New results of intersection numbers on moduli spaces of curves

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We present a series of results we obtained recently about the intersection numbers of tautological classes on moduli spaces of curves, including a simple formula of the *n*-point functions for Witten's τ classes, an effective recursion formula to compute higher Weil–Petersson volumes, several new recursion formulae of intersection numbers and our proof of a conjecture of Itzykson and Zuber concerning denominators of intersection numbers. We also present Virasoro and KdV properties of generating functions of general mixed κ and ψ intersections.

recursion formulae

Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne–Mumford moduli stack of stable curves of genus g with n marked points. Let ψ_i be the first Chern class of the line bundle whose fiber over each pointed stable curve is the cotangent line at the ith marked point. Let λ_i be the ith Chern class of the Hodge bundle $\mathbb{E}$, whose fiber over each pointed stable curves is $H^0(C, \omega_c)$.

We also have the κ classes originally defined by Mumford (1), Morita (2), and Miller (3). A more natural variation was later given by Arbarello–Cornalba (4). It is known that the κ and ψ classes generate the tautological cohomology ring of the moduli spaces, and most of the known cohomology classes are tautological.

The following intersection numbers

$$\left\langle \tau_{d_1}\cdots\tau_{d_n}\prod_{j\geq 1}\kappa_j^{b_j}\right\rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n}\prod_{j\geq 1}\kappa_j^{b_j}$$

are called the higher Weil–Petersson volumes (5). These are important invariants of moduli spaces of curves.

In 1990, Witten (6) made the remarkable conjecture that the generating function of intersection numbers of ψ classes on moduli spaces are governed by KdV hierarchy. Witten's conjecture (first proved by Kontsevich; ref. 7) is among the deepest known properties of moduli spaces of curves and motivated a surge of subsequent developments.

The intersection theory of tautological classes on the moduli space of curves is a very important subject and has close connections to string theory, quantum gravity and many branches of mathematics.

The *n*-Point Functions for Intersection Numbers

Definition 1: We call the following generating function

$$F(x_1,\ldots,x_n)=\sum_{g=0}^{\infty}\sum_{\sum d_j=3g-3+n}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g\prod_{j=1}^{n}x_j^{d_j}$$

the n-point function.

Consider the following "normalized" n-point function

$$G(x_1,\ldots,x_n)=\exp\left(\frac{-\sum_{j=1}^n x_j^3}{24}\right)\cdot F(x_1,\ldots,x_n).$$

Starting from 1-point function $G(x) = 1/x^2$, we can obtain any n-point function recursively by the following theorem.

Theorem 1 (8). *For $n \geq 2$,*

$$G(x_1,\ldots,x_n)=\sum_{r,s\geq 0}\frac{(2r+n-3)!!P_r(x_1,\ldots,x_n)\Delta(x_1,\ldots,x_n)^s}{4^s(2r+2s+n-1)!!},$$

where P_r and Δ are homogeneous symmetric polynomials defined by

$$\Delta(x_1,\ldots,x_n)=\frac{\left(\sum_{j=1}^n x_j\right)^3-\sum_{j=1}^n x_j^3}{3},$$

$$P_r(x_1,\ldots,x_n)=\left(\frac{1}{2\sum_{j=1}^n x_j}\sum_{\underline{n}=I\amalg J}\left(\sum_{i\in I}x_i\right)^2\left(\sum_{i\in J}x_i\right)^2 G(x_I)G(x_J)\right)_{3r+n-3}$$

$$=\frac{1}{2\sum_{j=1}^n x_j}\sum_{\underline{n}=I\amalg J}\left(\sum_{i\in I}x_i\right)^2\left(\sum_{i\in J}x_i\right)^2\sum_{r'=0}^{r}G_{r'}(x_I)G_{r-r'}(x_J),$$

where $I, J \neq \theta$, $\underline{n} = \{1, 2, \ldots, n\}$, and $G_g(x_I)$ denotes the degree $3g + |I| - 3$ homogeneous component of the normalized $|I|$-point function $G(x_{k_1},\ldots,x_{k_{|I|}})$, where $k_j \in I$.

Thus, we have an elementary and more efficient algorithm to calculate all intersection numbers of ψ classes other than the celebrated Witten–Kontsevich theorem.

