Proc. Natl. Acad. Sci. USA
Vol. 94, pp. 773–776, February 1997
Applied Mathematics

Scaling laws for fully developed turbulent flow in pipes: Discussion of experimental data

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Contributed by A. J. Chorin, November 26, 1996

ABSTRACT **We compare mean velocity profiles measured in turbulent pipe flows (and also in boundary layer flows) with the predictions of a recently proposed scaling law; in particular, we examine the results of the Princeton "superpipe" experiment and assess their range of validity.**

1. Introduction

For a number of years, it has been widely believed that the mean velocity profile in the intermediate region of turbulent pipe flow is adequately described by the von Kármán-Prandtl universal logarithmic law of the wall (1, 2):

$$\phi \equiv \frac{u}{u_*} = \frac{1}{\kappa} \ln \frac{u_* y}{\nu} + A, \qquad \textbf{[1.1]}$$

where $u_* = \sqrt{\tau/\rho}$ is the "friction" or "dynamical" velocity that determines the velocity scale, τ is the shear stress at the wall, y is the distance from the wall, ρ is the fluid's density, and ν is its kinematic viscosity. The Reynolds number Re is defined in the case of a pipe as $Re = \bar{u}d/\nu$, where $\bar{u}$ is the mean velocity (discharge rate divided by the cross-section's area), and d is the pipe's diameter. The parameters κ and A are assumed to be universal, Re-independent constants. In a previous paper (ref. 3, in which the references to our previous work can also be found), it was argued that the von Kármán-Prandtl law is not appropriate and that a correct description is given by the scaling (power) law:

$$\phi = (B_0 \ln Re + B_1)\left(\frac{u_* y}{\nu}\right)^{\beta_1/\ln Re}. \qquad \textbf{[1.2]}$$

A key point in Eq. **1.2** is that the exponent is inversely proportional to the logarithm of Re.

By the very logic of the derivations of Eqs. **1.1** and **1.2**, the parameters κ ("von Kármán's constant"), A, B_0, B_1, and β_1 are universal constants, i.e., the same for all developed turbulent flows in circular smooth pipes. Nevertheless, when it comes to κ and A in Eq. **1.1**, there is a definite, non-negligible scatter in the values found in the literature; depending on the source, κ ranges between 0.40 and 0.44, and A ranges between 5.0 and 6.3. For the parameters B_0, B_1, and β_1 in Eq. **1.2**, we have obtained earlier, from a comparison with the data of Nikuradze (4), the values $B_0 = 0.577 \cdots \cong 1/\sqrt{3}$, $B_1 = 5/2$, and $\beta_1 = 3/2$ so that the scaling law (Eq. **1.2**) takes the definite form

$$\phi = \left(\frac{1}{\sqrt{3}} \ln Re + \frac{5}{2}\right)\left(\frac{u_* y}{\nu}\right)^{3/2 \ln Re}. \qquad \textbf{[1.3]}$$

The difference between the laws in Eq. **1.1** and the laws in Eqs. **1.2** and **1.3** is essential; if the universal logarithmic law (Eq. **1.1**) were valid, the experimental data in the ($\ln \eta$, ϕ) plane, where $\eta = u_* y/\nu$, would concentrate on a single, universal, Reynolds-number-independent straight line. On the contrary, if Eqs. **1.2** and **1.3** hold, the experimental points in the ($\ln \eta$, ϕ) plane should cover an area, bounded by the envelope of the family of scaling law curves, having Re as parameter. Concomitantly, if one defines the variable

$$\psi = \frac{1}{\alpha} \ln \left[\frac{\alpha\phi}{B_0\beta_1 + B_1\alpha}\right], \quad \alpha \equiv \frac{\beta_1}{\ln Re}, \qquad \textbf{[1.4]}$$

then, in the ($\ln \eta$, ψ) plane, the roles of the two laws are reversed; the experimental points that correspond to Eqs. **1.2** and **1.3** for various values of Re lie on a single line, the bisectrix of the first quadrant $\psi = \ln \eta$, whereas the data points that correspond to Eq. **1.1** would appear as a family of curves parametrized by Re.

The goal of this paper is to examine what the experimental data, in particular recent experimental data, tell us about the validity of one or the other of the contrasting laws **1.1** and **1.3**.

