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Hyers–Ulam stability of monomial functional equations on a general domain

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ABSTRACT **In the present paper the Hyers–Ulam stability of monomial functional equations for functions defined on a power-associative, power-symmetric groupoid is proved.**

1. Introduction

The basic problem of the stability of functional equations was proposed by S. Ulam in 1940 in the following form. Suppose that a function f satisfies the so called Cauchy (or additive) functional equation $f(x+y) = f(x) + f(y)$ only approximately. Then does there exist an additive function which approximates f? (Cf. also ref. 1.) In 1941 D. H. Hyers gave the following answer to this question. If B_1 and B_2 are Banach spaces and for a nonnegative real number ε and a function f: $B_1 \to B_2$ we have $\|f(x+y)-f(x)-f(y)\| \le \varepsilon$ $(x, y \in B_1)$, then there exists a unique function a: $B_1 \to B_2$ satisfying $a(x+y) - a(x) - a(y) = 0$ $(x, y \in B_1)$ and $\|f(x) - a(x)\| \le \varepsilon$ $(x \in B_1)$ (2). There are a lot of contributions in the literature of functional equations on this type of stability (cf., e.g., ref. 3). Generalizations of Hyers' result for functions defined on nonassociative and noncommutative structures were investigated, among others, by J. Rätz (4), G. L. Forti (5), and recently, by R. D. Luce, Z. Páles, and P. Volkmann (refs. 6, 7, and 8, respectively). The stability of the Cauchy equation on power-associative, power-symmetric groupoids was proved in ref. 4, while, in refs. 6 and 7, it was shown even without assuming power-associativity. Motivated by these results we study the Hyers–Ulam stability of monomial functional equations on a power-associative, power-symmetric domain.

Our main result reads as follows. If n is a positive integer, $(S, \circ)$ is a power-associative, power-symmetric groupoid, B is a Banach space, f: $S \to B$ is a function, and for a nonnegative real number ε we have

$$\|\Delta_y^n f(x) - n! f(y)\| \le \varepsilon \quad (x, y \in S),$$

then there exists a unique monomial function g: $S \to B$ of degree n such that

$$\|f(x) - g(x)\| \le \frac{1}{n!}\varepsilon \quad (x \in S)$$

holds. [The methods used in the paper also work for a more general range (cf. *Remark 2*), for technical simplicity we consider functions mapping into Banach spaces.] In the special case when S is a linear normed space (or an Abelian group), the result above yields the well-known Hyers–Ulam stability of monomial functional equations (see, e.g., refs. 9–11), furthermore, if $n = 1$, we get the stability of the Cauchy equation (cf. refs. 2, 4, 6, and 7), if $n = 2$, we obtain that of the so called quadratic (or square-norm) functional equation (cf. ref. 12).

2. Notation and Terminology

Throughout the paper $(S, \circ)$ denotes a groupoid, that is, a nonempty set S with a binary operation $\circ$: $S \times S \to S$. The powers of an element $x \in S$ are defined by $x^1 = x$ and for a positive integer m by $x^{m+1} = x^m \circ x$. To simplify the notation, we use the convention $x_1 \circ x_2 \circ x_3 \circ \cdots \circ x_{m-1} \circ x_m = (\cdots((x_1 \circ x_2) \circ x_3) \cdots \circ x_{m-1}) \circ x_m$ for integers $m \ge 3$ and $x_1, \ldots, x_m \in S$.

An operation $\circ$ [or the groupoid $(S, \circ)$] is called *power-associative* if $x^{k+m} = x^k \circ x^m$ for all positive integers k, m and each $x \in S$. (Concerning the role of power-associative operations in ring theory, we refer to ref. 13; such operations in connection with the stability of the Cauchy equation were first studied in ref. 4.) It can be simply verified by induction that in a power-associative groupoid $(S, \circ)$, we have $(x^k)^m = x^{km}$ $(k, m \in \mathbb{N}, x \in S)$. It is easy to see that power-associativity does not imply associativity: e.g., the operation $x \circ y = |x - y|$ on $S = \mathbb{R}_+$ is power-associative but not associative.

