shown [first by Souriau (2) in the mathematics literature and then by Gurses and Gursey (3) (independently) in the physics literature] that the Einstein–Infeld–Hoffmann (EIH) method can be used to derive the equations of a string provided that some special additional assumptions are made about its energy momentum tensor. In view of the recent interest in "branes" as a physical theory, I thought it would be useful to work out the general case of a submanifold of arbitrary dimension, without making any special assumptions about the nature of the associated tensorial distribution other than its nondegeneracy. The EIH condition implies that the energy momentum tensor be tangential to the submanifold, and our nondegeneracy assumption means that it induces a semi-Riemann metric on Q. In all dimensions except two we can rescale the energy momentum tensor so that the density with respect to which we are integrating is the volume measure of the associated metric. (In two dimensions this is an additional assumption.) When this is done, the EIH condition says that the map giving the submanifold is "harmonic" relative to the ambient metric and the intrinsic metric coming from the energy momentum tensor. Here we use the word "harmonic" to mean the obvious generalization to the semi-Riemannian case of the standard notion of harmonicity in Riemannian geometry. The precise statements will be given below. This elementary result does not seem to be in the mathematical literature, and some of my mathematician friends have urged me to publish it. It is hard for me to believe that it is not somewhere in the physics literature, and I apologize in advance if I have not cited the appropriate references. The method of proof follows the fundamental interpretation of the EIH condition using the theory of distributions due to Souriau in his groundbreaking paper (4). We will also see how these equations are modified in the presence of a Yang–Mills field, using the method of ref. 5.

We begin by recalling two standard formulas in Riemannian geometry.

2. Preliminaries

2.1. Divergence of a Vector Field on a Semi-Riemannian Manifold. Suppose that $\mathbf{g}$ is a semi-Riemann metric on an n-dimensional manifold, M. Then $\mathbf{g}$ determines a density, call it g, which assigns to every n tangent vectors, $\xi_1, \ldots, \xi_n$ at a point the "volume" of the parallelepiped that they span:

$$g\colon \xi_1, \ldots, \xi_n \mapsto |\det(\langle \xi_i, \xi_j \rangle)|^{\frac{1}{2}}. \qquad [1]$$

If X is a vector field on M, and ω is any (smooth) nowhere vanishing density on M, we can define the divergence of X with respect to ω as

$$\mathrm{div}_\omega X \cdot \omega = L_X \omega. \qquad [2]$$

The meaning of this definition is as follows: the right-hand side of **2** is the Lie derivative of the density, ω, with respect to the vector field, X. It is a density. Since the density ω vanishes nowhere, it follows that the density $L_X\omega$ must be some function times ω. This function is defined to be the divergence of X with respect to the density ω. We want a formula for div_g where g is the volume density of $\mathbf{g}$.

For this we form the covariant differential of X with respect to the connection determined by $\mathbf{g}$,

$$\nabla X.$$

It assigns an element of Hom (TM_p, TM_p) to each $p \in M$ according to the rule

$$\xi \mapsto \nabla_\xi X.$$

The trace of this operator is a number, assigned to each point, p, i.e., a function known as the "contraction" of ∇X, so

$$C(\nabla X) := f, \quad f(p) := \mathrm{tr}(\xi \mapsto \nabla_\xi X).$$

The standard formula for the divergence of a vector field with respect to the volume form g of a Riemann metric $\mathbf{g}$ is

$$\mathrm{div} X = C(\nabla X). \qquad [3]$$

2.2. The Lie Derivative of of a Semi-Riemann Metric. The second standard formula we will use is

$$L_V \mathbf{g} = \mathcal{S}\nabla(V\downarrow). \qquad [4]$$

The left-hand side of this equation is the Lie derivative of the metric $\mathbf{g}$ with respect to the vector field V. It is a rule that assigns a symmetric bilinear form to each tangent space. By definition, it is the rule that assigns to any pair of vector fields, X and Y, the value

$$(L_V\mathbf{g})(X, Y) = V\langle X, Y\rangle - \langle [V, X], Y\rangle - \langle X, [V, Y]\rangle.$$