Because $P_0(x, y) = \frac{1}{x+y}$, $P_r(x, y) = 0$ for $r > 0$ and

$$P_r(x,y,z)=\frac{(xy)^r(x+y)^{r+1}+(yz)^r(y+z)^{r+1}+(zx)^r(z+x)^{r+1}}{4^r(2r+1)!!(x+y+z)},$$

we recover Dijkgraaf's 2-point function and Zagier's 3-point function obtained years ago.

There is another slightly different formula of the n-point functions. When $n = 3$, this has also been obtained by Zagier.

Theorem 2 (8). *For $n \geq 2$,*

$$F(x_1,\ldots,x_n)=\exp\frac{\left(\sum_{j=1}^n x_j\right)^3}{24}\times\sum_{r,s\geq 0}\frac{(-1)^s P_r(x_1,\ldots,x_n)\Delta(x_1,\ldots,x_n)^s}{8^s(2r+2s+n-1)s!},$$

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where P_r and Δ are the same polynomials as defined in Theorem 2.

Okounkov (9) obtained an analytic expression of the n-point functions using n-dimensional error-function-type integrals. Brézin and Hikami (10) use correlation functions of GUE ensemble to find explicit formulae of n-point functions.

Higher Weil–Petersson Volumes

We have discovered a general recursion formula of higher Weil–Petersson volumes (11), which is a vast generalization of the Mirzakhani's recursion formula (12).

First we fix notations as in ref. 5. Consider the semigroup N^∞ of sequences $\mathbf{m} = (m_1, m_2, \ldots)$, where m_i are nonnegative integers and $m_i = 0$ for sufficiently large i.

Let $\mathbf{m}, \mathbf{t}, \mathbf{a_1}, \ldots, \mathbf{a_n} \in N^\infty$ and $\mathbf{s} := (s_1, s_2, \ldots)$ be a family of independent formal variables.

$$|\mathbf{m}| := \sum_{i\geq 1} i m_i,\ \|\mathbf{m}\| := \sum_{i\geq 1} m_i,\ \mathbf{s}^{\mathbf{m}} := \prod_{i\geq 1} s_i^{m_i},\ \mathbf{m}! := \prod_{i\geq 1} m_i!,$$

$$\binom{\mathbf{m}}{\mathbf{t}} := \prod_{i\geq 1}\binom{m_i}{t_i},\ \binom{\mathbf{m}}{\mathbf{a_1}, \ldots, \mathbf{a_n}} := \prod_{i\geq 1}\binom{m_i}{a_1(i), \ldots, a_n(i)}.$$

Let $\mathbf{b} \in N^\infty$, we denote a formal monomial of κ classes by

$$\kappa(\mathbf{b}) := \prod_{i\geq 1} \kappa_i^{b_i}.$$

Theorem 3 (11). *Let $\mathbf{b} \in N^\infty$ and $d_j \geq 0$.*

$$(2d_1+1)!!\left\langle \kappa(\mathbf{b}) \prod_{j=1}^{n} \tau d_j \right\rangle_g$$

$$= \sum_{j=2}^{n} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2(|\mathbf{L}|+d_1+d_j)-1)!!}{(2d_j-1)!!}$$

$$\times \left\langle \kappa(\mathbf{L}')\tau_{|L|+d_1+d_j-1} \prod_{i\neq 1,j} \tau_{d_i} \right\rangle_g$$

$$+ \frac{1}{2} \sum_{\mathbf{L}+\mathbf{L}'=b} \sum_{r+s=|\mathbf{L}|+d_1-2} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} (2r+1)!!(2s+1)!!$$

$$\times \left\langle \kappa(\mathbf{L}')\tau_r\tau_s \prod_{i\neq 1} \tau_{d_i} \right\rangle_{g-1}$$

$$+ \frac{1}{2} \sum_{\substack{\mathbf{L}+\mathbf{e}+\mathbf{f}=\mathbf{b} \\ I\amalg J=\{2,\ldots,n\}}} \sum_{r+s=|\mathbf{L}|+d_1-2} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}, \mathbf{e}, \mathbf{f}} (2r+1)!!(2s+1)!!$$

$$\times \left\langle \kappa(\mathbf{e})\tau_r \prod_{i\in I} \tau_{d_i} \right\rangle_{g'} \left\langle \kappa(\mathbf{f})\tau_s \prod_{i\in J} \tau_{d_i} \right\rangle_{g-g'},$$

where the tautological constants $\alpha_{\mathbf{L}}$ can be determined recursively from the following formula

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \frac{(-1)^{\|\mathbf{L}\|}\alpha_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!} = 0,\ \mathbf{b}\neq 0,$$

namely

$$\alpha_b = \mathbf{b}! \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \mathbf{L}'\neq 0}} \frac{(-1)^{\|\mathbf{L}'\|-1}\alpha_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!},\ \mathbf{b}\neq 0,$$

with the initial value $\alpha_0 = 1$.