2. Chevron Profiles and the Scaling Law

In a previous paper (3), we found a dramatic new feature of the velocity profile in the ($\ln \eta$, ϕ) plane at large Reynolds numbers. By considering the asymptotin as $\nu \to 0$ in the power law, we discovered that as that limit was approached, each individual curve approached a chevron (broken line), one of whose legs was approximated by the envelope of the family of scaling law curves, while the other rose above that envelope; the difference in the slopes of the two segments was substantial, more than $\sqrt{e} \cong 1.65$. The kink in the profiles is a property of the power law and is not a consequence of an external forcing. In ref. 3, we cited a paper (5) in which the Princeton group of Zagarola *et al.* presented data that exhibit this kink. The results of the Princeton group will be discussed in greater detail below.

We note that the chevron-like behavior of the type we predicted for velocity profiles in pipes was noticed repeatedly (but never properly interpreted) in experimental data for the related (but not identical) problem of boundary layers with a zero pressure gradient. A few examples should suffice: the experimental data of Schubauer and Klebanoff (see ref. 6, p. 273, figure 25), of Wieghardt and Tilliman (see ref. 7), and, particularly instructive, of Fernholz and Finley (ref. 8; see especially figures 28–30). Here we will display the remarkable new results by Nagib and Hites (9, 10) for a zero pressure gradient boundary layer (Fig. 1). The chevron-like form of the velocity profiles in the ($\ln \eta$, ϕ) plane and splitting of the profile for different Reynolds numbers is clearly seen in this figure. These curves are of special importance for the adequate understanding and modeling of zero pressure gradient boundary layers. Indeed, they show that the scaling law holds all the way to the edge of the external homogeneous flow, in the ($\ln \eta$, ϕ) plane, until its intersection with the horizontal line ϕ

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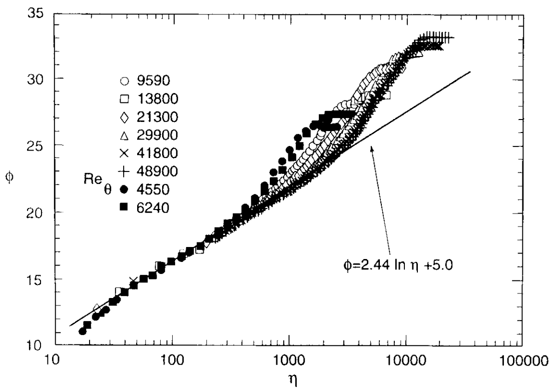

FIG. 1. Experimental velocity profiles in a turbulent boundary layer [reproduced with permission from ref. 9 (Copyright 1995, H. Nagib & M. Hites)].

$= U/u_*$, where U is the external velocity. Of course, there is a matching with the external flow, but the transition region is very small. It can be concluded that the logarithmic straight line applies after the departure of the curve from the envelope up to the intersection with the external flow; the straight line is an approximation of the upper part of the power law at large Reynolds numbers.

3. Discussion of the Princeton "Superpipe" Experiments

Now we come to a more detailed quantitative comparison of the proposed scaling law (Eq. **1.3**) with the recent, widely publicized results of the Princeton group (5, 11, 12), which include many new data for pipe flow obtained in a high-pressure pipe ("superpipe") proposed by G. Brown (13). High pressure increases the density ρ and also increases the dynamic viscosity μ, although at a much smaller rate, decreasing the kinematic viscosity $\nu = \mu/\rho$ substantially and thus increasing the Reynolds number Re. The Princeton group claims that in this way they increased the range of the Reynolds numbers for which they obtained reliable data (up to $Re = 3.53 \cdot 10^7$) by an order of magnitude over the range of Re achieved in the classical benchmark experiments of Nikuradze (4) with water flow (up to $Re = 3.24 \cdot 10^6$). The appearance of a chevron structure and the splitting of the velocity curves according to their Reynolds number discussed in ref. 3 are very plain in the Princeton data.

The advantage of the data set of the Princeton group for comparisons with theory is that, like the data of Nikuradze, they are presented in tabular form. In ref. 11, one can find results of 26 runs (series of experiments), each run containing data from the measurements of the velocity distribution over the cross-section of the pipe, as well as measured drag coefficients. The pressure varied from ≈1 to ≈190 atm (1 atm = 101.3 kPa). The kinematic viscosity of air under normal conditions is ≈0.15 cm²/s, and that of water is ≈0.01 cm²/s; therefore, the Princeton investigators had to compress air to roughly 15 atm to reach the kinematic viscosity of water.

