Prime type III factors

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It is shown that for each $0 < \lambda < 1$, the free Araki–Woods factor of type III_λ cannot be written as a tensor product of two diffuse von Neumann algebras (i.e., is prime) and does not contain a Cartan subalgebra.

A von Neumann algebra M is called *prime* if it cannot be written as a tensor product of two diffuse von Neumann algebras. Using Voiculescu's free entropy theory (1), Ge (2) and later Stefan (3) gave examples of prime factors of type II_1 (and hence of type II_∞). An example of a separable prime factor of type III is given here: I show that for each $0 < \lambda < 1$, the type III_λ free Araki–Woods factor T_λ introduced in ref. 4 is prime. The main idea of the proof is to interpret the decomposition $T_\lambda = A \otimes B$ as a condition on its core, which is of type II. I then use Stefan's result (3) showing that $L(\mathbb{F}_\infty)$ cannot be written as the closure of the linear span of $N{\cdot}C_1{\cdot}C_2$ where N is a II_1 factor, which is not prime, and C_i are abelian von Neumann algebras.

I also prove the existence of separable type III factors that do not have Cartan subalgebras by showing that T_λ, $0 < \lambda < 1$ has no Cartan subalgebras. The key ingredient is Voiculescu's result on the absence of Cartan subalgebras in $L(\mathbb{F}_\infty)$ (1).

Although the proofs are based on a reduction to the case of type II algebras (for which free entropy methods are available), I believe that the results of this paper should be viewed as an indication that free entropy theory should have an extension to algebras of type III.

T_λ Is Prime

We use the following theorem, due to Connes (see sections 4.2 and 4.3 of ref. 5):

THEOREM 2.1. *Let M be a separable type III_λ factor with $0 < \lambda < 1$. Then there exists a faithful normal state ϕ on M, for which:*

1. *The centralizer $M^\phi = \{m \in M : \phi(mn) = \phi(nm)\ \forall n \in M\}$ is a factor of type II_1;*
2. *The modular group σ_t^ϕ of ϕ is periodic, of period exactly $2\pi/\log\lambda$;*
3. *M is generated as a von Neumann algebra by M^ϕ and an isometry V, satisfying:*
   (a) *$V^*V = 1$, $V^k(V^*)^k \in M^\phi$ for all k;*
   (b) *$\sigma_t^\phi(V) = \lambda^{-it}(V)$; thus, $\phi(V^k(V^*)^k) = \lambda^k\phi((V^*)^kV^k) = \lambda^k\phi(1) = \lambda^k$;*
   (c) *V normalizes M^ϕ: VmV^* and V^*mV are both in M^ϕ if $m \in M^\phi$.*

The weight $\phi \otimes Tr(B(\ell^2))$ is unique up to scalar multiples and up to conjugation by (inner) automorphisms of $M \cong M \otimes B(\ell^2)$.

Moreover, $2 \Rightarrow 1$ and 3; $1 \Rightarrow 2$ and 3. In particular, if ϕ_1 and ϕ_2 satisfy either 1 or 2, the centralizers M^{ϕ_1} and M^{ϕ_2} are stably isomorphic: $M^{\phi_1} \otimes B(\ell^2) \cong M^{\phi_2} \otimes B(\ell^2)$.

The existence of such a state can be easily seen by writing M as the crossed product of a type II_∞ factor C by a trace-scaling action of $\mathbb{Z}$: set $\hat{\phi}$ to be the crossed-product weight (where C is taken with its semifinite trace). Next, compress to a finite projection $p \in C$ and set $\phi = \hat{\phi}(p{\cdot}p)$. The isometry V is precisely the compression of the unitary U, implementing the trace-scaling action of $\mathbb{Z}$.

Recall that a von Neumann algebra M is called *full* if its group of inner automorphisms is closed in the u-topology inside its group of all automorphisms (see ref. 6).

LEMMA 2.2. *Let M be a full type III_λ factor. Assume that $M = A_1 \otimes A_2$, where A_1 and A_2 are von Neumann algebras. Then A_1 and A_2 are both full factors, and exactly one of the following must hold true:*

1. *A_1 and A_2 are both of type III_{λ_1} and III_{λ_2}, respectively, and λ_1, λ_2 satisfy:*
   (i) $0 < \lambda_i < 1$, $i = 1, 2$, (ii) $\lambda_1^{\mathbb{Z}}\lambda_2^{\mathbb{Z}} = \lambda^{\mathbb{Z}}$;
2. *For some $i \neq j$, A_i is of type III_λ and A_j is of type II;*
3. *For some $i \neq j$, A_i is of type III_λ and A_j is of type I.*

In particular, if we require that A_1 and A_2 must both be diffuse, only 1 and 2 can occur. Moreover, if 2 occurs, we may assume that one of the algebras A_1, A_2 is of type II_1.