The right-hand side of **4** means the following: $V\downarrow$ denotes the linear differential form whose value at any vector field Y is

$$(V\downarrow)(Y) := \langle V, Y\rangle.$$

In tensor calculus terminology, $\downarrow$ is the "lowering operator," and it commutes with covariant differential. Since $\downarrow$ commutes with ∇, we have

$$\nabla(V\downarrow)(X, Y) = \nabla_X(V\downarrow)(Y) = \langle \nabla_X V, Y\rangle.$$

The symbol $\mathcal{S}$ in **4** denotes symmetric sum, so that the right-hand side of **4** when applied to X, Y is

$$\langle \nabla_X V, Y\rangle + \langle \nabla_Y V, X\rangle.$$

3. The principle of general covariance

See for example refs. 4 or 5. This principle says the following: let the group $\mathcal{G}$ act differentiably on the manifold $\mathcal{M}$. let $\mathcal{O} = \mathcal{G} \cdot \mathbf{g}$ be the $\mathcal{G}$ orbit of $\mathbf{g} \in \mathcal{M}$. Then the principle of general covariance (the EIH condition) applied to a covector μ is that

$$\mu \text{ vanishes when restricted to } T\mathcal{O}_{\mathbf{g}}. \qquad [5]$$

The case we are interested in is where $\mathcal{G}$ is the group of diffeomorphisms of compact support of a manifold M and $\mathcal{M}$ is the space of all semi-Riemann metrics on M. The tangent space $T\mathcal{M}_{\mathbf{g}}$ can be identified with the space

$$\Gamma(S^2(T^*M))$$

of smooth symmetric covariant two tensor fields on M (independently of **g**) and we will be interested in continuous linear functions μ on the subspace $\Gamma_0(S^2(T^*M))$ of tensor fields of compact support. Condition **5** then says that

$$\mu(L_V\mathbf{g}) = 0$$

for all vector fields V of compact support, or, in view of (4) that

$$\mu(\mathcal{S}(\nabla V \downarrow) = 0$$

for all V of compact support. We will be interested in μ provided by the following data:

1. a k dimensional manifold Q and a proper map $f\colon Q \to M$,
2. a smooth section $\mathbf{t}$ of $f^\sharp S^2(TM)$, so $\mathbf{t}$ assigns to each $q \in Q$ an element $\mathbf{t}(q) \in S^2TM_{f(q)}$, and
3. a density ω on Q.

For any section s of S^2T^*M and any $q \in Q$ we can form the "double contraction" $s(q) \bullet \mathbf{t}(q)$ since $s(q)$ and $\mathbf{t}(q)$ take values in dual vector spaces, and since f s proper, if s has compact support then so does the function $q \mapsto s(q) \bullet \mathbf{t}(q)$ on Q. We can then form the integral

$$\mu[s] := \int_Q s(\cdot) \bullet \mathbf{t}(\cdot)\omega. \qquad [6]$$

We observe (and this will be important in what follows) that μ depends on the tensor product $\mathbf{t} \otimes \omega$ as a section of $f^\sharp S^2TM \otimes \mathbf{D}$, where $\mathbf{D}$ denotes the line bundle of densities of Q rather than on the individual factors.

We apply the equation $\mu(\mathcal{S}(\nabla V \downarrow) = 0$ to this μ and to $v = \phi W$, where ϕ is a function of compact support and W a vector field of compact support on M. Since

$$\nabla(\phi W) = d\phi \otimes W + \phi \nabla W$$

and $\mathbf{t}$ is symmetric, this becomes

$$\int_Q \mathbf{t} \bullet (d\phi \otimes W \downarrow + \phi \nabla W \downarrow)\omega = 0. \qquad [7]$$

We first apply this to a ϕ that vanishes on $f(Q)$, so that the term $\phi \nabla W$ vanishes when restricted to Q. We conclude that the "single contraction" $\mathbf{t} \cdot \theta$ must be tangent to $f(Q)$ at all points for all linear differential forms θ and hence that

$$\mathbf{t} = df_* \mathbf{h}$$

for some section $\mathbf{h}$ of $S^2(TQ)$.

