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Mathematics

# The curvature of a Hilbert module over $\mathbb{C}[z_1,\ldots,z_d]$

**(curvature invariant / Gauss–Bonnet–Chern formula / multivariable operator theory)**

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**ABSTRACT** **A notion of curvature is introduced in multivariable operator theory. The curvature invariant of a Hilbert module over $\mathbb{C}[z_1, \ldots, z_d]$ is a nonnegative real number which has significant extremal properties, which tends to be an integer, and which is hard to compute directly. It is shown that for graded Hilbert modules, the curvature agrees with the Euler characteristic of a certain finitely generated algebraic module over the appropriate polynomial ring. This result is a higher dimensional operator-theoretic counterpart of the Gauss–Bonnet formula which expresses the average Gaussian curvature of a compact oriented Riemann surface as the alternating sum of the Betti numbers of the surface, and it solves the problem of calculating the curvature of graded Hilbert modules. The proof of that result is based on an asymptotic formula which expresses the curvature of a Hilbert module in terms that allow its comparison to a corresponding asymptotic expression for the Euler characteristic.**

## 1. Introduction

By a Hilbert module over the polynomial algebra $A = \mathbb{C}[z_1, \ldots, z_d]$ of dimension $d$, we mean a Hilbert space $H$ together with a commuting $d$-tuple of bounded operators $T_1, \ldots, T_d$ acting on $H$. The $A$-module structure is defined on $H$ by

$$f \cdot \xi = f(T_1, \ldots, T_d)\xi, \qquad f \in A, \quad \xi \in H.$$

A Hilbert module $H$ is said to be *contractive* if for every $\xi_1, \ldots, \xi_d \in H$ we have

$$\|T_1\xi_1 + \cdots + T_d\xi_d\|^2 \leq \|\xi_1\|^2 + \cdots + \|\xi_d\|^2, \qquad \textbf{[1]}$$

and the *rank* of a contractive Hilbert module is defined as the dimension of the range of the defect operator $\Delta = (\mathbf{1} - T_1T_1^* - \cdots - T_dT_d^*)^{1/2}$. The rank of $H$ is written rank$(H)$.

This note concerns finite rank contractive Hilbert modules. These are Hilbert space counterparts of finitely generated modules over polynomial rings. We introduce a numerical invariant $K(H)$ for such Hilbert modules $H$, called the curvature invariant of $H$. $K(H)$ is a real number satisfying $0 \leq K(H) \leq \text{rank}(H)$. We give almost no proofs here, but full details can be found in ref. 1, which is available at the author's web site (http://math.berkeley.edu/~arveson.)

The curvature invariant carries key information about the operator theory and structure of Hilbert modules. For example, it is sensitive enough to detect exactly when $H$ is a "free Hilbert module of finite rank" in the sense that $H$ is unitarily equivalent to a finite direct sum of copies of the free Hilbert module $H^2(\mathbb{C}^d)$ (to be described more pecisely below) if, and only if, $K(H) = \text{rank}(H)$. This is the first extremal property of $K(H)$.

The second extremal property of $K(H)$ is related to the structure of the invariant subspaces of the rank-one free Hilbert module $H^2$ in the following sense: a closed submodule $M \subseteq H^2(\mathbb{C}^d)$ is associated with an "inner sequence" iff

$$K(H^2(\mathbb{C}^d)/M) = 0,$$

$H^2(\mathbb{C}^d)/M$ denoting the quotient of contractive Hilbert modules.

To make use of these extremal properties for a given finite rank contractive Hilbert module $H$, one must know the value of $K(H)$. But direct computation appears to be difficult for most of the natural examples, and in the few cases where the computations can be explicitly carried out, the curvature turns out, to be an integer. Thus we were led to ask if the curvature invariant can be expressed in terms of some other invariant which is (*i*) obviously an integer and (*ii*) easier to calculate.

We establish such a formula in *Theorem B* below, which applies to Hilbert $A$-modules $H$ which are "graded" in the sense that their operator $d$-tuple is circularly symmetric. This formula is analogous to the Chern–Fenchel generalization of the Gauss–Bonnet formula to compact oriented Riemannian $2n$-manifolds (2); it asserts that $K(H)$ agrees with the Euler characteristic of a certain finitely generated *algebraic* module that is associated with it in a natural way. Because the Euler characteristic of a finitely generated module over $\mathbb{C}[z_1, \ldots, z_d]$ is relatively easy to compute by using conventional algebraic methods, the problem of calculating the curvature can be considered solved for graded Hilbert modules.