Proof: If one of A_1, A_2 fails to be a factor, then their tensor product would fail to be a factor, hence both A_1 and A_2 must be factors. Similarly, if at least one of A_1 and A_2 fails to be full, their tensor product would fail to be full.

If, say, A_1 is of type I or type II, then A_2 must be type III, because otherwise $A_1 \otimes A_2$ would be of type II or type I. Hence if at least one of A_1 and A_2 is not type III, the situation described in 2 or 3 must occur.

If A_1 and A_2 are both type III, so that A is type III_{λ_1} and A_2 is type III_{λ_2}, we must prove that $\lambda^{\mathbb{Z}} = \lambda_1^{\mathbb{Z}}\lambda_2^{\mathbb{Z}}$. Neither λ_1 nor λ_2 can be zero, because then at least one of A_1, A_2 would then fail to be full, and hence $A_1 \otimes A_2$ would fail to be full.

Denote by $T(M)$ the T invariant of Connes (see section 1.3 of ref. 5). Since

$$\frac{2\pi\mathbb{Z}}{\log\lambda} = T(A_1 \otimes A_2) = T(A_1) \cap T(A_2)$$

[ref. 5, Theorem 1.3.4(c)] and $T(A_j) = (2\pi\mathbb{Z}/\log\lambda_j)$, statement 1 must hold.

PROPOSITION 2.3. *Let M be a type III_λ factor, and assume that $M = A_1 \otimes A_2$, where A_1 is a type III_{λ_1} factor, A is a type III_{λ_2} factor, and $\lambda^{\mathbb{Z}} = \lambda_1^{\mathbb{Z}}\lambda_2^{\mathbb{Z}}$. Let ϕ_i be a normal faithful state on A_i as in Theorem 2.1, and let $\phi = \phi_1 \otimes \phi_2$ be a normal faithful state on M.*

Then the centralizer M^ϕ of ϕ in M is a factor, which can be written as a closure of the linear span of $N{\cdot}C_1{\cdot}C_2$, where N is a tensor product of two type II_1 factors, and C_i are abelian von Neumann algebras. In particular, M^ϕ is not isomorphic to $L(\mathbb{F}_\infty)$.

Proof: Since the modular group of $\phi_1 \otimes \phi_2$ is $\sigma_t^{\phi_1} \otimes \sigma_t^{\phi_2}$, it follows that $\sigma_t^{\phi_1 \otimes \phi_2}$ has period exactly $2\pi/\log\lambda$. Hence the centralizer of $\phi_1 \otimes \phi_2$ is a factor.

Choose now a decreasing sequence of projections $p_k^{(1)} \in A_1^{\phi_1}$, $p_k^{(2)} \in A_2^{\phi_2}$, with $\phi_i(p_k^{(i)}) = \lambda_i^k$, and isometries $V_i \in A_i$, so that $V_i^*V_i = 1$, $V_i^k(V_i^*)^k = p_k^{(i)}$, so that V_i normalizes $A_i^{\phi_i}$, and $A_i = W^*(A_i^{\phi_i}, V_i)$. Then $A_1 \otimes A_2$ is densely spanned by elements of the form

$$W = V_1^{m_1} \otimes V_2^{n_1}{\cdot}a_1^{(1)} \otimes a_1^{(2)}{\cdot}V_1^{m_2} \otimes V_2^{n_2}a_j^{(i)} \in A_i^{\phi_i},\ m_i, n_i \in \mathbb{Z},$$

with the convention that $V_i^{-n} = (V_i^*)^n$ if $n \geq 0$.

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MATHEMATICS

Using the fact that $V_i^* a V_i,\ V_i a V_i^* \in A_i^{\phi_i}$ whenever $a \in A_i^{\phi_i}$, we can rewrite W as

$$W = (V_1^*)^m \otimes (V_2^*)^n \cdot a^{(1)} \otimes a^{(2)} \cdot V_1^l \otimes V_2^k,$$
$$a^{(i)} \in A_i^{\phi_i},\ m, n, k, l \geq 0.$$

