Proc. Natl. Acad. Sci. USA
Vol. 94, pp. 33–35, January 1997
Mathematics

L^p estimates for the bilinear Hilbert transform

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Communicated by Charles Fefferman, Princeton University, Princeton, NJ, November 6, 1996 (received for review July 12, 1996)

ABSTRACT **For the bilinear Hilbert transform given by:**

$$\boldsymbol{Hfg(x) = \text{p.v.}\int f(x-y)g(x+y)\frac{dy}{y}},$$

we announce the inequality $\|Hfg\|_{p_3} \le K_{p_1,p_2}\|f\|_{p_1}\|g\|_{p_2}$, provided $2 < p_1, p_2 < \infty$, $1/p_3 = 1/p_1 + 1/p_2$ and $1 < p_3 < 2$.

We announce a partial resolution to long standing conjectures concerning the operator known as the bilinear Hilbert transform, defined as follows:

$$H\,fg(x) = \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} f(x-y)g(x+y)\frac{dy}{y}.$$

This operation is initially defined only for certain functions f and g, for instance those in the Schwartz class on $\mathbb{R}$. The conjectures concern the extension of H to a bounded operator on L^p spaces. We have proved:

THEOREM 1. H *extends to a bounded operator on* $\mathrm{L}^{p_1} \times \mathrm{L}^{p_2}$ *into* L^{p_3}, *provided* $2 < p_1, p_2 < \infty$ *and* $1 < p_3 < 2$, *where* $1/p_3 = 1/p_1 + 1/p_2$.

Some 30 years ago, in connection with the Cauchy integral on Lipschitz curves, Calderón (1) raised the question of H mapping $L^2 \times L^2$ into L^1; this inequality is true. Indeed, the bilinear Hilbert transform maps into L^{p_3} provided only that $p_3 > 2/3$.

Study of the bilinear Hilbert transform is intimately related to Carleson's theorem (2) asserting the pointwise convergence of Fourier series. A seminal result, it has received two proofs, with the alternative proof provided by Fefferman (3). These proofs have provided us with ingenious and complementary methods of time frequency analysis. A similar analysis seems necessary to understand H, and so our proof entails significant aspects of both Carleson's and Fefferman's proofs. We give a description of our proof, with details presented in their most concrete form. Complete proofs, which appear in ref. 4, require definitions and constructions somewhat more general than those presented here.

The bilinear Hilbert transform must be broken into scales and the frequency behavior of each scale understood. Hence we replace the kernel $1/y$ with $\sum_{j=-\infty}^{\infty} 2^j\rho(2^jy)$, where ρ is a Schwartz function with Fourier transform $\hat\rho(\xi) = \int e^{-2\pi ix\xi}\rho(x)\,dx$ supported on $[1/2, 2)$. For each j, consider:

$$H_j\,fg(x) = \int f(x-y)g(x+y)2^j\rho(2^jy)dy,$$

which has bilinear symbol $\hat\rho(2^{-j}(\xi - \theta))$. More specifically,

$$H_j\,fg(x) = \int\int \hat f(\theta)\hat g(\xi)e^{2\pi i(\xi+\theta)x}\hat\rho(2^{-j}(\xi-\theta))\,d\xi d\theta.$$

Therefore, if f is supported in frequency on the interval $[n2^j, (n+1)2^j]$, then $H_j\,fg(x)$ acts on the inverse Fourier transform of $\hat g(\xi)\mathbf{1}[(n+1/2)2^j, (n+3)2^j](\xi)$, and is supported in frequency on the interval $[(2n+1/2)2^j, (2n+4)2^j]$. The differing rates of translation make these three intervals distinct.

It is important to note that the location of the intervals is arbitrary, and therefore, for all j and j', the inner product of $H_j\,fg$ and $H_{j'}\,fg$ need not tend to zero as $|j - j'|$ tends to infinity. The analysis of H must be done in terms of both time and frequency.