We call an operation $\circ$: $S \times S \to S$ *lth-power-symmetric* (or if it is not confusing, simply *power-symmetric*) if $l \ge 2$ is a given integer such that $(x \circ y)^l = x^l \circ y^l$ for all $x, y \in S$ (in the case when $l = 2$, we also use the term *square-symmetric*). Algebraic properties of such operations were considered by several authors; their role in the stability of functional equations was investigated in ref. 4 and, for square-symmetric operations, in refs. 6 and 7. Obviously, commutativity does not follow from power-symmetry: for example, the operation $x \circ y = y$ is associative, lth-power-symmetric for each integer $l \ge 2$ but not commutative on an arbitrary set S with at least two elements (concerning the connection between commutativity and power-symmetry, we refer to ref. 14). Moreover, power-symmetry is a "weaker property" than bisymmetry: the power-associative and commutative operation above $x \circ y = |x - y|$ is also square-symmetric, but not bisymmetric. [An operation on S is called bisymmetric if $(x \circ \bar{x}) \circ (y \circ \bar{y}) = (x \circ y) \circ (\bar{x} \circ \bar{y})$ for all $x, \bar{x}, y, \bar{y} \in S$; cf. ref. 15.]

Finally, we consider the difference operator Δ, which is defined, for a function f mapping from a groupoid $(S, \circ)$ into a linear normed space X, by $\Delta_y^1 f(x) = f(x \circ y) - f(x)$ $(x, y \in S)$ and for $n \in \mathbb{N}$ by $\Delta_y^{n+1} f(x) = \Delta_y^1 \Delta_y^n f(x)$ $(x, y \in S)$. It can be easily verified by induction that, for an arbitrary positive integer n, we have

$$\Delta_y^n f(x) = (-1)^n f(x) + \sum_{j=1}^{n} (-1)^{n-j} \binom{n}{j} f(x \circ \underbrace{y \circ y \circ \cdots \circ y}_{j-times}). \qquad \textbf{[1]}$$

Using this notation we call f a *monomial function of degree n* if $\Delta_y^n f(x) - n! f(y) = 0$ for all $x, y \in S$. (Cf. refs. 15 and 16.)

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3. Results and Proofs

LEMMA 1. *Let $n \geq 1$ and $\lambda \geq 2$ be integers and consider the matrix*

$$A = \begin{pmatrix} \alpha_0^{(0)} & \dots & \alpha_0^{(\lambda n)} \\ \vdots & \ddots & \vdots \\ \alpha_{(\lambda-1)n}^{(0)} & \dots & \alpha_{(\lambda-1)n}^{(\lambda n)} \end{pmatrix}$$

with elements

$$\alpha_i^{(i+j)} = \begin{cases} (-1)^{n-j}\binom{n}{j}, & \text{if } 0 \leq j \leq n \\ 0, & \text{otherwise} \end{cases}$$

for $i = 0, \dots, (\lambda-1)n$, $j = -i, \dots, \lambda n - i$. Let a_i denote the i^{th} row in A ($i = 0, \dots, (\lambda-1)n$) and let $b = (\beta^{(0)} \dots \beta^{(\lambda n)})$, where

$$\beta^{(j)} = \begin{cases} (-1)^{n-\frac{j}{\lambda}}\binom{n}{\frac{j}{\lambda}}, & \text{if } \lambda \mid j, \\ 0, & \text{if } \lambda \nmid j, \end{cases} \quad (j = 0, \dots, \lambda n).$$

Then there exist positive integers $K_0, \dots, K_{(\lambda-1)n}$ such that

$$K_0 a_0 + \dots + K_{(\lambda-1)n} a_{(\lambda-1)n} = b$$

and

$$K_0 + \dots + K_{(\lambda-1)n} = \lambda^n.$$

Proof: Cf. refs. 17 and 18. Q.E.D.

LEMMA 2. *Let $(S, \circ)$ be a power-associative groupoid, X be a linear normed space, n be a positive integer, and $f: S \to X$ be a function. If, for a nonnegative real number ε, we have*

$$\|\Delta_y^n f(x) - n! f(y)\| \leq \varepsilon \quad (x, y \in S), \qquad \textbf{[2]}$$

then, for any positive integer l,

$$\|f(x^l) - l^n f(x)\| \leq \frac{l^n + 1}{n!}\varepsilon \quad (x \in S). \qquad \textbf{[3]}$$