Again, let us apply condition **7**, but no longer assume that ϕ vanishes on $f(Q)$. For any vector field Z on Q let us, by abuse of language, write

$$Z\phi \quad \text{for} \quad Zf^*\phi,$$

for any function ϕ on M, write

$$\langle Z, W \rangle \quad \text{for} \quad \langle df_* Z, W \rangle_M,$$

where W is a vector field on M, and

$$\nabla_Z W \quad \text{for} \quad \nabla_{df_* Z} W.$$

Write

$$\mathbf{h} = \sum h^{ij} e_i e_j$$

in terms of a local frame field $e_1, \ldots, e_k$ on Q. Then

$$\mathbf{t} \bullet (\nabla V \downarrow) = \sum h^{ij} \big[e_i(\phi)\langle e_j, W \rangle + \phi \langle \nabla_{e_i} W, e_j \rangle \big].$$

Now

$$\langle \nabla_{e_i} W, e_j \rangle = e_i \langle W, e_j \rangle - \langle W, \nabla_{e_i} e_j \rangle$$

so

$$\mathbf{t} \bullet \nabla V \downarrow = \sum_{ij} \big[h^{ij} e_i(\phi \langle e_j, W \rangle) - \phi \langle W, h^{ij} \nabla_{e_i} e_j \rangle \big].$$

Also,

$$\int_Q \sum h^{ij} e_i \big(\phi \langle e_j, W \rangle\big)\omega = -\int_Q \phi \langle e_j, W \rangle L_{\sum_i h^{ij} e_i} \omega.$$

Let us write

$$z^j = \operatorname{div}_\omega \Big(\sum h^{ij} e_i\Big)$$

so

$$L_{\sum_i h^{ij} e_i} \omega = w^j \omega.$$

If we set

$$Z := \sum z^j e_j,$$

then condition **7** becomes

$$\sum_{ij} h^{ij\,M} \nabla_{e_i} e_j = -Z, \qquad [8]$$

where we have used ${}^M\nabla$ to emphasize that we are using the covariant derivative with respect to the Levi–Civita connection on M, i.e.,

$${}^M\nabla_{e_i} e_j := \nabla_{df_* e_i}(df_* e_j).$$

To understand **8**, suppose that we assume that $\mathbf{h}$ is nondegenerate, and so induces a semi-Riemannian metric $\check{\mathbf{h}}$ on Q, and let us *assume* that ω is the volume form associated with $\check{\mathbf{h}}$. (In all dimensions except $k = 2$ this second assumption is harmless, since we can rescale $\mathbf{h}$ to arrange it to be true.) Let ${}^{\mathbf{h}}\nabla$ denote covariant differential with respect to $\check{\mathbf{h}}$. Let us choose the frame field $e_1, \ldots, e_k$ to be "orthonormal" with respect to $\check{\mathbf{h}}$, i.e.,

$$h^{ij} = \epsilon_j \delta_{ij}, \quad \text{where } \epsilon_j = \pm 1$$

so that

$$\sum_i h^{ij} e_i = \epsilon_j e_j.$$

Then

$$L_{e_j} \omega = C({}^{\mathbf{h}}\nabla e_j)\omega$$

and

$$C({}^{\mathbf{h}}\nabla e_j) = \sum_i \epsilon_i \langle {}^{\mathbf{h}}\nabla_{e_i} e_j, e_i \rangle_{\check{\mathbf{h}}} = -\sum_i \langle e_j, \epsilon_i^{\mathbf{h}} \nabla_{e_i} e_i \rangle_{\check{\mathbf{h}}},$$

so

$$Z = -\sum_j \sum_i \epsilon_j \langle e_j, \epsilon_i{}^{\mathbf{h}}\nabla_{e_i} e_i \rangle_{\check{\mathbf{h}}} e_j$$
$$= -\sum_i \epsilon_i{}^{\mathbf{h}}\nabla_{e_i} e_i = -\sum_{ij} h^{ij\mathbf{h}}\nabla_{e_i} e_j.$$

Given a semi-Riemann metric $\check{\mathbf{h}}$ on Q, a semi-Riemann metric $\mathbf{g}$ on M, the **second fundamental form of a map** $f : Q \to M$, is defined as

$$B_f(X, Y) := {}^{\mathbf{g}}\nabla_{df(X)}(df(Y)) - df({}^{\mathbf{h}}\nabla_X Y). \qquad \mathbf{[9]}$$

Here X and Y are vector fields on Q and $df(X)$ denotes the "vector field along f" that assigns to each $q \in Q$ the vector $df_q(X_q) \in TM_{f(q)}$.