Instead of proceeding with a decomposition of H, we define a model of it adapted to the combinatorics of the time–frequency plane. Let $\mathcal{D}$ be a dyadic grid in $\mathbb{R}$. Call $I \times \omega \in \mathcal{D} \times \mathcal{D}$ a tile if $|I|\cdot|\omega| = 1$. The interval ω is a union of four dyadic subintervals of equal length, ω_1, ω_2, ω_3, and ω_4, which we list in ascending order. Thus, $\xi_i < \xi_j$ for all $1 \le i < j \le 4$ and $\xi_j \in \omega_j$. (We will only use ω_j for $j = 1, 2, 3$.) We adopt the notation $t = I_t \times \omega_t$ and $tj = I_t \times \omega_{tj}$ for $j = 1, 2, 3$. Fix a Schwartz function ϕ with $\hat\phi$ supported on $[-1/8, 1/8]$, in addition require that $\int \phi(x - 16n)\phi(x)\,dx = 0$ for all integers n. Set for all tiles t and $j = 1, 2, 3$,

$$\phi_{tj}(x) = \frac{e^{2\pi ic(\omega_{tj})x}}{\sqrt{|I_t|}}\,\phi\!\left(\frac{x - c(I_t)}{|I_t|}\right),$$

where $c(J)$ denotes the center of the interval J.

Then our model of the bilinear Hilbert transform is

$$\mathcal{M}f_1f_2(x) = \sum_t \frac{\langle f_1, \phi_{t1}\rangle}{\sqrt{|I_t|}}\langle f_2, \phi_{t2}\rangle\phi_{t3}(x),$$

which is initially defined only for Schwartz functions f_1 and f_2. We emphasize that the sum extends over all tiles, and hence all scales. The analogue of *Theorem 1* is

THEOREM 2. $\mathcal{M}$ extends to a bounded operator on $\mathrm{L}^{p_1} \times \mathrm{L}^{p_2}$ *into* $\mathrm{L}^{p_3'}$, *provided* $2 < p_1, p_2 < \infty$ *and* $1 < p_3' = (1/p_1 + 1/p_2)^{-1} < 2$.

With more liberal notions of "grid," "tile," and "ϕ_{tj}," the bilinear Hilbert transform is in the convex hull of terms like our model $\mathcal{M}$.

In the present situation we can give a proof by way of duality. Thus we take $f_3 \in L^{p_3}$ and show that:

$$\sum_t \left|\frac{\langle f_1, \phi_{t_1}\rangle}{\sqrt{|I_t|}}\langle f_2, \phi_{t_2}\rangle\langle f_3, \phi_{t_3}\rangle\right| \le K\prod_{j=1}^{3}\|f_j\|_{p_j}.$$

The sum is over positive quantities; namely, the decomposition above already captures all of the cancellation necessary for convergence of the sums. It also shows that the sum defining $\mathcal{M}$ is unconditionally convergent in t. And as each $f_j \in L^{p_j}$,

0027-8424/97/9433-3$2.00/0
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Abbreviation: CF, Carleson–Fefferman.

where $p_j > 2$, it follows that each function is locally square integrable. As it turns out, L^2 arguments are decisive in proving *Theorem 2.*

We localize the sum above in the x variable by setting:

$$F_t(x) = \prod_{j=1}^{3} \frac{|\langle f_j,\, \phi_{tj}\rangle|}{\sqrt{|I_t|}}\, \mathbf{1}[I_t](x).$$

Certainly $\int F_t(x)\, dx = |I_t|^{-1/2} \Pi_{j=1}^{3} |\langle f_j,\, \phi_{tj}\rangle|$. And so we show that $\mathbf{F}(x) = \Sigma_t F_t(x)$ is integrable. This follows from a weak-type result: for p_j as above, there is a $\delta > 0$ so that for all $|r_j - p_j| < \delta$, $1 \le j \le 3$, the operator $\mathbf{F}(x)$ maps $L^{r_1} \times L^{r_2} \times L^{r_3}$ into $L^{r,\infty}$, where $1/r = 1/r_1 + 1/r_2 + 1/r_3$. Then a variant of the Marcinkiewicz interpolation theorem due to Janson (5) implies the strong-type inequality.

