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On Calderón's conjecture for the bilinear Hilbert transform

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ABSTRACT **We show that the bilinear Hilbert transform defined by**

$$\boldsymbol{Hfg}(\boldsymbol{x}) = \text{p.v.} \int \boldsymbol{f}(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{g}(\boldsymbol{x}+\boldsymbol{y})\,\frac{\boldsymbol{dy}}{\boldsymbol{y}}$$

maps $L^p \times L^q$ into L^r for $1 < p, q \leq \infty$, $1/p + 1/q = 1/r$, and $2/3 < r < \infty$.

Section 1. Introduction

We continue the investigation begun in ref. 1 concerning the bilinear Hilbert transform defined by

$$H_\alpha fg(x) := \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} f(x - \alpha y) g(x - y)\,\frac{dy}{y}, \quad \alpha \neq 1.$$

This singular integration is initially defined only for certain functions f and g, for instance those in the Schwartz class on $\mathbb{R}$. But H_α can be extended to a bounded operator on certain L^p classes, as ref. 1 showed. We extend the theory of that paper with this result:

THEOREM 1.1. *For any $\alpha \neq 1$, H_α extends to a bounded operator on $L^{p_1} \times L^{p_2}$ into L^{p_3}, provided $1 < p_1, p_2 \leq \infty$, $1/p_3 = 1/p_1 + 1/p_2$, and $2/3 < p_3 < \infty$.*

A special instance of this theorem was conjectured by A. P. Calderón in connection with ref. 2, namely $L^2 \times L^\infty$ into L^2 or, by duality, $L^2 \times L^2$ into L^1. A special feature of the result is that the index p_3 for the image space need only be bigger than 2/3. We do not know that this is necessary for the theorem, although it is necessary for our proof.

The proof refines the method in ref. 1, with a more effective organization of the elements of the proof and an extension of certain almost orthogonality results in that paper to L^p functions for $1 < p < 2$.

Section 2. The Model Sums

As in ref. 1, the essence of the matter concerns an analysis of a analogue of H_α that is more suited to the methods we employ. In particular, we utilize combinatorial features of the space–frequency plane.

Let $\mathcal{D}$ be a dyadic grid in $\mathbb{R}$. Call $I \times \omega \in \mathcal{D} \times \mathcal{D}$ a *tile* if $|I|\cdot|\omega| = 1$. The interval ω is a union of four dyadic subintervals of equal length, ω_1, ω_2, ω_3, and ω_4, which we list in ascending order. Thus, $\xi_i < \xi_j$ for all $1 \leq i < j \leq 4$ and $\xi_j \in \omega_j$. We adopt the notation $t = I_t \times \omega_t$ and $t_j = I_t \times \omega_{tj}$ for $j = 1, 2, 3$. Fix a Schwartz function ϕ with $\hat{\phi}$ supported on $[-1/8, 1/8]$. Set for all tiles t, and $j = 1, 2, 3$,

$$\phi_{tj}(x) = \frac{e^{-2\pi i c(\omega_{tj})x}}{\sqrt{|I_t|}}\,\phi\!\left(\frac{x - c(I_t)}{|I_t|}\right),$$

where $c(J)$ denotes the center of the interval J.

Then our model of the bilinear Hilbert transform is any of the following sums

$$\mathcal{M}_{\mathbf{S}} f_1 f_2(x) = \sum_{t \in \mathbf{S}} |I_t|^{-1/2} \langle f_1, \varphi_{t\sigma(1)}\rangle \langle f_2, \varphi_{t\sigma(2)}\rangle \varphi_{t\sigma(3)}(x),$$

in which $\mathbf{S}$ is a finite subset of $\mathbf{S}_{\text{all}}$, the set of all tiles and $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$ is any one-to-one map. Including this map σ emphasizes the symmetry of these model sums under duality. We comment that the sum above makes sense only *a priori* for finite sums. We provide estimates for $\mathcal{M}_{\mathbf{S}}$ that are independent of the number of elements in $\mathbf{S}$, and hence the sum can be extended to $\mathbf{S}_{\text{all}}$. It is in this sense that several statements below should be interpreted.