The **tension field** $\tau(f)$ of the map f (relative to a given $\mathbf{g}$ and $\check{\mathbf{h}}$) is the trace of the second fundamental form so

$$\tau(f) = \sum_{ij} h^{ij}\left[{}^{\mathbf{g}}\nabla_{df(e_i)}(df(e_j)) - df({}^{\mathbf{h}}\nabla_{e_i} e_j)\right]$$

in terms of local frame field.

A map f such that $\tau(f) \equiv 0$ will be called **harmonic**. This reduces to the standard definitions if both $\mathbf{g}$ and $\check{\mathbf{h}}$ are positive definite. For these definitions, see for example, ref. 6, pages 12 and 13. We thus see that under the above assumptions about $\mathbf{h}$ and ω:

THEOREM 1. *Condition* **5** *says that f is harmonic relative to* $\mathbf{g}$ *and* $\check{\mathbf{h}}$.

Suppose that we make the further assumption that $\check{\mathbf{h}}$ is the metric induced from $\mathbf{g}$ by the map f. Then

$$df({}^{\mathbf{h}}\nabla_X Y) = ({}^{\mathbf{g}}\nabla_{df(X)} df(Y))^{\tan},$$

the tangential component of ${}^{\mathbf{g}}\nabla_{df(X)} df(Y)$, and hence

$$B_f(X, Y) = ({}^{\mathbf{g}}\nabla_{df(X)} df(Y))^{\mathrm{nor}},$$

the normal component of ${}^{\mathbf{g}}\nabla_{df(X)} df(Y)$. This is just the classical second fundamental form vector of Q regarded as an immersed submanifold of M. Taking its trace gives kH, where H is the mean curvature vector of the immersion. Thus if in addition to the above assumptions we make the assumption that the metric $\check{\mathbf{h}}$ is induced by the map f, then we conclude that **5** says that $H = 0$, i.e., that the immersion f must be a minimal immersion. (This is a standard argument, see for example ref. 6, page 15.) We have recovered the result of Gurses and Gursey (3), which says that the EIH condition, together with the assumption that the energy momentum tensor is the Kalb–Ramond tensor, yields the Nambu string equations.

4. Modifications in the Presence of a Yang–Mills Field

We follow the definition and notations of ref. 5 and 7 so $\mathcal{G}$ is taken as the group of automorphisms of compact support of a principal bundle $P \to M$ and $\mathcal{M}$ consists of all pairs $(\mathbf{g}, \Theta)$ where $\mathbf{g}$ is a semi-Riemann metric on M and Θ is a connection on P. The tangent space to $\mathcal{M}$ at any point can be identified with the space of all pairs (s, A) where s is a covariant symmetric two tensor field on M and A is a one form on M with values in the associated vector bundle $k(P)$ where k is the Lie algebra of structure group K of P. Any $\xi \in \mathrm{aut}_0(P)$, the Lie algebra of $\mathcal{G}$ projects onto a vector field V of compact support on M, and the tangent space to the $\mathcal{G}$-orbit $\mathcal{O}$ at $(\mathbf{g}, \Theta)$ consists of all $(L_V\mathbf{g}, L_\xi\Theta)$.

We want to consider covectors μ associated with a submanifold Q, so we now need the following data:

1. a k dimensional manifold Q and a proper map f: $Q \to M$,
2. a smooth section $\mathbf{t}$ of $f^\sharp S^2(TM)$ so $\mathbf{t}$ assigns to each $q \in Q$ an element $\mathbf{t}(q) \in S^2TM_{f(q)}$,
3. a smooth section J of $f^\sharp(TM \otimes k^*(P)$, and
4. a density ω on Q.