Another important advantage of the Princeton work is that the data contain many experimental points far from the envelope. In the experiments reported by Nikuradze (4), there were no such data. Therefore, the most interesting step is the comparison of the Princeton data with the scaling law (Eq **1.3**) by the same procedure as the one previously used for comparison with the Nikuradze data. All the data of the Princeton group were plotted in the $(\ln \eta, \psi)$ plane, where

$$\psi = \frac{1}{\alpha} \ln \frac{2\alpha\phi}{\sqrt{3 + 5\alpha}}, \quad \alpha = \frac{3}{2 \ln Re}, \quad Re = \frac{\bar{u}d}{\nu}. \qquad \textbf{[3.1]}$$

For the first 10 runs listed in ref. 11 ($Re = 3.16 \cdot 10^4, 4.17 \cdot 10^4, 5.67 \cdot 10^4, 7.43 \cdot 10^4, 9.88 \cdot 10^4, 1.46 \cdot 10^5, 1.85 \cdot 10^5, 2.30 \cdot 10^5, 3.09 \cdot 10^5$, and $4.09 \cdot 10^5$), the data are presented in Fig. 2. It is seen that, as in the case of Nikuradze's data (4), the experimental points after $\eta = 25$ concentrate near the bisectrix of the first quadrant. The points close to the pipe axis should be removed because the scaling law should be invalid for them. It was enough to remove only the points where $2y/d$ was more than 0.95.

The situation is different, however, for the last six runs in ref. 11 ($Re = 1.02 \cdot 10^7, 1.36 \cdot 10^7, 1.82 \cdot 10^7, 2.40 \cdot 10^7, 2.99 \cdot 10^7$, and $3.52 \cdot 10^7$). The experimental points for all these runs are concentrated (for $2y/d < 0.95$) along straight lines, parallel to the bisectrix, but not on the bisectrix itself (Fig. 3). In fact, these straight lines are close to each other (because the corresponding values of log Re are close). The noticeable deviation from the bisectrix started from run 13 ($Re = 1.02 \cdot 10^6$).

Some hint as to what happens was given by a comparison of the experiments of Nikuradze and the Princeton group performed at roughly equal Reynolds numbers. There are six such experiments, and for five of them, at moderate Reynolds numbers, a satisfactory coincidence was found. This coincidence means that our scaling law (Eq. **1.3**) is also confirmed by the Princeton experiments. However, for run 16 by the Princeton group (ref. 11; $Re = 2.345 \cdot 10^6$), a noticeable disagreement was found with the Nikuradze run at $Re = 2.35 \cdot 10^6$ (4), which, like the other Nikuradze runs, verifies quite satisfactorily the scaling law (Eq. **1.3**). In the main part of the graph in the $(\ln \eta, \phi)$ plane, there is a uniform shift along the $\ln \eta$ axis between the Nikuradze and Princeton data. What can be the meaning of such a shift? If both u_* and y were measured correctly, the most likely source of the discrepancy is in the values of the viscosity. It is of importance that the pressure gradients in these experiments are small enough not to create a variation of the viscosity along the pipe, and thus in each run the viscosity can be viewed as constant.

We conjectured therefore that something happened in the high Reynolds number (high pressure) experiments of the Princeton group which shifted the viscosity that determines the velocity profile from its actual value ν to a "shifted" effective value ν', so that

$$\ln \eta = \ln \frac{u_* y}{\nu} = \ln \frac{u_* y}{\nu'} + \ln \frac{\nu'}{\nu}, \qquad \textbf{[3.2]}$$

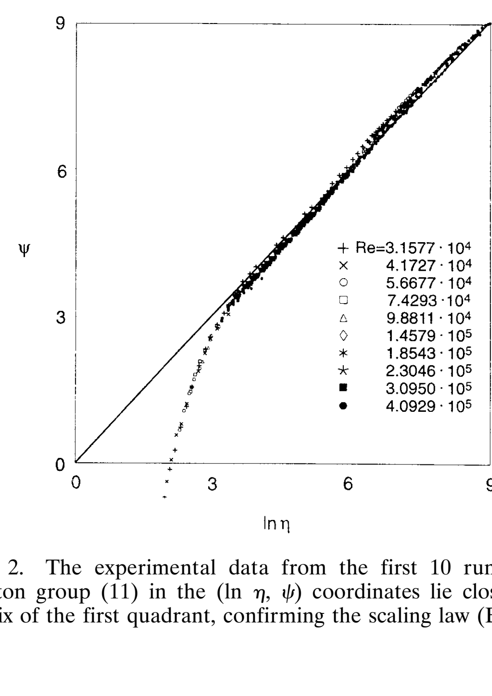

FIG. 2. The experimental data from the first 10 runs of the Princeton group (11) in the $(\ln \eta, \psi)$ coordinates lie close to the bisectrix of the first quadrant, confirming the scaling law (Eq. **1.3**).