The proof of the above theorem is to use Witten–Kontsevich theorem, a combinatorial formula in ref. 5 expressing κ classes by ψ classes and the following elementary but crucial lemma (11).

Lemma 1. *Let $F(\mathbf{L}, n)$ and $G(\mathbf{L}, n)$ be two functions defined on $N^\infty \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of nonnegative integers. Let α_L and β_L be real numbers depending only on $\mathbf{L} \in N^\infty$ that satisfy $\alpha_0\beta_0 = 1$ and*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_{\mathbf{L}}\beta_{\mathbf{L}'} = 0,\ \mathbf{b}\neq 0.$$

Then the following two identities are equivalent.

$$G(\mathbf{b}, n) = \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_{\mathbf{L}} F(\mathbf{L}', n+|\mathbf{L}|), \quad \forall\ (\mathbf{b}, n) \in N^\infty \times \mathbb{N}$$

$$F(\mathbf{b}, n) = \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \beta_{\mathbf{L}} G(\mathbf{L}', n+|\mathbf{L}|), \quad \forall\ (\mathbf{b}, n) \in N^\infty \times \mathbb{N}$$

When $\mathbf{b} = (l, 0, 0, \ldots)$, Theorem 3 recovers Mirzakhani's recursion formula of Weil–Petersson volumes for moduli spaces of bordered Riemann surfaces (12–17).

Theorem 3 also provides an effective algorithm to compute higher Weil–Petersson volumes recursively.

In fact, we can use the main formula in ref. 5 to generalize almost all pure ψ intersections to identities of higher Weil–Petersson volumes that share similar structures as Theorem 3. For example, the identities in the following theorem are generalizations of the string and dilation equations.

Theorem 4 (11). *For $\mathbf{b} \in N^\infty$ and $d_j \geq 0$,*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{\|\mathbf{L}\|} \binom{\mathbf{b}}{\mathbf{L}} \left\langle \tau_{|\mathbf{L}|} \prod_{j=1}^{n} \tau_{d_j}\kappa(\mathbf{L}') \right\rangle_g$$

$$= \sum_{j=1}^{n} \left\langle \tau_{d_{j-1}} \prod_{i\neq j} \tau_{d_i}\kappa(\mathbf{b}) \right\rangle_g,$$

and

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{\|\mathbf{L}\|} \binom{\mathbf{b}}{\mathbf{L}} \left\langle \tau_{|\mathbf{L}|+1} \prod_{j=1}^{n} \tau_{d_j}\kappa(\mathbf{L}') \right\rangle_g$$

$$= (2g-2+n) \left\langle \prod_{j=1}^{n} \tau_{d_j}\kappa(\mathbf{b}) \right\rangle_g.$$

Note that Theorem 4 generalizes the results in ref. 18.

New Identities of Intersection Numbers

The next two theorems follow from a detailed study of coefficients of the n-point functions in Theorem 1.

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Theorem 5 (8). *We have*

1. *Let $k > 2g$, $d_j \geq 0$ and $\sum_{j=1}^{n} d_j = 3g + n - k$.*

$$\sum_{\underline{n}=I \amalg J}\sum_{j=0}^{k}(-1)^j\left\langle \tau_j\tau_0^2\prod_{i\in I}\tau_{d_i}\right\rangle_{g'}\left\langle \tau_{k-j}\tau_0^2\prod_{i\in J}\tau_{d_i}\right\rangle_{g-g'}=0.$$

