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Kostant polynomials and the cohomology ring for G/B

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ABSTRACT The Schubert calculus for G/B can be completely determined by a certain matrix related to the Kostant polynomials introduced in section 5 of Bernstein, Gelfand, and Gelfand [Bernstein, I., Gelfand, I. & Gelfand, S. (1973) *Russ. Math. Surv.* 28, 1–26]. The polynomials are defined by vanishing properties on the orbit of a regular point under the action of the Weyl group. For each element w in the Weyl group the polynomials also have nonzero values on the orbit points corresponding to elements which are larger in the Bruhat order than w. The main theorem given here is an explicit formula for these values. The matrix of orbit values can be used to determine the cup product for the cohomology ring for G/B, using only linear algebra or as described by Lascoux and Schützenberger [Lascoux, A. & Schützenberger, M.-P. (1982) *C. R. Seances Acad. Sci. Ser. A* 294, 447–450]. Complete proofs of all the theorems will appear in a forthcoming paper.

Section 1. Introduction

Let G be a semisimple Lie group, H be a Cartan subgroup, W be its corresponding Weyl group with generators $\sigma_1, \sigma_2, \ldots \sigma_n$, and B be a Borel subgroup. Let $R = \mathbb{C}[\mathbf{h}^*]$ be the algebra of polynomial functions on the Cartan subalgebra $\mathbf{h}$. Fix a regular element $\mathbf{o} \in \mathbf{h}$ such that $\alpha_i(\mathbf{o})$ is a positive integer, for all simple roots α_i. Any Weyl group element v acts on the right on $\mathbf{o}$ by the action on the Cartan subalgebra. We define the following interpolating polynomials by their values on the orbit of $\mathbf{o}$.

Definition: The *Kostant polynomial* K_w is an element of R of degree $l(w)$ (nonhomogeneous) such that

$$K_w(\mathbf{o}v) = \begin{cases} 1 & v = w \\ 0 & l(v) \le l(w) \text{ and } v \ne w. \end{cases} \qquad \textbf{[1.1]}$$

K_w is unique modulo the ideal of all elements of R which vanish on the orbit of $\mathbf{o}$ under the Weyl group action.

These polynomials were defined originally by Kostant and appear in theorem 5.9 of ref. 1 for the finite case; they were later generalized by Kostant and Kumar in ref. 2, denoted $\xi^{w^{-1}}$ in their notation. In the case G is SL_n, the Kostant polynomials are the double Schubert polynomials (multiplied by a scalar) introduced by Lascoux and Schützenberger (3); see also ref. 4.

The object of study for this announcement is not precisely the Kostant polynomials themselves but instead the values of the Kostant polynomials on the points in the orbit of $\mathbf{o}$ under the Weyl group action. From the definition of K_w, we know K_w ($\mathbf{o}v$) is 0 if $l(v) \le l(w)$ and $v \ne w$. However, the orbit values $K_w(\mathbf{o}v)$ if $l(v) \ge l(w)$ are not specified (though completely determined). The main result of this paper, stated in *Theorem 2*, is an explicit formula for computing these orbit values, namely

$$K_w(\mathbf{o}v) = \frac{1}{\pi_w} \sum_{b_{i_1} b_{i_2} \cdots b_{i_k} \in R(w)} \prod_{j=1}^{k} \sigma_{b_1}\sigma_{b_2} \cdots \sigma_{(b_{i_j}-1)}\alpha_{b_{i_j}}(\mathbf{o}), \qquad \textbf{[1.2]}$$

where $\mathbf{b} = b_1 b_2 \cdots b_p$ is any fixed reduced word for v, $R(w)$ is the set of all reduced words for w, and the sum is over all sequences $1 \le i_1 < i_2 < \ldots < i_k \le p$ such that $b_{i_1} b_{i_2} \cdots b_{i_k} \in R(w)$. The scalar factor π_w^{-1} appears to normalize $K_w(\mathbf{o}w) = 1$; see *Section 4* for the definition. This formula is independent of the choice of reduced word for v, and it exhibits the strong connection between the Kostant polynomials, the Schubert calculus, and the Bruhat order. In *Section 5*, we show how the matrix of orbit values is related to the cup product in the cohomology ring for the flag manifold G/B, following ref. 2. In fact, the cup product can be determined simply by studying the vectors of orbit values.