The analogue of *Theorem 1.1* is

THEOREM 2.1. *For any choice of σ, $\mathcal{M}_{\mathbf{S}_{\text{all}}}$ extends to a bounded operator on $L^{p_1} \times L^{p_2}$ into L^{p_3}, provided the indices p_i are as in the previous theorem.*

Indeed, our proof supplies the additional information that the sum over tiles t is unconditionally convergent. And for this to hold an example shows that the inequality $p_3 \geq 2/3$ is necessary.

We shall take the prior result from ref. 1 for granted, namely that $\mathcal{M}_{\mathbf{S}_{\text{all}}}$ is a bounded operator on $L^{p_1} \times L^{p_2} \to L^{p_3}$ provided $2 < p_1, p_2, p_3/(p_3 - 1) < \infty$. Then, taking duality and interpolation into account, one can see that it suffices to prove *Theorem 2.1* in the case that $1 < p_1, p_2 < 2$ and $2/3 < p_3 < 1$. This we shall do, assuming that the map $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$ is the identity.

Indeed, this last case follows from a more precise statement that fully exploits the symmetry in the definition of the model sums.

LEMMA 2.2. *Let $1 < p_1, p_2, p_3 < 2$ satisfy $1 < 1/p_1 + 1/p_2 + 1/p_3 < 2$. Let $f_i \in L^{p_i}$ be of norm one and let*

$$E = \{x \in \mathbf{R} | M_{p_i}(Mf_i)(x) > 1\}.$$

Then for an absolute constant C

$$\sum_{s \in \mathbf{S}_E} |I_s|^{-1/2} \prod_{i=1}^{3} |\langle f_i, \varphi_{si}\rangle| \leq C,$$

where $\mathbf{S}_E = \{s \in \mathbf{S}_{all} \mid I_s \not\subset E\}$, Mf denotes the Hardy–Littlewood maximal function, and $M_r f = (M|f|^r)^{1/r}$.

Let us indicate how this proves *Theorem 2.1* for $1 < p_1, p_2 < 2$ and $1 < 1/p_1 + 1/p_2 < 3/2$. There is a simple reduction. By linearity and scaling invariance, the inequality

$$|\{\mathcal{M}_{\mathbf{S}_{\text{all}}} f_1 f_2 > 2\}| < K$$

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holding for all $\|f_1\|_{p_1} = \|f_2\|_{p_2} = 1$ will prove that $\mathcal{M}_{\mathbf{S}_{\rm all}}$ maps into the appropriate weak L^{p_3} space. Then Marcinceiwcz interpolation proves the theorem.

Fix f_1 and f_2 of norms one in their respective L^{p_i} spaces. We use the set

$$E_0 := \bigcup_{i=1}^{2} \{M_{p_i}(Mf_i) > 1\}$$

to split up $\mathbf{S}_{\rm all}$ and the set whose measure we are to estimate. Let $\mathbf{S}_{\rm in} := \{s \in \mathbf{S}_{\rm all} \mid I_s \subset E_0\}$, $\mathbf{S}_{\rm out} := \mathbf{S}_{\rm all} - \mathbf{S}_{\rm in}$, and define $E_{\rm in} := \{\mathcal{M}_{\mathbf{S}_{\rm in}} f_1 f_2 > 1\}$ and similarly $E_{\rm out} := \{\mathcal{M}_{\mathbf{S}_{\rm out}} f_1 f_2 > 1\}$. It suffices to bound the measures of these last two sets by constants.