2. *Let $d_j \geq 1$ and $\sum_{j=1}^{n} d_j = g + n$.*

$$\sum_{\underline{n}=I \amalg J}\sum_{j=0}^{2g}(-1)^j\left\langle \tau_j\tau_0^2\prod_{i\in I}\tau_{d_i}\right\rangle_{g'}\left\langle \tau_{2g-j}\tau_0^2\prod_{i\in J}\tau_{d_i}\right\rangle_{g-g'}$$

$$=\frac{(2g+n+1)!}{4^g(2g+1)!\prod_{j=1}^{n}(2d_j-1)!!}.$$

Theorem 6 (8, 19). *We have*

1. *Let $k > 2g$, $d_j \geq 0$ and $\sum_{j=1}^{n} d_j = 3g + n - k - 1$.*

$$\sum_{j=0}^{k}(-1)^j\left\langle \tau_{k-j}\tau_j\prod_{i=1}^{n}\tau_{d_i}\right\rangle_g=0.$$

2. *Let $d_j \geq 1$ and $\sum_{j=1}^{n}(d_j-1) = g - 1$.*

$$\sum_{j=0}^{2g}(-1)^j\left\langle \tau_{2g-j}\tau_j\prod_{i=1}^{n}\tau_{d_i}\right\rangle_g=\frac{(2g+n-1)!}{4^g(2g+1)!\prod_{j=1}^{n}(2d_j-1)!!}.$$

In fact, it's easy to see that Theorems 5 and 6 imply each other through the following proposition.

Proposition 7 (8). *Let $d_j \geq 0$ and $\sum_{j=1}^{n} d_j = g + n$.*

$$\sum_{\underline{n}=I \amalg J}\sum_{j=0}^{2g}(-1)^j\Bigg(\left\langle \tau_j\tau_0^2\prod_{i\in I}\tau_{d_i}\right\rangle\left\langle \tau_{2g-j}\tau_0^2\prod_{i\in J}\tau_{d_i}\right\rangle$$

$$+\left\langle \tau_j\tau_{2g-j}\tau_0^2\prod_{i\in I}\tau_{d_i}\right\rangle\left\langle \tau_0^2\prod_{i\in J}\tau_{d_i}\right\rangle\Bigg)$$

$$=(2g+n+1)\sum_{j=0}^{2g}(-1)^j\left\langle \tau_0\tau_j\tau_{2g-j}\prod_{i=1}^{n}\tau_{d_i}\right\rangle_g.$$

Because $\mathrm{ch}_k(\mathbb{E}) = 0$ for $k > 2g$, $\lambda_g\lambda_{g-1} = (-1)^{g-1}(2g-1)!\mathrm{ch}_{2g-1}(\mathbb{E})$, by Mumford's formula (1) of the Chern character of Hodge bundles, it's not difficult to see that Theorem 6 implies the following theorem.

Theorem 8 (8, 19). *Let k be an even number and $k \geq 2g$, $d_j \geq 0$, $\sum_{j=1}^{n} d_j = 3g + n - k - 2$.*

$$\left\langle\prod_{j=1}^{n}\tau_{d_j}\tau_k\right\rangle_g=\sum_{j=1}^{n}\left\langle\tau_{d_j+k-1}\prod_{i\neq j}\tau_{d_i}\right\rangle_g$$

$$-\frac{1}{2}\sum_{\underline{n}=I \amalg J}\sum_{j=0}^{k-2}(-1)^j\left\langle\tau_j\prod_{i\in I}\tau_{d_i}\right\rangle_{g'}\left\langle\tau_{k-2-j}\prod_{i\in J}\tau_{d_i}\right\rangle_{g-g'}.$$

Note that, when $k = 2g$, the above theorem is equivalent to the following Hodge integral identity (20) (also known as Faber's intersection number conjecture; ref. 21)

$$\int_{\overline{M}_{g,n}}\psi_1^{d_1}\cdots\psi_n^{d_n}\lambda_g\lambda_{g-1}=\frac{(2g-3+n)!|B_{2g}|}{2^{2g-1}(2g)!\prod_{j=1}^{n}(2d_j-1)!!},$$

where $\sum_{j=1}^{n}(d_j-1) = g-2$ and $d_j \geq 1$.

The above $\lambda_g\lambda_{g-1}$ integral follows from degree 0 Virasoro constraints for $\mathbb{P}^2$ announced by Givental (22). However it is very desirable to have a direct proof of Theorem 8 when $k = 2g$, possibly using our explicit formulae of the n-point functions (see also ref. 23).