Let now $p_k = p_k^{(1)} = V_1^k (V_1^*)^k \in A_1^{\phi_1}$ be as above. One can choose a diffuse commutative von Neumann algebra $\mathcal{A}$, containing p_k, $k \geq 0$, and so that $\mathcal{A} \subset A_1^{\phi_1}$ and $V_1 \mathcal{A} V_1^*$, $V_1^* \mathcal{A} V_1 \subset \mathcal{A}$. Choose a projection $\mathcal{A} \ni q_0 \leq p_1$, so that $q_0 \perp p_2$ and $\phi_1(q) = N\phi_1(1 - p_1) = N(1 - \lambda_1)$ for some integer N. Choose projections $\mathcal{A} \ni q_1, \ldots, q_n \leq 1 - p_1$, so that $\sum_{i=1}^N q_i = 1 - p_1$ and $\phi_1(q_i) = \phi_1(q_0) = (1/N)\phi_1(1 - p_1)$. Choose matrix units $\{e_{ij}\}_{0 \leq i,j \leq N} \subset A_1^{\phi_1}$, so that $e_{ii} = q_i$, $0 \leq i \leq N$. Let C be the von Neumann algebra generated by

$$\{V_1^k e_{ij} (V_1^*)^k : 1 \leq i, j \leq N,\ k \geq 0\}.$$

By our choice of e_{ij}, C is hyperfinite (notice that $V_1^k e_{ii} (V_1^*)^k \in \mathcal{A}$, and C is in fact the crossed product of $\mathcal{A} \cong L^\infty(X)$ by a singly generated equivalence relation). Let $R_1 = W^*(C, V) \subset A_1$. Then R_1 is also hyperfinite; in fact, it is the crossed product of C by the endomorphism $x \mapsto V_1 x V_1^*$. Notice that R_1 contains V_1. Furthermore, for all $k \geq 0$, there is a $d \geq 0$ and partial isometries $r_1, \ldots, r_d \in R_1 \cap A_1^{\phi_1}$, so that

$$1 - V_1^k (V_1^*)^k = \sum_{i=1}^d r_i V_1^k (V_1^*)^k r_i^*.$$

Construct in a similar way the algebra $R_2 \subset A_2$, in such a way that $V_2 \in R_2$ and for all $k \geq 0$, there is a $d \geq 0$ and partial isometries $r_1, \ldots, r_d \in R_2 \cap A_2^{\phi_2}$, so that

$$1 - V_2^k (V_2^*)^k = \sum_{i=1}^d r_i V_2^k (V_2^*)^k r_i^*.$$

Notice that $R_1 \otimes R_2 \subset A_1 \otimes A_2$ is globally fixed by the modular group of $\phi_1 \otimes \phi_2$. In particular, this means that

$$(R_1 \otimes R_2)^{\phi_1 \otimes \phi_2|_{R_1 \otimes R_2}} = (R_1 \otimes R_2) \cap (A_1 \otimes A_2)^{\phi_1 \otimes \phi_2}.$$

Assume now that $W \in (A_1 \otimes A_2)^{\phi_1 \otimes \phi_2}$. Then $\sigma_t^{\phi_1 \otimes \phi_2}(\mathrm{W}) = \mathrm{W}$. Hence $\lambda_1^{m-l} \cdot \lambda_2^{n-k} = 1$. It follows that W can be written in one of the following forms, using the fact that $V_i^* A_i^{\phi_i} V_i \subset A_i^{\phi_i}$:

$$W = (V_1^*)^m \otimes 1 \cdot a^{(1)} \otimes a^{(2)} \cdot 1 \otimes V_2^k, \text{ or}$$
$$W = 1 \otimes (V_2^*)^n \cdot a^{(1)} \otimes a^{(2)} \cdot V_1^l \otimes 1,$$

where $a^{(1)} \in A_1^{\phi_1}$, $a^{(2)} \in A_2^{\phi_2}$ and $\lambda_1^m = \lambda_2^k$, $\lambda_2^n = \lambda_1^l$. In the first case, choose $r_1, \ldots, r_d \in R_1 \cap A_1^{\phi_1}$ for which $1 - V_1^m (V_1^*)^m = \sum_{i=1}^d r_i V_1^m (V_1^*)^m r_i^*$. Then, writing

$$1 = V_1^m (V_1^*)^m + (1 - V_1^m (V_1^*)^m)$$
$$= V_1^m (V_1^*)^m + \sum r_i V_1^m (V_1^*)^m r_i^*$$

we obtain

$$W = (V_1^*)^m \otimes 1 \cdot a^{(1)} \otimes a^{(2)} \cdot 1 \otimes V_2^k$$
$$= [(V_1^*)^m a^{(1)} V_1^m \otimes a^{(2)}] \cdot (V_1^*)^m \otimes V_2^k$$
$$+ \sum_{i=1}^d [(V_1^*)^m a^{(1)} r_i V_1^m \otimes a^{(2)}] \cdot (V_1^*)^m r_i^* \otimes V_2^k$$
$$\in \operatorname{span}\{(A_1^{\phi_1} \otimes A_2^{\phi_2}) \cdot (R_1 \otimes R_2)^{\phi_1 \otimes \phi_2}\}.$$

Reversing the roles of A_1 and A_2, we get that in general, $\operatorname{span}\{(A_1^{\phi_1} \otimes A_2^{\phi_2}) \cdot (R_1 \otimes R_2)^{\phi_1 \otimes \phi_2}\}$ is dense in $(A_1 \otimes A_2)^{\phi_1 \otimes \phi_2}$.