A single instance of the weak-type inequality is:

$$|\{x|\mathbf{F}(x) > K\prod_{j=1}^{3}\|f_j\|_{r_j}\}| \le K, \qquad \textbf{[1]}$$

for some constant K. But this inequality implies the weak-type result, because $\mathbf{F}$ commutes with dilations by powers of 2, and so it suffices to establish this last inequality. These observations are useful since some of our estimates begin to break down on exceptional sets of small measure. Due the localization of F_t in the time variable and the fact that we only aim for a distributional inequality, we can delete tiles t whose time coordinate falls in a set of bounded measure.

The combinatorics of the time frequency plane enter in by way of the partial order on the tiles given by $t < t'$ if $I_t \subset I_{t'}$ and $\omega \supset \omega'$. Note that t and t' are not comparable with respect to $<$ if and only if $t \cap t' = \emptyset$. Being disjoint suggests orthogonality for the functions ϕ_{tj} and $\phi_{t'j'}$, the dominant theme of *Lemmas 1–3* we state below.

Call a collection of tiles $\mathbf{T}$ a Carleson–Fefferman (CF) set with top q if $t < q$ for all $t \in \mathbf{T}$. Thus $\omega_q \cap \omega_t \neq \emptyset$ for $t \in \mathbf{T}$. Call $\mathbf{T}$ a j-CF set if $\mathbf{T}$ is a CF-set for which the intervals ω_{tj} intersect for all $t \in \mathbf{T}$. Notice that if $\mathbf{T}$ is a 1-CF set, say, then the intervals $\{\omega_{tj}|t \in \mathbf{T}\}$ are pairwise disjoint for $j = 2, 3$. Therefore, by application of Cauchy–Schwartz:

$$|I_q|^{-1}\left\|\sum_{t\in\mathbf{T}} F_t(x)\right\|_1 \le \sup_{t\in\mathbf{T}} \frac{|\langle f_1,\, \phi_{t1}\rangle|}{\sqrt{|I_t|}} \times \prod_{j=2}^{3}\left[\left|I_q\right|^{-1}\sum_{t\in\mathbf{T}}\left|\langle f_j,\, \phi_{tj}\rangle\right|^2\right]^{1/2}. \qquad \textbf{[2]}$$

Notice that the last two square functions are Littlewood–Paley g functions, albeit conjugated by an exponential to account for the location of the CF set in frequency.

This estimate forms the motivation for *Lemma 1* below, which formalizes a decomposition of the set of tiles that is fundamental to our argument.

LEMMA 1. Fix $p_i > 2$. *There is a* $\delta > 0$ *and an* $\varepsilon_0 > 0$ *and a constant* K *so that for all* $|r_i - p_i| < \delta$ *and* $0 < \varepsilon < \varepsilon_0$, *the following holds. The collection of all tiles* $\mathbf{S}$ *is a union:*

$$\mathbf{S} = \mathbf{S}^0 \cup \bigcup_{n=0}^{\infty}\ \bigcup_{i,j=1}^{3} \mathbf{S}_{n,i,j},$$

with these properties. First, $\mathbf{S}^0$ *is trivial in that:*

$$\left|\bigcup_{s\in\mathbf{S}^0} I_s\right| \le K. \qquad \textbf{[3]}$$

Then $\mathbf{S}_{n,i,j}$ *is a union of disjoint* i-*CF sets* $\mathbf{T}_q$ *with tops* $q \in \mathbf{S}^*_{n,i,j}$, *and:*

$$\sum_{t\in\mathbf{T}_q} F_t(x) \le 2^{-n(1/r-\varepsilon)}\ \textit{for all}\ x,\ q\in\mathbf{S}^*_{n,i,j}. \qquad \textbf{[4]}$$