As pointed out in the last section, we can generalize all of the above new recursion formulae of ψ classes to identities of higher Weil–Petersson volumes. For example, we may generalize Proposition 7 and Theorem 8 to the following.

Proposition 9 (11). *Let $\mathbf{b} \in N^\infty$, $d_j \geq 0$.*

$$\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ \underline{n}=I \amalg J}}\sum_{j=0}^{2g}(-1)^j\binom{\mathbf{b}}{\mathbf{L}}\times\Bigg(\left\langle\tau_j\tau_0^2\prod_{i\in I}\tau_{d_i}\kappa(\mathbf{L})\right\rangle$$

$$\left\langle\tau_{2g-j}\tau_0^2\prod_{i\in J}\tau_{d_i}\kappa(\mathbf{L}')\right\rangle+\left\langle\tau_j\tau_{2g-j}\tau_0^2\prod_{i\in I}\tau_{d_i}\kappa(\mathbf{L})\right\rangle$$

$$\left\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\kappa(\mathbf{L}')\right\rangle\Bigg)=\sum_{j=0}^{2g}(-1)^j\left\langle\tau_0\tau_1\tau_j\tau_{2g-j}\prod_{j=1}^{n}\tau_{d_j}\kappa(\mathbf{b})\right\rangle_g.$$

Theorem 10 (11). *Let $\mathbf{b} \in N^\infty$, $M \geq 2g$ be an even number and $d_j \geq 0$.*

$$\sum_{j=1}^{n}\left\langle\tau_{d_j+M-1}\prod_{i\neq j}\tau_{d_i}\kappa(\mathbf{b})\right\rangle_g-\frac{1}{2}\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ \underline{n}=I \amalg J}}\sum_{j=0}^{M-2}(-1)^j\binom{\mathbf{b}}{\mathbf{L}}$$

$$\left\langle\tau_j\prod_{i\in I}\tau_{d_i}\kappa(\mathbf{L})\right\rangle\left\langle\tau_{M-2-j}\prod_{i\in J}\tau_{d_i}\kappa(\mathbf{L}')\right\rangle$$

$$=\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}(-1)^{\|\mathbf{L}\|}\binom{\mathbf{b}}{\mathbf{L}}\left\langle\tau_{|L|+M}\prod_{j=1}^{n}\tau_{d_j}\kappa(\mathbf{L}')\right\rangle_g.$$

We also found the following conjectural identity experimentally, which is amazing if compared with Theorems 6 and 8.

Conjecture 11 (19). *Let $g \geq 2$, $d_j \geq 1$, $\sum_{j=1}^{n}(d_j-1) = g$.*

$$\frac{(2g-3+n)!}{2^{2g+1}(2g-3)!\prod_{j=1}^{n}(2d_j-1)!!}=\left\langle\prod_{j=1}^{n}\tau_{d_j}\tau_{2g-2}\right\rangle_g$$

$$-\sum_{j=1}^{n}\left\langle\tau_{d_j+2g-3}\prod_{i\neq j}\tau_{d_i}\right\rangle_g$$

$$+\frac{1}{2}\sum_{\underline{n}=I \amalg J}\sum_{j=0}^{2g-4}(-1)^j\left\langle\tau_j\prod_{i\in I}\tau_{d_i}\right\rangle_{g'}\left\langle\tau_{2g-4-j}\prod_{i\in J}\tau_{d_i}\right\rangle_{g-g'}.$$

Because $(2g-3)!\mathrm{ch}_{2g-3}(\mathbb{E}) = (-1)^{g-1}(3\lambda_{g-3}\lambda_g - \lambda_{g-1}\lambda_{g-2})$, it's easy to see that the above identity is equivalent to the following identity of Hodge integrals.

Conjecture 12. *Let $g \geq 2$, $d_j \geq 1$, $\sum_{j=1}^{n}(d_j - 1) = g$.*

$$\frac{2g-2}{|B_{2g-2}|}\left(\left\langle \prod_{j=1}^{n}\tau_{d_j}\middle| \lambda_{g-1}\lambda_{g-2}\right\rangle_g - 3\left\langle \prod_{j=1}^{n}\tau_{d_j}\middle| \lambda_{g-3}\lambda_g\right\rangle_g\right)$$

$$= \frac{1}{2}\sum_{j=0}^{2g-4}(-1)^j\left\langle \tau_{2g-4-j}\tau_j\prod_{i=1}^{n}\tau_{d_i}\right\rangle_{g-1}$$

$$+ \frac{(2g-3+n)!}{2^{2g+1}(2g-3)!\prod_{j=1}^{n}(2d_j-1)!!}.$$

Virasoro Constraints and KdV Hierarchy

From Theorem 3, we found new Virasoro constraints and KdV hierarchy for generating functions of higher Weil–Petersson volumes that vastly generalize the Witten conjecture and the results of Mulase and Safnuk (15).