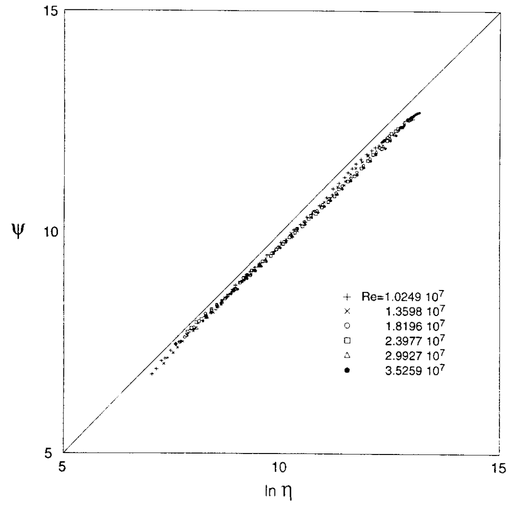

FIG. 3. The experimental data from the last six runs of the Princeton group (11) concentrate along lines parallel to the bisectrix, not on the bisectrix itself. The points with $2y/d > 0.95$ are excluded; they are in the near-axis region.

and the shift $\ln(\nu'/\nu)$ is constant for each run.

To check this assumption, the following procedure was used for the last six runs. For every experimental point of each run, the values of the difference

$$\chi = \ln \eta - \psi \quad \textbf{[3.3]}$$

(the shift from the bisectrix) and $\bar{\chi}$, the mean value of χ per run, were calculated. The dispersion of this quantity was also calculated and found to be small. Then every experimental point was shifted by $\bar{\chi}$ inward along the $\ln \eta$ axis. The results are presented in Fig. 3 (unshifted points) and Fig. 4 (after the shift). We see that there exists a single factor per run by which the viscosity is altered and which shifts the velocity profiles at high Reynolds numbers; this does not happen at moderate Reynolds numbers.

Three possible reasons for this shift were investigated.

***(i)* Incorrect Pressure and Temperature Measurement.** The density and viscosity were not measured directly, but were calculated by the Princeton group on the basis of measured pressures and temperatures. Therefore, an incorrect pressure

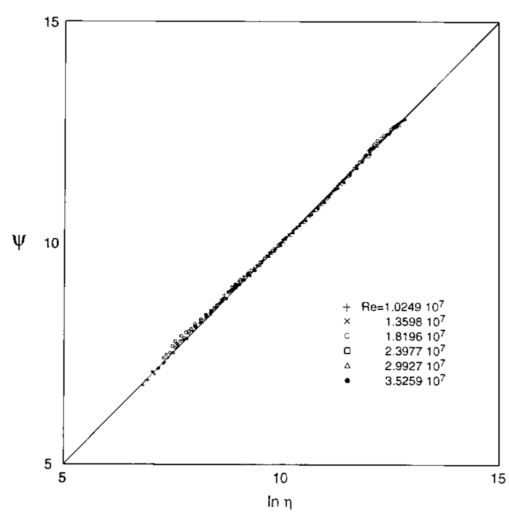

FIG. 4. After the viscosity correction (constant for each run), the large Reynolds numbers Princeton data (11) agree with the prediction of the scaling law in the $(\ln \eta, \psi)$ plane; the points are close to the bisectrix (except for the near-axis points).

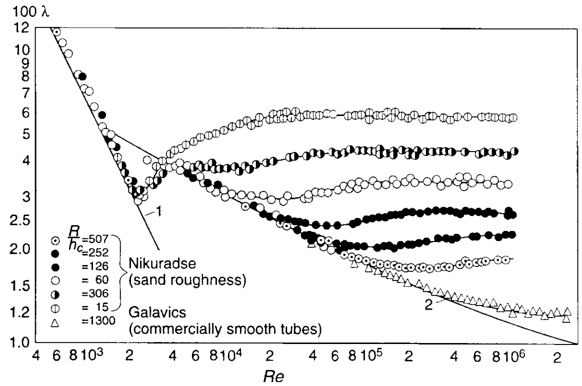

FIG. 5. The drag coefficient for pipes of various roughness [redrawn after Monin and Yaglom (6)]. 1, Laminar flow; 2, law for smooth pipes.

and temperature measurement could be the reason for the shift. It was clear from the beginning that the measurement of the temperature was not in doubt. After an inspection of the information presented in refs. 5 and 11, we came to the conclusion that errors in the pressure measurement were unlikely.