The problem of calculating the curvature of *ungraded* finite rank Hilbert modules remains open.

Our proof of *Theorem B* amounts to establishing appropriate asymptotic formulas for both the curvature and the Euler characteristic of arbitrary (i.e., perhaps ungraded) finite rank contractive Hilbert $A$-modules (*Theorems C* and *D* below). *Theorem D* is proved by showing that $K(H)$ is actually the trace of a certain self-adjoint trace class operator (the curvature *operator* of $H$), and an asymptotic formula for the trace of that operator is developed.

The problem of determining more precisely the distribution of eigenvalues of this curvature operator is an attractive one about which we as yet know very little.

## 2. Basic Definitions

We now describe these results in more detail, beginning with the definition of the curvature invariant $K(H)$. Let $T_1, \ldots, T_d \in \mathcal{B}(H)$ be the defining $d$-tuple of a contractive finite rank Hilbert $A$-module $H$. With every point $z = (z_1, \ldots, z_d)$ in complex $d$-space $\mathbb{C}^d$, we associate the operator

$$T(z) = z_1T_1 + \cdots + z_dT_d \in \mathcal{B}(H).$$

Because of the inequality **[1]**, we have

$$\|T(z)\| \leq |z| = \left(|z_1|^2 + \cdots + |z_d|^2\right)^{1/2}, \qquad z \in \mathbb{C}^d,$$

and in particular for every point $z$ in the open unit ball $B_d = \{z \in \mathbb{C}^d : |z| < 1\}$, we have $\|T(z)\| < 1$, and hence $\mathbf{1} - T(z)$ is invertible. Thus for every $z \in B_d$, we can define a positive operator $F(z)$ which acts on the finite dimensional Hilbert space $\Delta H$ as follows,

$$F(z)\xi = \Delta(\mathbf{1} - T(z)^*)^{-1}(\mathbf{1} - T(z))^{-1}\Delta\xi, \qquad \xi \in \Delta H.$$

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To define the curvature invariant, we require the boundary values of the function $z \in B_d \mapsto \operatorname{trace} F(z)$. These do not exist in a conventional sense because in all significant cases this function is unbounded. However, "renormalized" boundary values do exist almost everywhere relative to the natural rotation-invariant probability measure $\sigma$ defined on the unit sphere $\partial B_d$, in the following sense.

THEOREM A. *For $\sigma$-almost every $\zeta \in \partial B_d$, the limit*

$$K_0(\zeta) = \lim_{r\to 1}(1 - r^2)\operatorname{trace} F(r\zeta)$$

*exists and satisfies* $0 \leqslant K_0(\zeta) \leqslant \operatorname{rank}(H)$.

The curvature invariant of a finite rank contractive Hilbert module is defined as follows:

$$K(H) = \int_{\partial B_d} K_0(\zeta)\, d\sigma(\zeta), \qquad [2]$$

and we have $0 \leqslant K(H) \leqslant \operatorname{rank}(H)$.

## 3. Extremal Properties of $K(H)$

In a recent paper (3), the author developed the theory of a particular Hilbert module over $A = \mathbb{C}[z_1, \ldots, z_d]$, called $H^2(\mathbb{C}^d)$ or more simply $H^2$ when there is no possibility of confusion over the dimension. There is a natural inner product on $A$ and $H^2$ is obtained by completing this inner product space. The action of polynomials on $H^2$ extends the natural multiplication of $A$. The associated $d$-tuple of operators $S_1, \ldots, S_d$ acting on $H^2$ is called the $d$-shift. The $d$-shift satisfies **[1]** but does not satisfy the natural counterpart of von Neumann's inequality for the unit ball $B_d$. Indeed, there is no constant $K$ satisfying

$$\|f(S_1, \ldots, S_d)\| \leqslant K \sup_{z\in B_d} |f(z)|$$

for every polynomial $f \in A$ (see ref. 3, Theorem 3.3). It follows from this fact that the $d$-shift is not a subnormal $d$-tuple.