We begin with a review of a few results from the renowned paper by Bernstein, Gelfand, and Gelfand (1). In *Section 3*, the nil-Coxeter algebra is introduced to prove that the orbit value formula is independent of the choice of reduced word. The explicit formula for the orbit values is stated as a theorem and proved in *Section 4*. Finally, in *Section 5* our approach to computing the connection coefficients in the cohomology ring for G/B using vectors of the orbit values is outlined.

Complete proofs of all the statements in this announcement will appear in a forthcoming paper. Please contact the author for the expected publication date.

Section 2. Divided Difference Equations

The divided difference equations defined by Bernstein, Gelfand, and Gelfand are used to recursively compute the Schubert classes, starting from the unique Schubert class of codimension 0, and working up to higher codimension. In this section we show that these operators also act nicely on the Kostant polynomials. The divided difference equations for Kostant polynomials also lead to a recursive method for computing the vector of orbit values for these polynomials.

Recall that we have defined R to be the ring of polynomials over the Cartan subalgebra, $\mathbb{C}[\mathbf{h}^*]$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be simple roots in the root system Δ, which is contained in the ambient vector space with a symmetric bilinear form (α, β) which is positive definite over the real span of the roots. Let Δ_+ (Δ_-) be the positive roots (negative roots) with respect to this choice of simple roots. The generators $\sigma_1, \sigma_2, \ldots, \sigma_n$ of W are the reflections over the hyperplane perpendicular to the corresponding simple roots. The reflections act on vectors $v \in V$ by $\sigma_i(v) = v - \langle v, \alpha_i \rangle \alpha_i$ where

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‡This notation differs from the notation in ref. 1 by a sign. The result is that we have interchanged the positive and the negative roots from those used in ref. 1.

$$\langle\alpha, \beta\rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)}. \quad \textbf{[2.1]}$$

For each $1 \le i \le n$, the *divided difference operator* $\partial_i : R \to R$ acts on $f \in R$ by‡

$$\partial_i f = \frac{f - \sigma_i f}{-\alpha_i}. \quad \textbf{[2.2]}$$

Let I be the ideal generated by the Weyl group invariants of positive degree. The action of the divided difference operators preserves I. Hence, each ∂_i acts on the quotient R/I as well. If $\alpha : H^*(G/B, \mathbb{Q}) \to R/I$ is the Borel isomorphism, let $\mathfrak{S}_w$, the *Schubert class* of w, be the image of the Schubert cycle in $H^*(G/B, \mathbb{Q})$ corresponding to w_0w under this map, where w_0 is the longest element in the Weyl group.

PROPOSITION 2.1 (ref. 1, theorems 3.14 and 3.15). *The Schubert classes* $\mathfrak{S}_w \in R/I$ *have the property*

$$\partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{w\sigma_i} & l(w) < l(w\sigma_i) \\ 0 & l(w) > l(w\sigma_i). \end{cases} \quad \textbf{[2.3]}$$

Furthermore, the Schubert class $\mathfrak{S}_{w_0}$ *is given by*

$$\mathfrak{S}_{w_0} = \frac{(-1)^{|W|}}{|W|} \prod_{\gamma\in\Delta_+} \gamma (mod\ I). \quad \textbf{[2.4]}$$

The Schubert classes and the highest homogeneous component of the Kostant polynomials are the same modulo I up to a constant. Below the correct statement is given in our notation.

PROPOSITION 2.2 (ref. 1, theorem 5.9). *Let* K_w^0 *be the form of highest degree in* K_w*. Then, the image of* K_w^0 *in* R/I *is equal to*

$$\prod_{\gamma\in\Delta_+\cap w\Delta_-} (\gamma(\mathbf{o}))^{-1}\mathfrak{S}_w. \quad \textbf{[2.5]}$$

The next theorem shows that the divided difference operators satisfy a modified recursive formula. This theorem has also been given by Kostant and Kumar in ref. 2 for the case of an arbitrary Kac–Moody Lie algebra.