***(ii)* Incorrect Density and Viscosity Calculations.** Indeed, the Princeton group used rather old pressure–temperature/density–viscosity relations for their calculations. D. G. Friend (National Institute of Standards and Technology, Boulder, CO, personal communication) supplied us with the values of density and viscosity of air at pressures and temperatures recorded in the Princeton measurements (11). These data confirmed the Princeton calculations accurately. This confirmation has left only one possible explanation for the observed shift in the viscosity.

***(iii)* The Roughness of the Pipe Walls Is Revealed at Large Reynolds Numbers.** As is well known, if the walls of the pipe are not sufficiently smooth, the roughness protrudes from the viscous sublayer, and a shift in the velocity profile is observed in the intermediate region, exactly as if the viscosity of the fluid were changed. There is a well known formula for the "equivalent" viscosity (see ref. 6, p. 286, formula 5.25b).

To check the third possibility, we turn to the data concerning the Reynolds number dependence of the drag coefficient for flows in rough pipes (see Fig. 5, available in ref. 6, p. 308). The general situation is as follows. For a given mean height of the roughness, the data for rough and smooth pipes coincide up to

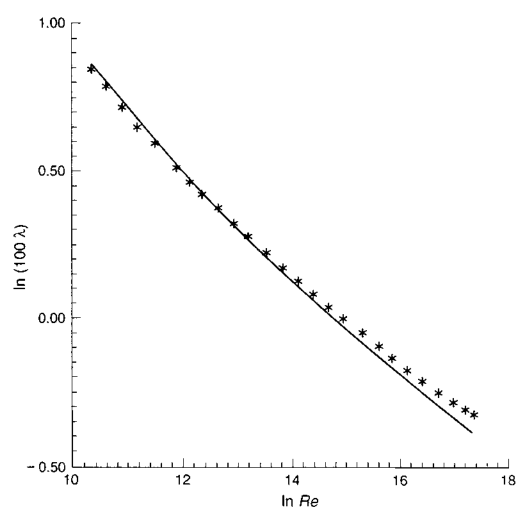

FIG. 6. The drag coefficient λ as a function of the Reynolds number for the Princeton (11) data. *, Princeton data; solid line, the law for smooth pipes.

a critical Reynolds number. When this is reached, the Reynolds number dependence of the drag coefficient for rough pipes deviates from that for smooth pipes. Clearly the critical Reynolds number depends on the mean height of the roughness: the lesser the height, the later the deviation begins.

The analog of Fig. 5 for the Princeton experiments is presented in our Fig. 6; the solid line corresponds to the theoretical relation for the drag coefficient corresponding to the scaling law (Eq. **1.3**). The graph shows that the deviation starts at approximately $Re = 10^6$. This is a sensitive indicator of the smoothness of pipes; it shows that starting from run 13 ($Re = 1.02 \cdot 10^6$), the velocity profiles presented by the Princeton group correspond to rough pipes rather than to smooth pipes. $Re = 10^6$ is where the deviation from the bisectrix in the ($\ln \eta$, ψ) plane began. It is possible that the problem of roughness can be healed by following the large-pipe suggestion of Hussain in ref. 14.

4. Conclusions

(*i*) The analysis of the new experimental data adduces additional arguments against the von Kármán-Prandtl universal logarithmic law and in favor of a specific power law (Eqs. **1.2** and **1.3**).

(*ii*) The Princeton group apparently did not surpass the range of Reynolds numbers for smooth pipes achieved by Nikuradze, and indeed, as far as we can see, did not reach its upper bound. The kinematic viscosity of air in the last run which corresponds to a smooth pipe according to the data in ref. 11 can be estimated as 10^{-2} cm^2/s—equal to the kinematic viscosity of water used in the Nikuradze experiments, but not less.

(*iii*) The ($\ln \eta$, ψ) procedure for processing the velocity profile used here was sensitive enough to detect the influence of roughness independently of the drag coefficient data.

We thank the following people for helpful discussions, suggestions, or help with the data: Dr. D. G. Friend, Prof. N. Goldenfeld, Prof. O. Hald, Prof. F. Hussain, Prof. H. Nagib, Mr. M. Hites, and Prof. C. Wark. This work was supported in part by the Applied Mathematical Sciences subprogram of the Office of Energy Research, U.S. Department of Energy, under contract DE-AC03-76-SF00098, and in part by the National Science Foundation under Grants DMS94-14631 and DMS89-19074.

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