Proof: Let $n, l \in \mathbb{N}$ be given and suppose that $f: S \to X$ satisfies **2**. In the case when $l = 1$, **3** holds trivially. If $l \geq 2$, we define, for $i = 1, \dots, (l-1)n+1$, the functions $F_i: S \to X$ by

$$F_i(z) = \Delta_z^n f(z^i) - n! f(z) \quad (z \in S)$$

and the function $G: S \to X$ by

$$G(z) = \Delta_{z^l}^n f(z) - n! f(z^l) \quad (z \in S).$$

If we replace (x, y) by $(z, z), (z^2, z), \dots, (z^{(l-1)n+1}, z)$ and by (z, z^l) in **2**, we get

$$\|F_i(z)\| \leq \varepsilon \quad (i = 1, \dots, (l-1)n+1,\ z \in S) \qquad \textbf{[4]}$$

and

$$\|G(z)\| \leq \varepsilon \quad (z \in S). \qquad \textbf{[5]}$$

Using **1** and the notation of *Lemma 1* for $\lambda = l$, the functions above can be written in the form

$$F_i(z) = \sum_{j=1}^{ln+1} \alpha_{i-1}^{(j-1)} f(z^j) - n! f(z)$$

for $i = 1, \dots, (l-1)n+1$, $z \in S$ and

$$G(z) = \sum_{j=1}^{ln+1} \beta^{(j-1)} f(z^j) - n! f(z^l) \quad (z \in S).$$

By *Lemma 1*, there exist positive integers $K_0, \dots, K_{(l-1)n}$ with the properties

$$K_0 + \dots + K_{(l-1)n} = l^n$$

and

$$G(z) = K_0 F_1(z) + \dots + K_{(l-1)n} F_{(l-1)n+1}(z) + l^n n! f(z) - n! f(z^l)$$

for all $z \in S$. The combination of these equations with **4** and **5** yields **3**. Q.E.D.

THEOREM. *Let $n \geq 1$ and $l \geq 2$ be given integers, $(S, \circ)$ be a power-associative, lth-power-symmetric groupoid, B be a Banach space, and $f: S \to B$ be a function. If there exists a nonnegative real number ε for which*

$$\|\Delta_y^n f(x) - n! f(y)\| \leq \varepsilon \quad (x, y \in S), \qquad \textbf{[6]}$$

then there exists a unique monomial function $g: S \to B$ of degree n such that

$$\|f(x) - g(x)\| \leq \frac{1}{n!}\varepsilon \quad (x \in S). \qquad \textbf{[7]}$$

Proof: Let $n, l \in \mathbb{N}$, $l \geq 2$ be given and let $f: S \to B$ satisfy **6**. By *Lemma 2*, we have

$$\|f(x^l) - l^n f(x)\| \leq \frac{l^n + 1}{n!}\varepsilon \quad (x \in S).$$

Using the triangle inequality, we get, for each $m \in \mathbb{N}$,

$$\begin{aligned} &\left\| f(x) - \frac{1}{l^{mn}} f(x^{l^m}) \right\| \\ &\leq \left\| f(x) - \frac{1}{l^n} f(x^l) \right\| + \left\| \frac{1}{l^n} f(x^l) - \frac{1}{l^{2n}} f(x^{l^2}) \right\| \\ &\quad + \dots + \left\| \frac{1}{l^{(m-1)n}} f(x^{l^{m-1}}) - \frac{1}{l^{mn}} f(x^{l^m}) \right\| \\ &\leq \frac{1}{l^n}\|l^n f(x) - f(x^l)\| + \frac{1}{l^{2n}}\|l^n f(x^l) - f(x^{l^2})\| \\ &\quad + \dots + \frac{1}{l^{mn}}\|l^n f(x^{l^{m-1}}) - f(x^{l^m})\| \\ &\leq \sum_{j=1}^{m} \frac{1}{l^{jn}} \frac{l^n + 1}{n!}\varepsilon \quad (x \in S). \end{aligned} \qquad \textbf{[8]}$$

Let us define the functions $g_m: S \to B$ by

$$g_m(x) = \frac{1}{l^{mn}} f(x^{l^m}) \quad (x \in S, m \in \mathbb{N}). \qquad \textbf{[9]}$$