THEOREM 1. *For* $w \in W$ *the divided difference operator* ∂_i *acts on* K_w *as follows*

$$\partial_i K_w = \begin{cases} \dfrac{K_{w\alpha_i}}{\alpha_i(\mathbf{o}w\sigma_i)} & l(w) < l(w\sigma_i) \\ 0 & l(w) > l(w\sigma_i). \end{cases} \quad \textbf{[2.6]}$$

Proof: The proof follows from evaluating $\partial_i K_w$ at the orbit points of length less than or equal to the length of $w\sigma_i$. □

COROLLARY 2.3. *The orbit values* $K_w(\mathbf{o}v)$ *can be computed recursively from the top down. Namely,* $K_{w_0}(\mathbf{o}v)$ *is* 1 *if* $v = w_0$ *and* 0 *otherwise. If* $w \neq w_0$ *there exists an* i *such that* $l(w) < l(w\sigma_i)$*, let* u *be* $w\sigma_i$*. Then for any* $v \in W$,

$$K_w(\mathbf{o}v) = \partial_i K_u(\mathbf{o}v) = \frac{K_u(\mathbf{o}v) - K_u(\mathbf{o}v\sigma_i)}{-\alpha_i(\mathbf{o}v)} \alpha_i(\mathbf{o}w). \quad \textbf{[2.7]}$$

Corollary 2.3 gives an algorithm to compute the values $K_w(\mathbf{o}v)$. We will use this corollary in *Section 4* to prove the formula for orbit values.

Section 3. The Nil-Coxeter Algebra

In this section, we allow W to be the Weyl group for an arbitrary Kac–Moody Lie algebra. Let $\mathcal{A} = \mathcal{A}_W$ be the nil-Coxeter algebra for W. In other words, if W is generated by $\sigma_1, \sigma_2, \ldots, \sigma_n$ with relations given by $(\sigma_i\sigma_j)^{m_{ij}} = 1$, then $\mathcal{A}$ is generated as an algebra over $R = \mathbb{C}[\mathbf{h}^*]$ by $u_1, u_2, \ldots, u_n$ with the relations

$$\underbrace{u_iu_ju_iu_j\cdots}_{m_{ij}\text{ factors}} = \underbrace{u_ju_iu_ju_i\cdots}_{m_{ij}\text{ factors}} \quad \text{for } i \neq j \quad \textbf{[3.1]}$$

$$u_i^2 = 0. \quad \textbf{[3.2]}$$

As a vector space over R, a basis for $\mathcal{A}$ is given by $\{u_w : w \in W\}$, where u_w represents the equivalent products $u_{a_1}u_{a_2}\cdots u_{a_p}$ for any $a_1a_2\cdots a_p \in R(w)$. The Weyl group acts on $\mathcal{A}$ by acting on the elements in R, and the generators u_i are fixed by all elements in the Weyl group.

Following the notation of Fomin and Kirillov (section 1 of ref. 5), we define the Yang–Baxter operators $h_i: R \to \mathcal{A}$ by

$$h_i(x) = e^{xu_i} = 1 + xu_i. \quad \textbf{[3.3]}$$

The relations among the Weyl group generators impose relations on the $h_i(x)$'s as well. It is well known that a minimal set of relations among the generators of a Weyl group are of the form $(\sigma_i\sigma_j)^{m_{ij}} = 1$. If W is the Weyl group of a semisimple Lie algebra, then the only possibilities for m_{ij} are 2, 3, 4, or 6.