Here, recall that $1/r = \Sigma_i 1/r_i$, *which can be taken arbitrarily close to* 1. *And, most significantly, for* $t = min_i\{p_i/2\} - \varepsilon$,

$$\|N_{n,i,j}\|_t = \left\|\sum_{q\in\mathbf{S}^*_{n,i,j}} \mathbf{1}_{I_q}\right\|_t \le K2^{n(1/t+K\varepsilon)}. \qquad \textbf{[5]}$$

With *Lemma 1* in place, we estimate:

$$\sum_{n=0}^{\infty}\sum_{i,j=1}^{3}\left\|\sum_{t\in\mathbf{S}_{n,i,j}} F_t(x)\right\|_t \le \sum_{n=0}^{\infty} 2^{-n(1/r-\varepsilon)} \sum_{i,j=1}^{3}\|N_{n,i,j}\|_t \le K\sum_{n=0}^{\infty} 2^{-n(1/r-1/t-C\varepsilon)} \le K'.$$

The last sum is finite as r is arbitrarily close to one, while $t + \varepsilon = \min\{p_i/2\} > 1$ is a fixed distance from one. Therefore, with Eq. **3**, Eq. **1** holds.

We cannot give the complete construction of the $\mathbf{S}_{n,i,j}$, but rather the initial steps, in which the nearly orthogonal classes of ϕ_{ti} are identified. First, we make an important comparison to a maximal function. If $\mathbf{T}_q$ is an i-CF set with top q, we have for $j \neq i$,

$$\Delta(\mathbf{T}_q, j) = \left[\frac{1}{|I_q|}\sum_{t\in\mathbf{T}_q}\left|\langle f_j,\, \phi_{tj}\rangle\right|^2\right]^{1/2} \le C \inf_{x\in I_q} M_2 f_j(x).$$

Here $M_2 g$ is the maximal function $(M|g|^2)^{1/2}$. Thus the set $F = \cup_i\{M_2 f_i > C^{-1}\}$ has bounded measure and we define $\mathbf{S}^0 = \{s|I_s \subset F\}$, making Eq. **4** trivial. For all i-CF sets $\mathbf{T}$ with top q, and $\mathbf{T} \subset \mathbf{S}\backslash\mathbf{S}^0$, we have $\Delta(\mathbf{T}, j) \le 1$, for $j \neq i$.

The remaining construction is inductive. Assume that the $\mathbf{S}_{m,i,j}$ are defined for all $m < n$ and all i, j, in such a way that for $\mathbf{S}^r = \mathbf{S}\backslash\cup_{m<n}\cup_{i,j}\mathbf{S}_{m,i,j})$ we have:

$$\frac{|\langle f_i,\, \phi_{ti}\rangle|}{\sqrt{|I_t|}} \le 2^{-n/r_i} \in \mathbf{S}^r, \qquad i = 1, 2, 3, \qquad \textbf{[6]}$$

and for any i-CF set $\mathbf{T}_q \subset \mathbf{S}^r$ with top q, $\Delta(\mathbf{T}_q, j) \le 2^{-n/r_j+2}$, for $j \neq i$. As the same inequality applies to each sub-CF set of T, we conclude that:

$$\left\|\left[\sum_{t\in\mathbf{T}_q}\left|\frac{\langle f_j,\, \phi_{tj}\rangle}{\sqrt{|I_t|}}\right|^2 \mathbf{1}_{I_t}\right]^{1/2}\right\|_{Dyadic\ BMO} \le K2^{-n/r_j}. \qquad \textbf{[7]}$$