Then we define

$$\mu[(s, A)] := \int_Q \big(s(\cdot) \bullet \mathbf{t}(\cdot) + J(\cdot) \bullet A\big)\omega. \qquad \mathbf{[10]}$$

We will first show that the general covariance condition shows that $\mathbf{t}$ and J are "tangent to Q" as before. For this we first look at ξ which are supported in a neighborhood over which P is trivialized and so Θ can be identified with a k valued one form B on M. Also ξ can be identified with a pair $\xi = (V, v)$ where V is the projected vector field on M and v is a k valued function on M, in which case the associated tangent vector to the orbit is

$$(L_V\mathbf{g}, L_VB + dv + [v, B]).$$

First choose $V = 0$ and $v = \phi w$ where ϕ vanishes on Q so the only term left in the above expression upon restriction to Q is $A = d\phi \otimes w$. The fact that $J \bullet A$ must vanish for all such A implies that $J = df(\mathbf{j})$ where $\mathbf{j}$ is a section of $TQ \otimes k^* f^\sharp P$.

If we take $v = 0$ and $V = \phi W$ where ϕ vanishes on Q we have

$$L_VA = d\iota(V)A + \iota(V)dA = d\iota(V)A = d\phi \otimes \iota(W)A$$

and $d\phi = 0$ on Q so $\mathbf{j} \bullet L_VA = 0$ for this choice of ϕ. The only contribution comes from $L_V\mathbf{g}$ and we conclude that $\mathbf{t} = df_*\mathbf{h}$ as before.

Next take $V = 0$ and v arbitrary. The pairing between the section $\mathbf{j}$ of $T(Q) \otimes k^*$ and the restriction of dv to Q which is a section of $T^*Q \otimes k$ can be done in stages: first contract the TQ component of $\mathbf{j}$ with the T^*Q component of dv. This is just the Lie derivative $L_{\mathbf{j}}v$ of v with respect to the vector field $\mathbf{j}$ with coefficients in k^*. So it is a $k^* \otimes k$ valued function on Q. Then we can apply the evaluation map

$$ev : k^* \otimes k \to \mathbf{R}$$

to obtain a function that we must integrate with respect to ω. We can integrate this by parts to obtain

$$\int_Q \mathbf{j} \bullet dv\ \omega = -\int_Q ev(\mathrm{div}_\omega \mathbf{j} \otimes v)\omega.$$

On the other hand, $[v, B]$ is a section of $T(Q) \otimes k$ and we have the coadjoint action $\mathrm{ad}^\#$ of g on g^* so that we can write

$$\mathbf{j} \bullet [v, B] = -ev(\mathrm{ad}^\#(B)\mathbf{j})$$

where we have extended the $\mathrm{ad}^\#$ notation to include the pairing between T^*Q and TQ. We obtain

$$\mathrm{div}_\omega \mathbf{j} + \mathrm{ad}^\#(B)\mathbf{j} = 0. \qquad \mathbf{[11]}$$

Finally we will consider the case where $v = 0$ and V is arbitrary, and revert to more invariant notation. That is, we will start with a compactly supported vector field V on M and let $\tilde{V}$ be the corresponding horizontal vector field on P relative to the connection Θ. Thus

$$\iota(\tilde{V})\Theta = 0$$

so

$$L_{\tilde{V}}\Theta = \iota(\tilde{V})d\Theta + d\iota(\tilde{V})\Theta = \iota(\tilde{V})d\Theta.$$

Now the curvature, F of the connection Θ is that k-valued two form that assigns to every pair of tangent vectors ξ, η at a point of M the value $d\Theta(\tilde{\xi}, \tilde{\eta})$ where $\tilde{\xi}, \tilde{\eta}$ are their horizontal lifts. We thus see that the term $\mathbf{j} \bullet L_{\tilde{V}}\Theta$ contributes the term $[ev\iota(\mathbf{j})F](V)$ to the integral, in other words the term

$$[ev\iota(\mathbf{j})F] \downarrow$$

must be added to the right-hand side of **8** to give the effect of the Yang–Mills field.

5. Remarks

A number of obvious questions arise. What happens when we consider higher order distributions along Q, rather than just zero-th order energy momentum tensor fields? For the case of curves this was worked out for first order distributions by Souriau (4) with very interesting results. In (8) the interaction between spin and torsion was derived using Cartan soldering forms as a component of the geometrical object, again for the case of curves. It would be interesting to see what this yields for higher dimensional submanifolds.

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