PROPOSITION 3.1 (sections 1, 4, and 6 of ref. 5). *Given any Weyl group of a semisimple Lie algebra, the Yang–Baxter operators satisfy the following Yang–Baxter equations:*

$$h_i(x)h_j(y) = h_j(y)h_i(x). \qquad \text{if } (\sigma_i\sigma_j)^2 = 1, \quad \textbf{[3.4]}$$

$$h_i(x)h_j(x+y)h_i(y) = h_j(y)h_i(x+y)h_j(x) \qquad \text{if } (\sigma_i\sigma_j)^3 = 1, \quad \textbf{[3.5]}$$

$$h_i(x)h_j(x+y)h_i(x+2y)h_j(y) = h_j(y)h_i(x+2y)h_j(x+y)h_i(x) \qquad \text{if } (\sigma_i\sigma_j)^4 = 1, \quad \textbf{[3.6]}$$

and

$$h_i(x)h_j(3x+y)h_i(2x+y)h_j(3x+2y)h_i(x+y)h_j(y) = h_j(y)h_i(x+y)h_j(3x+2y)h_i(2x+y)h_j(3x+y)h_i(x) \qquad \text{if } (\sigma_i\sigma_j)^6 = 1. \quad \textbf{[3.7]}$$

It is well known (ref. 6, p. 14) that the set of roots $\{\sigma_{b_1}\sigma_{b_2}\cdots\sigma_{b_{k-1}}\alpha_{b_k} : 1 \le k \le p\}$ is equal to $\Delta_+ \cap v\Delta_-$, hence independent of the choice of reduced word. Next we define a family of polynomials which are closely related to this set.

Definition: For any $v \in W$ and any reduced word $\mathbf{a} = a_1a_2\cdots a_p$ for v, define a *root polynomial* for $\mathbf{a}$ in the nil-Coxeter algebra $\mathcal{A}$ by

$$\mathfrak{R}_{\mathbf{a}} = \prod_{i=1}^{p} h_{a_i}(\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_{i-1}}a_{a_i}). \quad \textbf{[3.8]}$$

For example, if the root system is of type A_2, the Weyl group is the symmetric group, S_3. Let α_1 and α_2 be the simple roots. For $i = 1$ or 2, $\sigma_i\alpha_i = -\alpha_i$ and $\sigma_i\alpha_j = \alpha_1 + \alpha_2$ for i different from j. The word 121 is a reduced word of the permutation [3, 2, 1] (written in one-line notation). Then R_{121} is given by

$$\mathfrak{R}_{121} = (1 + \alpha_1u_1)(1 + \sigma_1\alpha_2u_2)(1 + \sigma_1\sigma_2\alpha_1u_2) \quad \textbf{[3.9]}$$

$$= 1 + (\alpha_1 + \alpha_2)(u_{[2,1,3]} + u_{[1,3,2]}) + (\alpha_1^2 + \alpha_1\alpha_2)u_{[2,3,1]} + (\alpha_1\alpha_2 + \alpha_2^2)u_{[3,1,2]} + (\alpha_1^2\alpha_2 + \alpha_1\alpha_2^2)u_{[3,2,1]}. \quad \textbf{[3.10]}$$

In fact, we show in the next theorem that $\mathfrak{R}_{\mathbf{a}}$ for $\mathbf{a} \in R(v)$ depends only on the Weyl group element v and not on the choice of reduced word. Therefore, we can define the *root polynomial for v*, $\Re_v$ to be $\Re_{\mathbf{a}}$ for any $\mathbf{a} \in R(v)$.

THEOREM 2. *For any* $v \in W$, *choose any reduced word* $\mathbf{a} = a_1a_2 \cdots a_p \in R(v)$, *then*

$$\Re_v = \prod_{i=1}^{p} h_{a_i}(\sigma_{a_1}\sigma_{a_2} \cdots \sigma_{a_{i-1}}\alpha_{a_i}) \qquad \textbf{[3.11]}$$

is well defined.

Independently, J. Stembridge has shown that this theorem holds for all Coxeter groups (7). His proof does not depend on case-by-case computations.