The essential term, $|E_{\rm out}|$, is controlled by *Lemma 2.2*. Indeed, we may assume that $|E_{\rm out}| > 1$, for otherwise there is nothing to prove. Then choose p_3 with $1 < 1/p_1 + 1/p_2 + 1/p_3 < 2$ and we take the third function to be the indicator of $E_{\rm out}$, normalized in L^{p_3}; that is, we take $f_3(x) := |E_{\rm out}|^{-1/p_3} \chi_{E_{\rm out}}(x)$. Observe that this function is strictly less than one. Therefore, *Lemma 2.2* gives us the desired result,

$$|E_{\rm out}|^{1-1/p_3} \le \langle \mathcal{M}_{\mathbf{S}_{\rm out}} f_1 f_2, f_3 \rangle \le C.$$

The estimate of $E_{\rm in}$ begins by defining

$$E_1 = E_0 \cup \bigcup_{\substack{J \in \mathcal{D} \\ J \subset E}} 4J.$$

This set has measure $|E_1| \le 5|E_0| \le C'$, and we shall not estimate $\mathcal{M}_{\mathbf{S}_{\rm in}}$ on this set. And off of this set we have

$$\|\mathcal{M}_{\mathbf{S}_{\rm in}} f_1 f_2\|_{L^1(E_1^c)} \le C.$$

This inequality is, however, of a routine nature, and so we do not supply a proof of it here. This finishes the discussion of the proof of *Theorem 2.1* from *Lemma 2.2*.

Section 3. The Combinatorics of *Lemma 2.2*

We devote ourselves to the proof of *Lemma 2.2*, beginning with issues related to the combinatorics of the collection $\mathbf{S}_E$, and concluding with the issues related to almost orthogonality. View the functions f_i as fixed and set

$$F_{ti} := |\langle f, \varphi_{ti}\rangle|, \quad F_t := |I_t|^{-1/2} \prod_{i=1}^{3} F_{ti}.$$

There is a partial order on tiles given by $t < t'$ if $I_t \subset I_{t'}$, $\omega_t \supset \omega_{t'}$. By splitting the collection of tiles into two we can assume that if $s <_{\neq} t$ then $4|I_s| \le |I_t|$.

Call a collection of tiles $\mathbf{T}$ a *tree with top* q if $t < q$ for all $t \in \mathbf{T}$. For $1 \le i \le 4$, call $\mathbf{T}$ an i-tree if in addition to $\mathbf{T}$ being a tree with top q, $\omega_{ti} \cap \omega_q$ for all $t \in \mathbf{T}$. In this case, observe that for $s <_{\neq} t <_{\neq} q$ in the tree, we must have $\omega_{si} \supset \omega_t \supset \omega_{ti} \supset \omega_q$. Thus, for $j \ne i$ the intervals $\{\omega_{tj} \mid t \in \mathbf{T}\}$ not only are disjoint but are lacunary. And indeed, the Littlewood–Paley Theory applies to the collection of functions $\{\varphi_{tj} \mid t \in \mathbf{T}\}$.

There are comparisons to maximal functions. Observe that for a single tile we have

$$|I_t|^{-1/2} F_{tj} \le C_0 \inf_{x \in I_t} Mf_j(x) \le C_0, \qquad t \in \mathbf{S}_E, \qquad j = 1, 2, 3. \qquad \textbf{[3.1]}$$

The last inequality follows from the definition of $\mathbf{S}_E$.

At a deeper level, we have the following for a tree. For an i-tree $\mathbf{T} \subset \mathbf{S}_E$ with top q and $j \ne i$, observe that the Littlewood–Paley inequalities imply

$$\frac{1}{|I_q|} \int_{I_q} \left[\sum_{t \in \mathbf{T}} |I_t|^{-1} |\langle f, \varphi_{tj}\rangle|^2 \mathbf{1}_{I_t}(x) \right]^{1/2} dx \le C_1 \inf_{x \in I_t} M_{p_j} f_j(x) \le C_2.$$