Since each R_i is hyperfinite, the algebra $R_1 \otimes R_2$ is also hyperfinite; hence $(R_1 \otimes R_2)^{\phi_1 \otimes \phi_2}$ is hyperfinite. It follows that the centralizer $M^{\phi_1 \otimes \phi_2}$ of $M = A_1 \otimes A_2$ can be written as the closure of the span of NR, where N is a tensor product of two type II_1 factors, and R is a hyperfinite algebra. Since every hyperfinite algebra can be written as a linear span of the product $C_1 \cdot C_2$, where C_i are abelian von Neumann algebras, it follows that the centralizer M^ϕ is the closure of the span of $N \cdot C_1 \cdot C_2$, with N a tensor product of two type II_1 factors, and C_1, C_2 abelian von Neumann algebras. Hence by Stefan's result (3), we get that M^ϕ cannot be isomorphic to $L(\mathbb{F}_\infty)$.

Theorem 2.4. *Let T_λ be the free Araki–Woods factor constructed in ref. 4. Then $T_\lambda \not\cong A_1 \otimes A_2$, where A_1 and A_2 are any diffuse von Neumann algebras.*

Proof: Since T_λ is a full III_λ factor, we have by *Lemma 2.2* that the only possible tensor product decompositions with A_1 and A_2 diffuse are ones where either exactly one of A_1 and A_2 is type II_1 and the other is of type III_λ, or each A_i is of type III_{λ_i}, with $\lambda_1^{\mathbb{Z}} \lambda_2^{\mathbb{Z}} = \lambda^{\mathbb{Z}}$.

Denote by ψ the free quasifree state on T_λ. It is known (see ref. 4, Corollary 6.8) that T_λ^ψ is a factor, isomorphic to $L(\mathbb{F}_\infty)$. Let ϕ be an arbitrary normal faithful state on T_λ, such that T_λ^ϕ is a factor. Then (see *Theorem 2.1*), $T_\lambda^\phi \otimes B(\ell^2) \cong T_\lambda^\psi \otimes B(\ell^2) \cong L(\mathbb{F}_\infty) \otimes B(\ell^2)$. Since $L(\mathbb{F}_\infty)$ has $\mathbb{R}_+$ as its fundamental group (see ref. 7), it follows that whenever ϕ is a state on T_λ, and T_λ^ϕ is a factor, then $T_\lambda^\phi \cong L(\mathbb{F}_\infty)$.

Assume now that one of A_1, A_2 is of type II_1; for definiteness, assume that it is A_1. Choose on A_2 a normal faithful state ϕ_2 for which A^{ϕ_2} is a factor, and let τ be the unique trace on A_1. Let $\phi = \tau \otimes \phi_2$ on T_λ. Then $T_\lambda^\phi \cong A_1 \otimes A_2^{\phi_2}$, and hence cannot be isomorphic to $L(\mathbb{F}_\infty)$ by the results of Stephan (3) and Ge (2). This is a contradiction.

Assume now that A_i is type III_{λ_i}, with $0 < \lambda_i < 1$. Then by *Proposition 2.3* there is a state ϕ on T_λ, for which T_λ^ϕ is a factor, but is not isomorphic to $L(\mathbb{F}_\infty)$; contradiction.

3. T_λ Has No Cartan Subalgebras

Recall that a von Neumann algebra M is said to contain a *Cartan subalgebra A* if:

1. $A \subset M$ is a MASA (maximal abelian subalgebra).
2. There exists a faithful normal conditional expectation $E : M \to A$.
3. $M = W^*(\mathcal{N}(A))$, where $\mathcal{N}(A) = \{u \in M : uAu^* = A, u^*u = uu^* = 1\}$ is the normalizer of A.

For type II_1 factors M, condition 2 is automatically implied by condition 1.