Proof: One can easily verify that for any position i,

$$\Re_{a_1a_2 \ldots a_p} = \Re_{a_1 \ldots a_i}(\sigma_{a_1} \cdots \sigma_{a_i}\Re_{a_{i+1} \ldots a_p}), \qquad \textbf{[3.12]}$$

since $(1 + \sigma_{a_1} \cdots \sigma_i \cdots \sigma_{a_{k-1}}\alpha_k u_k) = \sigma_{a_1} \cdots \sigma_{a_i}(1 + \sigma_{i+1} \cdots \sigma_{k-1}\alpha_k u_k)$. Hence, the problem is reduced to showing that $\Re_s = \Re_{s'}$, where $s = ij \ldots$ and $s' = ji \ldots$ with $l(s) = l(s') = 2, 3, 4$, or 6. The proof follows from *Proposition 3.1* applied to the different possibilities for s and s'. □

Section 4. Orbit Value Formula

Let $v \in W$ and fix a reduced word $b_1b_2 \ldots b_p \in R(v)$. Recall that the roots in the set $\Delta_+ \cap v\Delta_-$ are given by $\{\sigma_{b_1}\sigma_{b_2} \cdots \sigma_{b_{i-1}}\alpha_{b_i} : i \leq i \leq p\}$. In other words, for each initial sequence of the chosen reduced word, $\sigma_{b_1}\sigma_{b_2} \cdots \sigma_{b_{j-1}}\alpha_{b_j}$ is a positive root in the set $\Delta_+ \cap v\Delta_-$. Associate a scalar to the jth root by evaluating it at the fixed orbit point $\mathbf{o}$,

$$r_{\mathbf{b}}(j) = r_{b_1b_2 \ldots b_p}(j) = \sigma_{b_1}\sigma_{b_2} \cdots \sigma_{b_{j-1}}\alpha_{b_j}(\mathbf{o}). \qquad \textbf{[4.1]}$$

Let π_v be the scalar obtained as follows,

$$\pi_v = \prod_{\gamma \in \Delta_+ \cap v\Delta_-} \gamma(\mathbf{o}). \qquad \textbf{[4.2]}$$

Note that π_v is equal to the product $r_{\mathbf{b}}(1)r_{\mathbf{b}}(2) \cdots r_{\mathbf{b}}(p)$.

THEOREM 3. *Let* $v, w \in W$ *and fix a reduced word* $\mathbf{b} = b_1b_2 \ldots b_p$ *for* v. *The orbit values of* K_w *are given by*

$$K_w(\mathbf{o}v) = \frac{1}{\pi_w} \sum_{b_{i_1}b_{i_2} \cdots b_{i_k} \in R(w)} r_{\mathbf{b}}(i_1)r_{\mathbf{b}}(i_2) \cdots r_{\mathbf{b}}(i_k), \qquad \textbf{[4.3]}$$

where $r_{\mathbf{b}}(j)$ *is defined by Eq.* **4.1** *and the sum is over all sequences* $1 \leq i_1 < i_2 < \ldots < i_k \leq p$ *such that* $b_{i_1}b_{i_2} \cdots b_{i_k} \in R(w)$. *Furthermore, the sum in Eq.* **4.3** *is independent of the choice of* $\mathbf{b} \in R(v)$.

Proof: First, note that the sum in Eq. **4.3** is the coefficient of u_w in the root polynomial $\Re_v$ from *Section 3*. It was shown in *Theorem 2* that $\Re_v$ is well defined for any choice of reduced word. Therefore, the coefficient of u_w in $\Re_v$ is also independent of our choice of reduced word for v.

Second, we show that Eq. **4.3** holds by decreasing induction on the length of w. For the longest element $w_0 \in W$ we know $K_{w_0}(\mathbf{o}w_0) = 1$ and $K_{w_0}(\mathbf{o}v) = 0$ for all $v \in W$ such that $l(v) < l(w_0)$. This agrees with Eq. **4.3**, since for any $\mathbf{b} = b_1b_2 \cdots b_p \in R(w_0)$, there is at most one term in the sum and $\pi_w = r_{\mathbf{b}}(1)r_{\mathbf{b}}(2) \cdots r_{\mathbf{b}}(p)$. Therefore, we can assume by induction that Eq. **4.3** holds for all $u \in W$ such that $l(u) > l(w)$. The proof follows by *Corollary 2.3* and the inductive hypothesis. □

COROLLARY 4.1. *The value* $\pi_w K_w(\mathbf{o}v)$ *is a nonnegative integer provided* $\alpha_i(\mathbf{o})$ *is a positive integer for each simple root* α_i.

The following corollaries are simple consequences of *Theorem 3.* They were also shown in ref. 2.