We define $\mathbf{S}^*_{n,1,1}$ to be the set of maximal tiles q with $|\langle f_1,\, \phi_{q,1}\rangle| \ge 2^{-n/r_1-1}\sqrt{|I_q|}$, and take $\mathbf{S}_{n,1,1}$ to consist of all tiles t so that $t1 < q$ for some $q \in \mathbf{S}^*_{n,1,1}$. These tiles are removed, and then $\mathbf{S}_{n,i,i}$ is defined similarly for $i = 2, 3$. After the deletion of the tiles $\mathbf{D}_0 = \cup_{i=1}^{3}\mathbf{S}_{n,i,i}$, we have $|\langle f_i,\, \phi_{ti}\rangle| \le 2^{-n/r_i-1}\sqrt{|I_t|}$ for all tiles $t \in \mathbf{S}^{r'} = \mathbf{S}^r\backslash\mathbf{D}_0$.

The set $\mathbf{S}_{n,1,2}$ has a slightly different construction. Consider 1-CF sets $\mathbf{T}_q \subset \mathbf{S}^{r'}$ with top q so that $\Delta(\mathbf{T}, 2) \ge 2^{-n/r_2+1}$. We take $\mathbf{T}_q$ to be the maximal 1-CF set with this property. Let $q(1)$ be such a top, which is maximal with respect to $<$, and in addition $\sup\{\xi|\xi \in \omega_q\}$ is maximal. Remove the tiles $\mathbf{T}_{q(1)}$, and repeat this procedure to define $\mathbf{T}_{q(2)}$ and so on. $\mathbf{S}_{n,1,2}$ is then $\cup_\ell \mathbf{T}_{q(\ell)}$ and $\mathbf{S}^*_{n,1,2} = \{q(\ell)|\ell \ge 1\}$. Observe that for any 1-CF set $\mathbf{T} \subset \mathbf{S}^{r'}\backslash\mathbf{S}_{n,1,2}$, we have $\Delta(\mathbf{T}, 2) \le 2^{-n/r_2+1}$. These procedures are repeated inductively to define the $\mathbf{S}_{n,i,j}$ for all n, i, j.

With the construction above it is elementary to check that these properties hold.

$$\frac{|\langle f_i,\, \phi_{qi}\rangle|}{\sqrt{|I_q|}} \ge 2^{-n/r_i-1}, \qquad q \in \mathbf{S}^*_{n,i,i}, \qquad \textbf{[8]}$$

And in the case of $i \neq j$, the collection $\mathbf{S}_{n,i,j}$ is a union of disjoint i-CF sets $\mathbf{T}_q$, with $q \in \mathbf{S}^*_{n,i,j}$, for which:

$$\frac{|\langle f_i, \phi_{tj}\rangle|}{\sqrt{|I_t|}} \leq 2^{-n/r_i-1},\ t \in \mathbf{T}_q \text{ and } \Delta(\mathbf{T}_q, j) \geq 2^{-n/r_j+1}. \quad [9]$$

These last two bounds differ by a factor of 4, which is relevant below. See the comments concerning the minimal tiles immediately following *Lemma 3* below. To achieve Eq. **4**, one must delete some tiles t, using Eq. **2**, the upper bounds Eqs. **6** and **7**, and the control on the number of trees given in Eq. **5**.

The essence of the matter lies in the control of the number of CF-sets, which is in the verification of Eq. **5**. Eq. **5** relies upon the inequalities in the previous paragraph and *Lemma 2* and *3* below, which address the issue of almost orthogonality.

Let us consider $\mathbf{S}_{n,1,1}$, say. The tiles $\mathbf{S}^*_{n,1,1}$ are maximal and therefore pairwise disjoint, which suggest weak orthogonality for the collection of functions $\{\phi_{q,1} | q \in \mathbf{S}^*_{n,1,1}\}$. If they were in fact orthogonal, Bessel's inequality and Eq. **8** implies:

$$\sum_{q\in\mathbf{S}^*_{n,1,1}} |I_q| \leq 2^{2n/r_i+2} \sum_{q\in\mathbf{S}^*_{n,1,1}} |\langle f_1, \phi_{q1}\rangle|^2 \leq 2^{2n/r_i+2}\|f_1\|_2^2.$$

While f_1 is not in L^2, this inequality can be strengthened to an analogous form for L^r for $r > 2$.