Nevertheless, finite direct sums of copies of the $d$-shift serve as an effective substitute for free modules in the algebraic theory of finitely generated modules over polynomial rings. To discuss this we require the notion of a *pure* Hilbert $A$-module. Let $H$ be a contractive Hilbert $A$-module with canonical operators $T_1, \ldots, T_d$ and consider the linear map of operators $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ defined by

$$\phi(A) = T_1AT_1^* + \cdots + T_dAT_d^*, \qquad A \in \mathcal{B}(H).$$

$\phi$ is completely positive map satisfying $\|\phi\| = \|\phi(\mathbf{1})\| \leqslant 1$. It follows that the sequence of positive operators $\phi^n(\mathbf{1})$, $n = 0, 1, 2, \ldots$ is decreasing and hence the limit $\lim_{n\to\infty} \phi^n(\mathbf{1})$ exists relative to the strong operator topology as a positive contraction. The Hilbert $A$-module $H$ is said to be *pure* if

$$\lim_{n\to\infty} \phi^n(\mathbf{1}) = 0.$$

Let $r$ be a positve integer. By a *free Hilbert module of rank r* we mean a direct sum $F = r \cdot H^2 = H^2 \oplus \cdots \oplus H^2$ of $r$ copies of the Hilbert module $H^2 = H^2(\mathbb{C}^d)$, where the module structure is defined by the $d$-shift, i.e.,

$$f \cdot (g_1, \ldots, g_d) = (f \cdot g_1, \ldots, f \cdot g_r),$$
$$f \in A, \quad g_1, \ldots, g_d \in H^2.$$

This definition has a natural and obvious interpretation when $r = \infty$, and $\infty \cdot H^2$ is called the *free Hilbert module of infinite rank*. Notice that if we are given any closed submodule $M$ of a (contractive) Hilbert $A$-module $H$, then the quotient $H/M$ is naturally a Hilbert space and it inherits the structure of a contractive Hilbert module from that of $H$. In general, one has $\operatorname{rank}(H/M) \leqslant \operatorname{rank}(H)$. Every such quotient $F/M$ of a finite rank *free* Hilbert $A$-module $F$ by a closed submodule $M \subseteq F$ is a *pure* Hilbert $A$-module of rank at most $r = \operatorname{rank}(F)$. Conversely, the basic result on the dilation theory of $d$-contractions (see ref. 3, theorem 8.5 for example) implies the following characterization of pure Hilbert $A$-modules.

DILATION THEOREM. *Every pure contractive Hilbert $A$-module $H$ of finite or infinite rank $r$ is unitarily equivalent to a quotient $F/M$, where $F = r \cdot H^2$ is the free Hilbert module of rank $r$ and $M \subseteq F$ is a closed submodule.*

We conclude that to understand finite rank Hilbert $A$-modules, one should focus attention on free Hilbert modules of finite rank, their closed submodules and quotients. In particular, how does one determine whether a given Hilbert module is free? What are the properties of submodules of free Hilbert modules? In the following discussion, we relate these questions to extremal properties of the curvature invariant. These are fundamental results in the subject.

FIRST EXTREMAL PROPERTY. *Let $H$ be a pure Hilbert $A$-module of finite rank $r$. Then $H$ is unitarily equivalent to a free Hilbert $A$-module iff $K(H) = r$, and in that case $H$ is equivalent to $r \cdot H^2$.*

To discuss the second extremal property of $K(H)$, we consider closed submodules $M \subseteq H^2$ of the free Hilbert $A$-module of rank 1. Such a submodule $M$ is itself a contractive Hilbert module, it is pure, but it is frequently of infinite rank (see *Theorem E* below). In any case, *the Dilation Theorem* readily implies that there is a (finite or infinite) sequence of bounded holomorphic functions $\phi_1, \phi_2, \ldots$ on the open unit ball $B_d$ of $\mathbb{C}^d$ whose multipliers define bounded operators on $H^2(\mathbb{C}^d)$, which in fact satisfy

$$M_{\phi_1}M_{\phi_1}^* + M_{\phi_2}M_{\phi_2}^* + \cdots = P_M, \qquad [3]$$

$P_M$ denoting the orthogonal projection of $H^2$ onto the subspace $M$. Formula **[3]** makes the following assertion. If we consider the row operator defined by the sequence $R = (M_{\phi_1}, M_{\phi_2}, \ldots)$ as an operator from a direct sum of copies of $H^2$ into $H^2$, then $R$ is a partially isometric homomorphism of Hilbert $A$-modules whose range is precisely the given invariant subspace $M \subseteq H^2$.