COROLLARY 4.2. *The orbit values* $K_w(\mathbf{o}v)$ *and* $K_w(\mathbf{o}v\sigma_i)$ *are equal if and only if* $l(w) < l(w\sigma_i)$.

COROLLARY 4.3. *The orbit value* $K_w(\mathbf{o}v)$ *is different from* 0 *if and only if* $w \leq v$ *in the Bruhat order*.

Section 5. Determination of the Cup Product in the Cohomology Ring of G/B

In this section the main application of *Theorem 3* is described. The highest homogeneous component of a Kostant polynomial represents a Schubert class. Therefore, the highest homogeneous component of the product of Kostant polynomials represents the product of Schubert classes. It will be shown that one can find the expansion of products of Kostant polynomials in the basis of Kostant polynomials by using the vectors of orbit values. This method for computing the cup product is much more efficient than previously known techniques, which involved multiplying polynomials and possibly reducing modulo the ideal of invariants. Also, since it extends to the exceptional root systems, it is more complete than the existing theory of Schubert polynomials defined by Lascoux and Schützenberger (8); see also refs. 4 and 9–11 and many more.

Fix a total order on the Weyl group elements which respects the partial order determined by length. We define the *orbit value vector* V_w in $\mathbb{Z}^{|W|}$ to be the vector with entry $K_w(\mathbf{o}v)$ in the vth component indexed by the chosen total order. For example, the orbit value vector for the longest element will be $(0, 0, \ldots, 0, 1)$. Note the set $\{V_w : w \in W\}$ is a basis for $\mathbb{Z}^{|W|}$, since $K_w(\mathbf{o}w) \neq 0$ and $K_w(\mathbf{o}v) = 0$ for all $v < w$ in the chosen total order.

LEMMA 5.1. *Let* $V_u \cdot V_v$ *be the coordinate-wise product of vectors. If*

$$V_u \cdot V_v = \sum p_{uv}^w V_w, \qquad \textbf{[5.1]}$$

then the product of Kostant polynomials K_uK_v *(modulo the ideal* J *of all polynomials which vanish on the orbit of* $\mathbf{o}$*) expands with the same coefficients,*

$$K_u \cdot K_v = \sum p_{uv}^w K_w \ (mod\ \mathbf{J}). \qquad \textbf{[5.2]}$$

Furthermore, if $l(w) = l(u) + l(v)$, *the coefficients* p_{uv}^w *are constants which are independent of the choice of orbit point provided* $\alpha_i(\mathbf{o})$ *is positive for each* i.

COROLLARY 5.2. *For* $u, v, w \in W$, *the coefficient* p_{uv}^w *from Eqs.* **5.1** *and* **5.2** *can be computed recursively by*

$$p_{uv}^w = K_u(\mathbf{o}w)K_v(\mathbf{o}W) - \sum_{t<w} p_{uv}^t K_t(\mathbf{o}w). \qquad \textbf{[5.3]}$$

It is easy to compute the expansion of any vector in $\mathbb{Z}^{|W|}$ into the sum of the vectors V_w because of their upper triangular form. The expansion involves only linear algebra. Therefore, I propose one compute the coefficients c_{uv}^w in the expansion of $\mathfrak{S}_u\mathfrak{S}_v = \Sigma c_{uv}^w \mathfrak{S}_w$ by computing $V_uV_v = \Sigma p_{uv}^w V_w$.

COROLLARY 5.3. *For* $w \in W$ *such that* $l(w) = l(u) + l(v)$, *the structure constant* c_{uv}^w *is equal to* $p_{uv}^w \pi_u \pi_v \pi_w^{-1}$.

Proof: This follows from *Lemma 5.1* and *Proposition 2.2*. □

Kostant and Kumar (2) have shown that the coefficients p_{uv}^w in *Lemma 5.1* can also be completely determined as coefficients in a product of matrices that depend only on the matrix of orbit values. Let $D = [d_{uv}]$ be the matrix with entries indexed by $u, v \in W$, and the entry d_{uv} is defined[§] to be the orbit value $\pi_u K_u(\mathbf{o}v)$, computed by either Eq. **2.7** or Eq. **4.3**.