However, disjointness of tiles does not imply orthogonality, because the functions $\phi_{q,1}$ are not compactly supported in the x variable. Indeed, by our choice of ϕ, for two tiles t and s we have $\langle\phi_{ti}, \phi_{si}\rangle = 0$ if $\omega_{ti} \cap \omega_{si} = \emptyset$ or $\omega_{ti} = \omega_{si}$. But if $\omega_{ti} \subset_{\neq} \omega_{si}$ then for all $n \geq 0$:

$$|\langle\phi_{ti}, \phi_{si}\rangle| \leq C_n \sqrt{\frac{|I_s|}{|I_t|}}\left(1 + \frac{\operatorname{dist}(I_t, I_s)}{|I_t|}\right)^{-n}. \quad [10]$$

If we assume that a stronger separation of the tiles in the x variable, then we would expect orthogonality. And in this direction we have:

LEMMA 2. *For* n $\geq$ 1 *there are constants* K *and* K_n *so that the following holds for all* A $\geq$ 1. *Let* **S** *be any collection of tiles so that:*

$$\{AI_t \times \omega_{ti} | t \in \mathbf{S}\} \text{ are pairwise disjoint.} \quad [11]$$

Here, for an interval I, AI *denotes the interval with the same center as* I *and length* A|I|. *Set* $N_{\mathbf{S}}(x) = \sum_{t\in\mathbf{S}} \mathbf{1}_{I_t}(x)$. *Then:*

$$\sum_{t\in\mathbf{S}} |\langle f, \phi_{ti}\rangle|^2 \leq K(1 + K_n A^{-n}\|N_{\mathbf{S}}\|_\infty)\|f\|_2^2.$$

A further combinatorial lemma asserts that if the tiles $\{ti | t \in \mathbf{S}'\}$ are merely disjoint, then after deleting tiles t for which I_t falls in an exceptional set of small measure, $\mathbf{S}'$ is a union of $O(A^3)$ collections of tiles **S** that satisfy the stronger disjointness condition (Eq. **11**).

The previous lemma is essential in obtaining Eq. **5** for the classes $S_{n,i,i}$. A corresponding lemma is necessary for the $S_{n,i,j}$, with $i \neq j$, with Eq. **9** replacing the role of Eq. **8**. It is:

LEMMA 3. For n $\geq$ 1 *there are constants* K *and* K_n *so that the following holds for all* A $\geq$ 1. *Let* **S** *be a union of* j-*CF sets* $\mathbf{T}_q$ *with tops* q $\in$ **S***. *Suppose that:*

$$AI_t \subset I_q \text{ for all } t \in \mathbf{T}_q \text{ and } q \in \mathbf{S}^*,$$

and for t $\in \mathbf{T}_q$, q $\in$ **S*** *and* i $\neq$ j *fixed,*

$$\text{if } \omega_{ti} \subset_{\neq} \omega_{si} \text{ for some } s \in \mathbf{S}, \text{ then } I_q \cap I_s = \emptyset. \quad [12]$$

Set $N_{\mathbf{S}}(x) = \sum_{q\in\mathbf{S}^*} \mathbf{1}_{I_q}(x)$. *Then:*

$$\sum_{t\in\mathbf{S}} |\langle f, \phi_{ti}\rangle|^2 \leq K(1 + K_n A^{-n}\|N_{\mathbf{S}}\|_\infty)\|f\|_2^2.$$

Notice that in a j-CF set $\mathbf{T}_q$, the tiles $\{\omega_{ti} | t \in \mathbf{T}_q\}$ are pairwise disjoint. Thus Eq. **12** is stronger than merely asserting that the tiles $\{\omega_{ti} | t \in \mathbf{T}\}$ are pairwise disjoint. With the construction of the $\mathbf{S}_{n,i,j}$ for $i \neq j$ given above, Eq. **12** is true after deleting the minimal tiles $\mathbf{S}^{\min}_{n,i,j}$ in $\mathbf{S}_{n,i,j}$. The minimal tiles are controlled with the first half of Eq. **9** and the observation that $\sum_{s\in\mathbf{S}^{\min}_{n,i,j}} \mathbf{1}_{I_s}(x) \leq N_{n,i,j}(x)$ for all x.