The sequence $(\phi_n)$ of multipliers appearing in **[3]** is certainly not uniquely determined by the invariant subspace $M$, but if it is carefully chosen, it is unique up to a "unitary rotation matrix." We also emphasize that it is usually impossible to find a single multiplier $\phi$ or even a finite sequence of multipliers $\phi_1, \ldots, \phi_r$ which satisfies **[3]**. Indeed, the minimum length $r$ of such a sequence turns out to be the rank of $M$ as a Hilbert $A$-module, and as we have pointed out above, the latter is usually infinite. Thus, while such sequences exist for arbitrary closed submodules $M \subseteq H^2$, they are typically *infinite* sequences.

In dimension $d = 1$ there is a single multiplier $\phi$ satisfying $M_\phi M_\phi^* = P_M$, and **[3]** reduces essentially to the assertion of Beurling's theorem: every invariant subspace of $H^2(\mathbb{C})$ is the range of an isometry which commutes with the unilateral shift. In turn, the multiplier $\phi$ is an inner function, that is, a bounded analytic function defined on the open unit disk whose boundary values satisfy $|\phi(e^{i\theta})| = 1$ almost everywhere $d\theta$ on the unit circle. This description of the boundary values of the multiplier $\phi$ is easily established by using subnormality: the unilateral shift is the restriction of a unitary operator (the bilateral shift) to an irreducible invariant subspace.

To what extent does such a function-theoretic description of invariant subspaces hold in higher dimensions? Alexandrov has shown that inner functions exist for the unit ball in dimension $d \geqslant 2$ (see ref. 4; also see refs. 5–7), but as **[3]** and the remarks following it imply, one cannot hope to achieve **[3]** with a single multiplier or even with a finite sequence of multipliers. Thus, we require a notion of innerness that can be applied to finite or infinite sequences.

Now Eq. **[3]** implies that the sequence of functions $(\phi_n)$ has the following property on the open unit ball

$$|\phi_1(z)|^2 + |\phi_2(z)|^2 + \cdots \leq 1, \qquad z \in B_d.$$

In particular, each $\phi_n$ is a bounded holomorphic function on $B_d$ and thus it has radial limits $\tilde{\phi}(\zeta)$ for $\sigma$-almost every point $\zeta$ on the sphere $\partial B_d$ (8). The preceding inequality implies that

$$|\tilde{\phi}_1(\zeta)|^2 + |\tilde{\phi}_2(\zeta)|^2 + \cdots \leq 1$$

holds almost everywhere $d\sigma(\zeta)$ on $\partial B_d$. $(\phi_n)$ is called an *inner sequence* if equality holds

$$|\tilde{\phi}_1(\zeta)|^2 + |\tilde{\phi}_2(\zeta)|^2 + \cdots = 1 \qquad \textbf{[4]}$$

for $\sigma$-almost every point $\zeta \in \partial B_d$.

We have already pointed out above that in dimension $d \geq 2$, the $d$-shift is not a subnormal $d$-tuple. Hence the argument alluded to above for establishing **[4]** in one dimension breaks down in a fundamental way, and the issue of whether or not an invariant subspace can be associated with an inner sequence becomes significant.

Second Extremal Property. Let $M$ be a closed submodule of the rank-one free Hilbert module $H^2(\mathbb{C}^d)$, and let $(\phi_n)$ be any sequence of holomorphic functions in the unit ball of $B_d$ whose multipliers satisfy **[3]**.

Then $(\phi_n)$ is an inner sequence iff $K(H^2(\mathbb{C}^d)/M) = 0$.

We will make use of these extremal properties at several points in the discussion to follow.

## 4. Euler Characteristic, Graded Hilbert Modules, Principal Formula

We have pointed out in the *Introduction* that the curvature invariant is not easily calculated directly, nor is it obviously an integer in even the simplest cases. In this section we introduce another invariant which is an integer by virtue of its definition, and which can be readily computed for many examples by using the basic tools of commutative algebra (cf. 9, 10).