PROPOSITION 5.4 (ref. 2). *Fix* $u \in W$. *Let* D_u *be the diagonal matrix with* d_{uv} *along the diagonal. Let* P_u *be the matrix of coefficients* $[p_{uv}^w]$ *from Eq.* **5.1.** *Then*

$$P_u = D \cdot D_u \cdot D^{-1}. \qquad \textbf{[5.4]}$$

[§]Note that our d_{uv} corresponds with $d_{u^{-1}v^{-1}}$ in ref. 2.

This proposition beautifully demonstrates the relationship between the cup product of the cohomology ring of G/B and the D matrix.

Section 6. Concluding Remarks

While formula **4.3** is presented only for the finite case (i.e., G semisimple), the formula for the orbit values is in fact true for all Kac–Moody Lie algebras. We have shown that the formula is independent of the choice of reduced word for any element of the Weyl group of a Kac–Moody algebra. The given proof of *Theorem 3* does not extend to the infinite case because it depends on the existence of a top element. However, S. Kumar has shown that there is an analog for *Corollary 2.3* which computes the orbit values starting from the identity and going up in length. He then proves the orbit value formula in the Kac–Moody case, using this recurrence and theorems from ref. 2. The complete version of the paper will contain an abstract outlining the details of Kumar's proof.

The Kostant polynomials and the double Schubert polynomials defined by Lascoux and Schützenberger (3) are closely related in the case of SL_n. Let $\mathfrak{S}_w(X, Y)$ be the double Schubert polynomial indexed by w on the two alphabets, X and Y. Then $K_w(x_1, x_2, \ldots, x_n) = (1/\pi_w)\mathfrak{S}_w(X, \mathbf{o})$. This fact can be proven by using the combinatorial interpretation for the terms in a double Schubert polynomial defined by Fomin and Kirillov (12). Unfortunately, the Kostant polynomials for the other classical groups do not appear to be related to the double Schubert polynomials as defined by Fulton (11).

It is well known that the structure constants for the product of Schubert classes are nonnegative integers. In fact, the coefficients in the entire P_u matrix appear to be nonnegative integers, provided the orbit point is chosen so that each $\alpha_i(\mathbf{o})$ is a positive integer. Can one give formulas analogous to **4.3** for these coefficients that would prove they are nonnegative integers? Note, the D and D_u matrices are also composed of nonnegative integers. However, the matrix D^{-1} contains negative rational numbers.

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1. Bernstein, I., Gelfand, I. & Gelfand, S. (1973) *Russ. Math. Surv.* **28,** 1–26.
2. Kostant, B. & Kumar, S. (1986) *Adv. Math.* **62,** 187–237.
3. Lascoux, A. & Schützenberger, M.-P. (1983–1984) *Interpolation de Newton A Plusieurs Variables*, Seminare D'Algebre, Lecture Notes in Mathematics (Springer, Berlin), Vol. 1146, pp. 161–175.
4. Macdonald, I. (1991) *Notes on Schubert Polynomials*, Publications du LACIM, (Univ. du Québec, Montreal), Vol. 6.
5. Fomin, S. & Kirillov, A. N. (1996) *Lett. Math. Phys.* **37,** 273–284.
6. Humphreys, J. E. (1990) *Reflection Groups and Coxeter Groups* (Cambridge Univ. Press, Cambridge, U.K.).
7. Stembridge, J. (1993) *Coxeter–Yang–Baxter Equations*, manuscript.
8. Lascoux, A. & Schützenberger, M.-P. (1982) *C. R. Seances Acad. Sci. Ser. I* **294,** 447–450.
9. Billey, S. & Haiman, M. (1995) *J. Am. Math. Soc.* **8,** 443–482.
10. Fomin, S. & Kirillov, A. N. (1996) *Trans. Am. Math. Soc.* **348,** 3591–3620.
11. Fulton, W. (1996) *J. Diff. Geometry* **43**, 276–290.
12. Fomin, S. & Kirillov, A. N. (1996) *Discrete Math.* **153,** 123–143.