Let $\mathbf{s} := (s_1, s_2, \ldots)$ and $\mathbf{t} := (t_0, t_1, t_2, \ldots)$, we introduce the following generating function

$$G(\mathbf{s}, \mathbf{t}) := \sum_g \sum_{\mathbf{m},\mathbf{n}} \left\langle \kappa_1^{m1}\kappa_2^{m2}\ldots\tau_0^{n0}\tau_1^{n1}\ldots\right\rangle_g \frac{\mathbf{s}^{\mathbf{m}}}{\mathbf{m}!}\prod_{i=0}^{\infty}\frac{t_i^{n_i}}{n_i!},$$

where $\mathbf{s}^{\mathbf{m}} = \prod_{i\geq 1} s_i^{m_i}$.

We introduce the following family of differential operators for $k \geq -1$,

$$V_k = -\frac{1}{2}\sum_{\mathbf{L}}(2(|\mathbf{L}|+k)+3)!!\gamma_{\mathbf{L}}\mathbf{s}^{\mathbf{L}}\frac{\partial}{\partial t_{|\mathbf{L}|+k+1}}$$

$$+\frac{1}{2}\sum_{j=0}^{\infty}\frac{(2(j+k)+1)!!}{(2j-1)!!}t_j\frac{\partial}{\partial t_{j+k}}$$

$$+\frac{1}{4}\sum_{d_1+d_2=k-1}(2d_1+1)!!(2d_2+1)!!\frac{\partial^2}{\partial t_{d_1}\partial t_{d_2}}$$

$$+\frac{\delta_{k,-1}t_0^2}{4}+\frac{\delta_{k,0}}{48},$$

where $\gamma_{\mathbf{L}}$ are defined by

$$\gamma_{\mathbf{L}} = \frac{(-1)^{\|\mathbf{L}\|}}{\mathbf{L}!(2|\mathbf{L}|+1)!!}.$$

Theorem 13 (11, 15). *We have $V_k\exp(G) = 0$ for $k \geq -1$ and the operators V_k satisfy the Virasoro relations*

$$[V_n, V_m] = (n-m)V_{n+m}.$$

The Witten–Kontsevich theorem states that the generating function for ψ class intersections

$$F(t_0, t_1, \ldots) = \sum_g \sum_{\mathbf{n}}\left\langle \prod_{i=0}^{\infty}\tau_i^{n_i}\right\rangle_g \prod_{i=0}^{\infty}\frac{t_i^{n_i}}{n_i!}$$

is a τ-function for the KdV hierarchy.

Because Virasoro constraints uniquely determine the generating functions $G(\mathbf{s}, t_0, t_1, \ldots)$ and $F(t_0, t_1, \ldots)$, we have the following theorem.

Theorem 14 (11, 15).

$$G(\mathbf{s}, t_0, t_1, \ldots) = F(t_0, t_1, t_2+p_2, t_3+p_3, \ldots),$$

where p_k are polynomials in s given by

$$p_k = \sum_{|\mathbf{L}|=k-1}\frac{(-1)^{\|\mathbf{L}\|-1}}{\mathbf{L}!}\mathbf{s}^{\mathbf{L}}.$$

In particular, for any fixed values of $\mathbf{s}$, *$G(\mathbf{s}, \mathbf{t})$ is a τ-function for the KdV hierarchy.*

Theorem 14 also generalized results in ref. 24.

Denominators of Intersection Numbers

Let *denom*(r) denote the denominator of a rational number r in reduced from (coprime numerator and denominator, positive denominator). We define

$$D_{g,n} = lcm\left\{denom\left(\left\langle\prod_{j=1}^{n}\tau_{d_j}\right\rangle_g\right)\middle|\sum_{j=1}^{n}d_j = 3g-3+n\right\}$$

and for $g \geq 2$,

$$\mathcal{D}_g = lcm\left\{denom\left(\int_{\overline{M}_g}\kappa(\mathbf{b})\right)\middle| |\mathbf{b}| = 3g-3\right\}$$

where *lcm* denotes least common multiple.