Throughout this section, $H$ will denote a Hilbert module over $A = \mathbb{C}[z_1, \ldots, z_d]$ of *finite rank* $r$. We will work not with $H$ itself but with the following linear submanifold of $H$

$$M_H = \text{span}\{f \cdot \Delta\xi : f \in A, \xi \in H\}.$$

The definition and basic properties of the Euler characteristic are independent of any topology associated with the Hilbert space $H$, and depend solely on the linear algebra of $M_H$. If we choose any linear basis $\zeta_1, \ldots, \zeta_r$ for the $r$-dimensional space $\Delta H$, then every element of $M_H$ is a finite sum of vectors of the form

$$f_1 \cdot \zeta_1 + \cdots + f_r \cdot \zeta_r,$$

where $f_1, \ldots, f_r$ are polynomials in $A$. Thus, $M_H$ is a finitely generated $A$-module. By Hilbert's Syzygy Theorem (refs. 11 and 12, or ref. 9 or 10), $M_H$ has finite free resolutions in the category of finitely generated $A$-modules

$$0 \to F_n \to \cdots \to F_2 \to F_1 \to M_H \to 0, \qquad \textbf{[5]}$$

each $F_k$ being a sum of a finite number $\beta_k$ of copies of the rank-one module $A$. The sequence **[5]** is an exact sequence of $A$-modules, and is called a finite free resolution of $M_H$. Free resolutions are not unique, but there do exist free resolutions whose length $n$ does not exceed the dimension $d$ of the underlying polynomial ring $A = \mathbb{C}[z_1, \ldots, z_d]$.

The alternating sum of the ranks $\beta_1 - \beta_2 + \beta_3 - + \cdots$ does not depend on the particular free resolution of $M_H$, and we define the *Euler characteristic* of $H$ by

$$\chi(H) = \sum_{k=1}^{n} (-1)^{k+1} \beta_k. \qquad \textbf{[6]}$$

For algebraic reasons one always has $\chi(H) \geq 0$.

We now describe the principal formula which evaluates the curvature invariant of *graded* Hilbert modules. By a *graded Hilbert space* we mean a pair $H, \Gamma$ where $H$ is a (separable) Hilbert space and $\Gamma$ is a strongly continuous unitary representation of the circle group $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. $\Gamma$ is called the *gauge group* of $H$. For each $n \in \mathbb{Z}$ we can define the associated spectral subspace

$$H_n = \{\xi \in H : \Gamma(\lambda)\xi = \lambda^n \xi, \quad \lambda \in \mathbb{T}\},$$

and thus $H$ can be regarded as a $\mathbb{Z}$-graded Hilbert space in the sense that we have an orthogonal decomposition

$$H = \cdots \oplus H_{-1} \oplus H_0 \oplus H_1 \oplus \cdots. \qquad \textbf{[7]}$$

Conversely, if we are given a $\mathbb{Z}$-graded Hilbert space **[7]**, then we can construct an associated gauge group simply by defining $\Gamma(\lambda)$ on the $n$th summand to be $\xi \mapsto \lambda^n \xi$ and then extending by linearity. Algebraists tend to prefer the description in terms of $\mathbb{Z}$-gradings because that formulation works for arbitrary scalar fields, but for our purposes it is more convenient to think in terms of gauge groups.

A graded Hilbert $A$-module is a contractive Hilbert module $H$, which is also a graded Hilbert space such that the gauge group $\Gamma$ is related as follows to the operators $T_1, \ldots, T_d$ associated with the module structure

$$\Gamma(\lambda) T_k \Gamma(\lambda)^{-1} = \lambda T_k, \qquad \lambda \in \mathbb{T}, \quad k = 1, \ldots, d.$$

Thus, graded Hilbert modules are precisely those whose underlying operator $d$-tuple is circularly symmetric. One can easily verify that the previous formula is satisfied if and only if the $T_k$ shift the terms $H_n$ of the associated $\mathbb{Z}$-grading **[7]** as follows

$$T_k H_n \subseteq H_{n+1}, \qquad n \in \mathbb{Z}, \quad k = 1, \ldots, d.$$

There are many natural examples of graded Hilbert modules, and in fact there is a "graded" variation of the *Dilation Theorem* above that characterizes them in much the same way as finite rank pure Hilbert $A$-modules are characterized.