The method of proof of both *Lemmas 2* and *3* is similar. For instance, in *Lemma 2*, one considers the operator:

$$\mathcal{S}_{\mathbf{S}}f(x) = \sum_{t\in\mathbf{S}} \langle f, \phi_{ti}\rangle \phi_{ti}(x).$$

If **S** is finite, this is a compact self-adjoint operator, with maximal eigenvalue B. It suffices to estimate B, as for all $f \in L^2$, $\sum_{t\in\mathbf{S}} |\langle f, \phi_{ti}\rangle|^2 = \langle f, \mathcal{S}_{\mathbf{S}}f\rangle \leq B\|f\|_2^2$. Consider a normalized extremal eigenfunction f of $\mathcal{S}_{\mathbf{S}}$. One then estimates:

$$B^2 = \|\mathcal{S}_{\mathbf{S}}f\|_2^2 = \sum_{t\in\mathbf{S}}\sum_{s\in\mathbf{S}} \langle f, \phi_{ti}\rangle\langle\phi_{ti}, \phi_{si}\rangle\langle\phi_{si}, f\rangle,$$

which is expanded in diagonal and off-diagonal terms. The diagonal term is $\sum_{t\in\mathbf{S}} |\langle f, \phi_{ti}\rangle|^2 = \langle f, \mathcal{S}_{\mathbf{S}}f\rangle \leq B$, which is an adequate estimate for B^2. The off-diagonal term is by Cauchy–Schwarz,

$$\mathbb{O} \leq 2\sum_{t\in\mathbf{S}} |\langle f, \phi_{ti}\rangle| \sum_{\substack{s\in\mathbf{S} \\ \omega_{ti}\subset_{\neq}\omega_{si}}} |\langle\phi_{ti}, \phi_{si}\rangle\langle\phi_{si}, f\rangle|$$

$$\leq 2B^{1/2}\left[\sum_{t\in\mathbf{S}}\left[\sum_{\substack{s\in\mathbf{S} \\ \omega_{ti}\subset_{\neq}\omega_{si}}} \left|\langle\phi_{ti}, \phi_{si}\rangle\langle f, \phi_{si}\rangle\right|\right]^2\right]^{1/2}.$$

The innermost sum is bounded by $C_n A^{-n}\sqrt{|I_t|}\, \inf_{x\in I_t} MMf(x)$. This is seen by invoking the estimate $|\langle f, \phi_{si}\rangle| \leq K\sqrt{|I_s|}\, \inf_{x\in I_s} Mf(x)$, using Eq. **10** and carefully exploiting the geometry of the tiles via the assumption **11**.

One then sees that the off-diagonal term is no more than:

$$\mathbb{O} \leq C_n B^{1/2} A^{-n} \sum_{t\in\mathbf{S}} |I_t| \inf_{x\in I_t} (MMf)^2(x) \leq C_n B^{1/2} A^{-n} \|N_{\mathbf{S}}\|_\infty.$$

This, with the diagonal estimate, proves that $B^2 \leq B + C_n B^{1/2} A^{-n} \|N_{\mathbf{S}}\|_\infty$, whence follows *Lemma 2.*

The research outlined herein is the product of several years of effort (6, 7).

The success of our efforts is due to the guidance and encouragement we have received from R. Coifman. M.L. has been supported by the National Science Foundation. M.L. and C.T. acknowledge the support of a North Atlantic Treaty Organization travel grant.

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