Because denominators of intersection numbers on $\bar{M}_{g,n}$ all come from orbifold quotient singularities, the divisibility properties of $D_{g,n}$ and $\mathcal{D}_g$ should reflect overall behavior of singularities.

We have the following properties of $D_{g,n}$ and $\mathcal{D}_g$.

Proposition 15 (25). *We have $D_{g,n} \mid D_{g,n+1}$, $D_{g,n} \mid \mathcal{D}_g$ and $\mathcal{D}_g = D_{g,3g-3}$.*

Theorem 16 (25). *For $1 < g' \leq g$, the order of any automorphism group of a Riemann surface of genus g' divides $D_{g,3}$.*

The following corollary of Theorem 16 is a conjecture raised by Itzykson and Zuber (26) in 1992.

Corollary 17. *For $1 < g' \leq g$, the order of any automorphism group of an algebraic curve of genus g' divides $\mathcal{D}_g$.*

The proof of Theorem 16 needs the following two lemmas (see ref. 25).

Lemma 2. *If $p \leq g+1$ is a prime number, then* $\mathrm{ord}(p, D_{g,3}) \geq 2$.

Lemma 3 (27). *Let X be a Riemann Surface of genus $g \geq 2$, then for any prime number p,*

$$\mathrm{ord}(p,|Aut(X)|) \leq \left[\log_p\frac{2pg}{p-1}\right] + \mathrm{ord}(p,2(g-1)).$$

We have also obtained conjectural exact values of $\mathcal{D}_g$ in ref. 19.

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1. Mumford D (1983) in *Arithmetic and Geometry,* eds Artin M, Tate J (Birkhauser, Basel), Part II, pp 271–328.
2. Morita S (1987) *Invent Math* 90:551–577.
3. Miller E (1986) *J Diff Geom* 24:1–14.
4. Arbarello E, Cornalba M (1996) *J Alg Geom* 5:705–709.
5. Kaufmann R, Manin Yu, Zagier D (1996) *Comm Math Phys* 181:763–787.

6. Witten E (1991) *Surv Diff Geom* 1:243–310.
7. Kontsevich M (1992) *Comm Math Phys* 147:1–23.
8. Liu K, Xu H (2007) http://arxiv.org/abs/math/0701319v1.
9. Okounkov A (2002) *Int Math Res Notices,* 933–957.
10. Brezin E, Hikami S (2007) http://arxiv.org/abs/0704.2044v1.
11. Liu K, Xu H (2007) http://arxiv.org/abs/0708.0565v1.
12. Liu K, Xu H (2007) http://arxiv.org/abs/0705.2086v1.
13. Mirzakhani M (2007) *Invent Math* 167:179–222.
14. Mirzakhani M (2007) *J Am Math Soc* 20:1–23.
15. Mulase M, Safnuk B (2006) http://arxiv.org/abs/math/0601194v3.
16. Safnuk B (2007) http://arxiv.org/abs/0704.2530v1.
17. Eynard B, Orantin N (2007) http://arxiv.org/abs/0705.3600v1.
18. Do N, Norbury P (2006) e-Print Archive, http://arxiv/org/abs/math/0603406v1.
19. Liu K, Xu H (2006) http://arxiv.org/abs/math/0609367v4.
20. Getzler E, Pandharipande R (1998) *Nuclear Phys B* 530:701–714.
21. Faber C (1999) In *Moduli of Curves and Abelian Varieties* (Vieweg, Braunschweig, Germany), pp 109–129.
22. Givental A (2001) *Mosc Math J* 1:551–568, 645.
23. Goulden I, Jackson D, Vakil R (2006) http://arxiv.org/abs/math/0611659v1.
24. Manin Yu, Zograf P (2000) *Ann Inst Fourier* 50:519–535.
25. Liu K, Xu H (2006) http://arxiv.org/abs/math/0608209v4.
26. Itzykson C, Zuber J (1992) *Int J Mod Phys A* 7:5661–5705.
27. Harvey W (1966) *Q J Math* 17:86–97.