We view the following result as an analogue of the the Gauss–Bonnet–Chern formula for graded Hilbert $A$-modules $H$. It implies that $K(H)$ is an integer in such cases, and because there are well-developed algebraic tools for computing the Euler characteristic of finitely generated modules over polynomial rings, it can also be viewed as a solution of the problem of calculating the curvature invariant of graded Hilbert modules.

Theorem B. *For every pure finite rank graded Hilbert $A$-module $H$,*

$$K(H) = \chi(H).$$

## 5. Something About the Proof: Asymptotics of $K(H)$ and $\chi(H)$

*Theorem B* is proved by establishing asymptotic formulas for both the curvature invariant and Euler characteristic of arbitrary (i.e., perhaps ungraded) finite rank Hilbert $A$-modules. We briefly discuss these results in this section and say something about the underlying curvature operator.

Let $H$ be a finite rank Hilbert $A$-module with canonical operators $T_1, \ldots, T_d$, and let $\phi$ be the corresponding completely positive map on $\mathcal{B}(H)$, $\phi(A) = T_1 A T_1^* + \cdots + T_d A T_d^*$,

$A \in \mathcal{B}(H)$. Because $\mathbf{1} - \phi(\mathbf{1}) = \Delta^2$ is a finite rank operator, $\phi^n(\mathbf{1}) - \phi^{n+1}(\mathbf{1}) = \phi^n(\Delta^2)$ is finite rank for every $n = 0, 1, 2, \ldots$ and hence so is

$$\mathbf{1} - \phi^{n+1}(\mathbf{1}) = \sum_{k=0}^{n} \phi^k(\Delta^2).$$

We have the following asymptotic formulas, the principal one being *Theorem D*.

THEOREM C. *For every finite rank Hilbert A-module H,*

$$\chi(H) = d! \lim_{n\to\infty} \frac{\operatorname{rank}(\mathbf{1} - \phi^n(\mathbf{1}))}{n^d}.$$

THEOREM D. *For every finite rank Hilbert A-module H, there is a naturally associated self-adjoint trace class operator* $d\Gamma$, *and*

$$K(H) = \operatorname{trace} d\Gamma = d! \lim_{n\to\infty} \frac{\operatorname{trace}(\mathbf{1} - \phi^n(\mathbf{1}))}{n^d}.$$

We will not define the operator $d\Gamma$ here, but we do offer the following comment. While $K(H) = \operatorname{trace} d\Gamma$ is always a nonnegative real number, $d\Gamma$ is never a positive operator. Indeed, $d\Gamma$ can be expressed as a difference $E - P$ where $E$ is a projection whose dimension is $\operatorname{rank}(H)$ and $P$ is a positive operator satisfying $\operatorname{trace} P \leq \operatorname{rank}(H)$. See ref. 1 for more detail.

Notice that because $\mathbf{1} - \phi^n(\mathbf{1})$ is a positive operator of norm at most 1 for every $n = 1, 2, \ldots$, we have

$$\operatorname{trace}(\mathbf{1} - \phi^n(\mathbf{1})) \leq \operatorname{rank}(\mathbf{1} - \phi^n(\mathbf{1})),$$

and hence *Theorems C* and *D* together imply that, for (perhaps ungraded) finite rank Hilbert modules $H$, we have the general inequality

$$0 \leq K(H) \leq \chi(H). \quad \textbf{[8]}$$

This inequality has significant implications (see *Corollaries 1* and *2* in the following section).

Finally, we point out that the following upper bound on the Euler number is a straightforward consequence of *Theorem C*:

$$\chi(H) \leq \operatorname{rank}(H). \quad \textbf{[9]}$$

## 6. Discussion: Problems and Applications

*Theorem B* implies that $K(H)$ is an integer for every pure finite rank Hilbert module $H$ which is *graded*. But we do not know if this is so for ungraded modules.

*Problem 1:* Is $K(H)$ an integer for every pure finite rank Hilbert $A$-module $H$?