Since

$$\sum_{j=1}^{\infty} \frac{1}{l^{jn}} = \frac{1}{l^n - 1},$$

we have

$$\|g_m(x) - g_k(x)\| \leq \frac{1}{l^{mn}} \frac{1}{l^n - 1} \frac{l^n + 1}{n!}\varepsilon \quad (x \in S)$$

for $k, m \in \mathbb{N}$, $k > m$. Thus, $(g_m(x))$ is a Cauchy sequence for each fixed $x \in S$. Because of the completeness of B, there exists the function $g: S \to B$

$$g(x) = \lim_{m \to \infty} g_m(x) \quad (x \in S).$$

It can be shown by induction and by using power-associativity that the lth-power-symmetry yields

$$(x \circ y)^{l^m} = x^{l^m} \circ y^{l^m} \quad (x, y \in S,\ m \in \mathbb{N}). \qquad \textbf{[10]}$$

Property **6** implies

$$\|\Delta^n_{y^{l^m}} f(x^{l^m}) - n!f(y^{l^m})\| \le \varepsilon \quad (x, y \in S,\ m \in \mathbb{N}).$$

Dividing this inequality by l^{mn}, letting m approach infinity, and using **1** and **10**, we obtain

$$\Delta^n_y g(x) - n!g(y) = 0 \quad (x, y \in S),$$

that is, g is a monomial function of degree n. Furthermore, **8** gives

$$\left\| f(x) - \frac{f(x^{l^m})}{l^{mn}} \right\| \le \frac{1}{l^n - 1}\frac{l^n + 1}{n!}\varepsilon \quad (x \in S,\ m \in \mathbb{N}),$$

thus,

$$\|f(x) - g(x)\| \le \frac{1}{l^n - 1}\frac{l^n + 1}{n!}\varepsilon \quad (x \in S). \qquad \textbf{[11]}$$

Eq. **10** implies that $(S, \circ)$ is, for an arbitrary positive integer r and $s = l^r$, also sth-power-symmetric. Therefore, writing l^r instead of l, the proof can be completed in a similar way as above. In this case we define functions $g_{m,r}$: $S \to B$, similarly to those in **9**, by

$$g_{m,r}(x) = \frac{1}{(l^r)^{mn}} f(x^{(l^r)^m}) \quad (x \in S, m \in \mathbb{N}),$$

and, for the function $g^{(r)}$: $S \to B$ defined by

$$g^{(r)}(x) = \lim_{m\to\infty} g_{m,r}(x) \quad (x \in S),$$

we get the inequality

$$\|f(x) - g^{(r)}(x)\| \le \frac{1}{(l^r)^n - 1}\frac{(l^r)^n + 1}{n!}\varepsilon \quad (x \in S),$$

instead of **11**. Obviously, $(g_{m,r}(x))$ is a subsequence of $(g_m(x))$ for each fixed $x \in S$, so we have $g(x) = g^{(r)}(x)$ for all $x \in S$ and $r \in \mathbb{N}$, which implies **7**.

Finally, we prove the uniqueness of g. Let us suppose that there exists a monomial function $\tilde{g}$: $S \to B$ of degree n which is different from g and satisfies

$$\|f(x) - \tilde{g}(x)\| \le c\varepsilon \quad (x \in S),$$

where $c \in \mathbb{R}$ is a constant. Then, using the triangle inequality, we get

$$\|g(x) - \tilde{g}(x)\| \le \left(\frac{1}{n!} + c\right)\varepsilon \quad (x \in S). \qquad \textbf{[12]}$$

Since g and $\tilde{g}$ are different, there exists an $x_0 \in S$ for which $g(x_0) \ne \tilde{g}(x_0)$. Thus, there exists an $l \in \mathbb{N}$ such that

$$l^n > \left(\frac{1}{n!} + c\right)\frac{\varepsilon}{\|g(x_0) - \tilde{g}(x_0)\|}. \qquad \textbf{[13]}$$

Lemma 2 yields $g(x_0^l) = l^n g(x_0)$ and $\tilde{g}(x_0^l) = l^n \tilde{g}(x_0)$, therefore, **13** implies

$$\|g(x_0^l) - \tilde{g}(x_0^l)\| > \left(\frac{1}{n!} + c\right)\varepsilon,$$

which contradicts **12**. Q.E.D.