Proposition 3.1. *Let M be a factor of type III_λ, $0 < \lambda < 1$. Then there exists a normal faithful state ψ on M, so that $\sigma^\psi_{2\pi/\log\lambda} = \mathrm{id}$, and that the centralizer M^ψ is a II_1 factor containing a Cartan subalgebra.*

Proof: Let $A \subset M$ be a Cartan subalgebra. Let $E : M \to A$ be a normal faithful conditional expectation. Let ϕ be a normal faithful state on $A \cong L^\infty[0, 1]$, and denote by θ the state $\phi \circ E$ on M. Then θ is a normal faithful state. Furthermore, $M^\theta \supset A$, because E is θ-preserving and hence $\sigma^\theta|_A = \sigma^{\theta|_A} = \mathrm{id}$. Since M is type III_λ, it follows that $\sigma^\theta_{t_0}$ is inner if $t_0 = 2\pi/\log\lambda$. Let $u \in M$ be a unitary for which $\sigma^\theta_{t_0}(m) = umu^*$, $\forall m \in M$. Then $uxu^* = x$ for all $x \in A$, since $\sigma^\theta|_A = \mathrm{id}$. It follows that $u \in A' \cap M =$

A', since A is a MASA. Choose $d \in A$ positive so that $d^{it_0} = u$. Note that d is in the centralizer of θ (which contains A). Set $\psi(m) = \theta(d^{-1}m)$ for all $m \in M$. Then the modular group of ψ at time t_0 is given by $\mathrm{Ad}_{u^*} \circ \sigma_{t_0}^{\theta} = \mathrm{id}$. It follows that ψ is a normal faithful state on M, so that $\sigma_{t_0}^{\psi} = \mathrm{id}$. It furthermore follows from *Theorem 2.1* that the centralizer of M^{ψ} is a factor of type II_1. By the choice of ψ, its modular group fixes A pointwise, hence $A \subset M^{\psi}$.

I claim that A is a Cartan subalgebra in $N = M^{\psi}$. First, $A' \cap N \subset A' \cap M = A$, hence A is a MASA. Since A is a Cartan subalgebra in M, M is densely linearly spanned by elements of the form $f{\cdot}u$, where $u \in \mathcal{N}(A)$ is a unitary and $f \in A$. The map

$$E(m) = \frac{1}{2\pi}\int_0^{2\pi} \sigma_t^{\psi}(m)dt$$

is a normal faithful conditional expectation from M onto N. If $u \in \mathcal{N}(A)$ is a unitary, so that $ufu^* = \alpha(f)$ for all $f \in A$ and $\alpha \in \mathrm{Aut}(A)$, then $uf = \alpha(f)u$. Hence

$$E(u)f = E(uf) = E(\alpha(f)u) = \alpha(f)E(u).$$

It follows that N is densely linearly spanned by elements of the form $E(f{\cdot}u) = f{\cdot}E(u)$ for $f \in A$ and $u \in \mathcal{N}(A)$. Let $w(u)$ be the polar part of $E(u)$, and let $p(u) = E(u)^*E(u)$ be the positive part of $E(u)$, so that $E(u) = w(u)p(u)$ is the polar decomposition of $E(u)$. Since

$$\begin{aligned} E(u)^*E(u)\alpha^{-1}(f) &= E(u)^*fE(u) \\ &= \alpha^{-1}(f)E(u)^*E(u), \end{aligned}$$

it follows that $p(u)$ commutes with A and hence is in A. Moreover, we then have that

$$w(u)fw(u)^* = \alpha(f),$$

so that $w(u) \in \mathcal{N}(A) \cap N$. Thus N is densely linearly spanned by elements of the form $f{\cdot}u$ for $f \in A$ and $u \in \mathcal{N}(A) \cap N$, hence A is a Cartan subalgebra of N.

COROLLARY 3.2. *For each* $0 < \lambda < 1$ *the* III_λ *free Araki–Woods factor* T_λ *does not have a Cartan subalgebra.*

Proof: If T_λ were to contain a Cartan subalgebra, it would follow that for a certain state ψ on T_λ, the centralizer of ψ is a factor containing a Cartan subalgebra. Let ϕ be the free quasifree state on T_λ. Then by *Theorem 2.1*, one has

$$(T_\lambda)^{\phi} \otimes B(\ell^2) \cong (T_\lambda)^{\psi} \otimes B(\ell^2).$$

Since $(T_\lambda)^{\phi} \cong L(\mathbb{F}_\infty)$ (see Corollary 6.8 of ref. 5), and because the fundamental group of $L(\mathbb{F}_\infty)$ is all of $\mathbb{R}_+$ (see ref. 7) we conclude that $L(\mathbb{F}_\infty)$ contains a Cartan subalgebra. But this is in contradiction to a result of Voiculescu that $L(\mathbb{F}_\infty)$ has no Cartan subalgebras (see ref. 1).

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