We conclude with a discussion of issues and problems related to *Problem 1*. Consider the simplest nontrivial case $\operatorname{rank}(H) = 1$. The *Dilation Theorem* implies that $H$ can be realized as a quotient $H^2/M$ of the free Hilbert module $H^2$ by a closed submodule $M$, and the general inequalities **[8]** and **[9]** imply that

$$0 \leq K(H^2/M) \leq \chi(H^2/M) \leq \operatorname{rank}(H^2/M) = 1.$$

Because $\chi(H^2/M)$ must be either 0 or 1 and we conclude that either

$$K(H^2/M) = \chi(H^2/M) = 0 \quad \textbf{[10]}$$

or

$$0 \leq K(H^2/M) < \chi(H^2/M) = 1. \quad \textbf{[11]}$$

Consider the first alternative **[10]**. Using a general result of Auslander and Buchsbaum (see refs. 9 or 10) it is not hard to show that $\chi(H^2/M) = 0$ iff $M$ contains a nonzero polynomial. On the other hand, recalling the Second Extremal property of $K(H)$, we see that when **[10]** holds, the closed submodule $M \subseteq H^2$ must be associated with an inner sequence. Thus we may conclude as follows.

COROLLARY 1. *Every closed invariant subspace of* $H^2$ *which contains a nonzero polynomial is associated with an inner sequence.*

More generally, one has *Corollary 2.*

COROLLARY 2. *Let H be a finite rank Hilbert A-module whose canonical operators* $T_1, \ldots, T_d$ *satisfy an equation* $f(T_1, \ldots, T_d) = 0$ *for some nonzero polynomial* $f \in A$. *Then* $K(H) = \chi(H) = 0$.

Returning to the rank-one case, we are led to ask if the second alternative **[11]** can occur, and the answer is yes. Indeed, if $M$ is any closed submodule of $H^2$ which (*i*) contains no nonzero polynomials and (*ii*) satisfies $\{0\} \neq M \neq H^2$, then from condition (*i*) one easily shows that $\chi(H^2/M) = \operatorname{rank}(H^2/M) = 1$. Condition (*ii*) implies that the quotient $H^2/M$ cannot be a free Hilbert module, hence from the First Extremal property of the curvature invariant we have $K(H^2/M) < \operatorname{rank}(H) = 1$. Thus **[11]** holds for all such quotients $H = H^2/M$. Explicit examples are described in ref. 1.

These remarks also show that *Theorem B* does not hold for ungraded Hilbert modules. Nevertheless, we still do not know if $K(H^2/M)$ must be 0 in the context of alternative **[11]**. By making use of the Second Extremal property of the curvature invariant, this special case of *Problem 1* (i.e., the case in which $\operatorname{rank}(H) = 1$) becomes the following more concrete question.

*Problem 2:* Is every nonzero closed invariant subspace $M$ of $H^2$ associated with an inner sequence?[†]

Finally, we offer some comments about the rank of submodules of the free rank-one Hilbert module $H^2$. We have already pointed out that every closed submodule $M$ of $H^2$ is a (contractive, pure) Hilbert $A$-module, and that when $M \neq H^2$ the *quotient* module $H^2/M$ is of rank 1. What about the rank of $M$ itself? Because $H^2$ itself is of finite rank, it is easy to see that any closed submodule $M \subseteq H^2$ which is of finite *codimension* in $H^2$ must also be of finite rank; and the ranks of such finite codimensional submodules can be arbitrarily large. The following result concerns infinite codimensional submodules $M$ which are generated as closed submodules by some set of *homogeneous* polynomials in $A$ (perhaps of varying degrees).

*Theorem E. Let* $\Gamma$ *be the natural gauge group acting on the rank-one free Hilbert module* $H^2$, *and let M be any closed submodule which is graded in the sense that* $\Gamma(\lambda)M \subseteq M$ *for every* $\lambda \in \mathbb{T}$, *and which is nonzero and of infinite codimension in* $H^2$. *Then* $\operatorname{rank}(M) = \infty$.

Notice that this result stands in rather stark contrast with the assertion of Hilbert's basis theorem (9). Indeed, we conjecture that the answer to the following question is yes.

*Problem 3:* Must the rank of every closed submodule of $H^2$, which is nonzero and of infinite codimension in $H^2$, be infinite?

Additional problems and applications are discussed in ref. 1.

[†]I have received a letter from Stefan Richter and Carl Sundberg, which implies an affirmative solution of *Problem 2*.

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