Remark 1: If $(S, \circ)$ is a semigroup (that is, $\circ$ is associative) then formula **1** can be written in the form

$$\Delta^n_y f(x) = (-1)^n f(x) + \sum_{j=1}^{n} (-1)^n \binom{n}{j} f(x \circ y^j) \quad (x, y \in S).$$

Obviously, this well known equation is not valid without assuming associativity. This fact gives another possible generalization of the concept of monomial functions for nonassociative groupoids. For a positive integer n and a function f mapping from a groupoid $(S, \circ)$ into a linear normed space X, we introduce

$$\Delta^{*n}_y f(x) = (-1)^n f(x) + \sum_{j=1}^{n} (-1)^n \binom{n}{j} f(x \circ y^j) \quad (x, y \in S),$$

and we call f a **-monomial function of degree* n if $\Delta^{*n}_y f(x) - n!f(y) = 0$ for all $x, y \in S$. It can be shown in a similar way as above that our Theorem also holds for the operator Δ^* and for *-monomial functions. Moreover, in this case the lth-power-symmetry of S can be replaced by the weaker assumption (cf. ref. 4) that, for all positive integers m,

$$f((x \circ y)^{l^m}) = f(x^{l^m} \circ y^{l^m}) \quad (x, y \in S).$$

Remark 2: It is easy to see that with an appropriate modification of the methods used in the proofs above our main result can be generalized in the following form. Let $n \ge 1$ and $l \ge 2$ be integers, $(S, \circ)$ be a power-associative, lth-power-symmetric groupoid, X be a sequentially complete Hausdorff topological vector space over the field of the rationals and let f: $S \to X$ be a function. If, for a nonempty, $\mathbb{Q}$-konvex, $\mathbb{Q}$-balanced, sequentially closed, bounded set $V \subseteq X$, we have

$$\Delta^n_y f(x) - n!f(y) \in V \quad (x, y \in S),$$

then there exists a unique monomial function g: $S \to X$ of degree n such that

$$f(x) - g(x) \in \frac{1}{n!}V \quad (x \in S).$$

(Concerning the terminology used here, cf., e.g., refs. 19 or 4 and the references given there.)

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1. Ulam, S. (1964) *Problems in Modern Mathematics* (Wiley, New York).
2. Hyers, D. H. (1941) *Proc. Natl. Acad. Sci. USA* **27**, 222–224.
3. Hyers, D. H., Isac, G. I. & Rassias, T. M. (1998) *Stability of Functional Equations in Several Variables* (Birkhäuser, Basel).
4. Rätz, J. (1980) in *General Inequalities 2*, ed. Walter, W. (Birkhäuser, Basel), pp. 233–251.
5. Forti, G. L. (1980) *Stochastica* **4**, 23–30.
6. Páles, Z., Volkmann, P. & Luce, R. D. (1998) *Proc. Natl. Acad. Sci. USA* **95**, 12772–12775.
7. Volkmann, P. (1998) *Proceedings of the International Conference Numbers, Functions, Equations '98*, Leaflets in Mathematics, ed. Páles, Z. (Janus Pannonius University, Pécs), pp. 150–151.
8. Páles, Z. (1998) *Aequationes Math.* **56**, 222–232.
9. Hyers, D. H. (1961) *Pacific J. Math.* **11**, 592–602.
10. Székelyhidi, L. (1981) *C. R. Math. Rep. Acad. Sci. Canada* **3**, 63–67.
11. Badora, R., Páles, Z. & Székelyhidi, L. (1999) *Aequationes Math.*, in press.
12. Cholewa, P. W. (1984) *Aequationes Math.* **27**, 76–86.
13. Albert, A. A. (1948) *Trans. Am. Math. Soc.* **64**, 552–593.
14. Johnson, E. C., Outcalt, D. L. & Yagub, A. (1968) *Am. Math. Monthly* **75**, 288–289.
15. Aczél, J. (1966) *Lectures on Functional Equations and Their Applications* (Academic, New York).
16. Székelyhidi, L. (1991) *Convolution Type Functional Equations on Topological Abelian Groups* (World Scientific, Singapore).
17. Gilányi, A. (1997) *Aequationes Math.* **54**, 289–307.
18. Gilányi, A. (1997) *Publ. Math. Debrecen* **51**, 343–361.
19. Bourbaki, N. (1953) *Espaces Vectoriels Topologiques* (Hermann, Paris).