This inequality hold for each subtree of $\mathbf{T}$, whence we conclude that the dyadic bounded mean oscillation (BMO) norm of the integrand above is at most C_2. The BMO structure then gives us the formally stronger assertion that

$$\Delta(\mathbf{T}, j) := \left[|I_q|^{-1} \sum_{t \in \mathbf{T}} |\langle f, \varphi_{tj}\rangle|^2 \right]^{1/2} \le C_0. \qquad \textbf{[3.2]}$$

The square functions $\Delta(\mathbf{T}, j)$ are relevant here, due to the following estimate valid for an i-tree $\mathbf{T}$. By Cauchy–Schwartz,

$$\sum_{t \in \mathbf{T}} F_t \le |I_q| \sup_{t \in \mathbf{T}} \frac{F_{ti}}{\sqrt{|I_t|}} \prod_{j \ne i} \Delta(\mathbf{T}, j). \qquad \textbf{[3.3]}$$

We summarize the combinatorics of $\mathbf{S}_E$ in the following decomposition. Set $n_0 = 2 \log_2 C_0$, the constant C_0 appearing in the previous two paragraphs. The collection $\mathbf{S}_E$ is a union of subcollections $\mathbf{S}_{ni}$ for $i = 1, 2, 3$ and $n \ge n_0$ so that

$$\Delta(\mathbf{T}, j') \le 2^{-n/p_j'}, \qquad 1/p_j' = 1 - 1/p_j, \qquad \textbf{[3.4]}$$

for all j-trees $\mathbf{T} \subset \mathbf{S}_{ni}$, $j \ne j'$. Furthermore, denote by $\mathbf{S}_{ni}^*$ those elements of $\mathbf{S}_{ni}$ that are maximal with respect to the partial order '<'. The critical property is

$$\sum_{q \in \mathbf{S}_{ni}^*} |I_q| \le C 2^{n(1+\eta)}, \qquad \textbf{[3.5]}$$

where $0 < \eta < 2 - \Sigma\, 1/p_i$.

Once this decomposition is established, the proof of *Lemma 2.2* is easily accomplished. For each $n \ge n_0$ and $i = 1, 2, 3$, the collection $\mathbf{S}_{ni}$ is a union of trees $\mathbf{T}_q$ with tops $q \in \mathbf{S}_{ni}^*$. Each tree is a union of four trees to which the estimate **3.3** applies. Hence, the properties of $\mathbf{S}_{ni}$ imply that

$$\sum_{t \in \mathbf{S}_{ni}} F_t \le 2^{-n \Sigma_i 1/p_i'} \sum_{q \in \mathbf{S}_{ni}^*} |I_q| \le C 2^{n(1+\eta-\Sigma_i 1/p_i')}.$$

But, by the choice of η and the requirements on the p_i, the exponent on n is negative. And so this estimate has a finite sum over $n \ge n_0$.

We turn to the task of achieving this decomposition of $\mathbf{S}_E$. It is inductive and best done by defining some auxiliary collections. Assume that the $\mathbf{S}_{mi}$ are defined for all $m < n$ and all $1 \le i \le 3$, in such a way that for $\mathbf{S}^r = \mathbf{S} \backslash (\cup_{m<n} \cup_i \mathbf{S}_{mi})$ we have

$$\frac{|\langle f_i, \phi_{ti}\rangle|}{\sqrt{|I_t|}} \le 2^{-n/p_i'}\, t \in \mathbf{S}^r, \qquad i = 1, 2, 3, \qquad \textbf{[3.6]}$$

and for any i-tree $\mathbf{T} \subset \mathbf{S}^r$, $\Delta(\mathbf{T}, j) \le 2^{-n/p_j'+2}$, for $j \ne i$.

The collections $\mathbf{S}_{ni}$ will be a union of four subcollections denoted $\mathbf{S}_{nij}$ for $1 \le j \le 4$. We define $\mathbf{S}_{n11}^*$ to be the set of maximal tiles q with $|I_q|^{-1/2} |\langle f_1, \phi_{q1}\rangle| \ge 2^{-n/p_1'-1}$, and take $\mathbf{S}_{n11}$ to consist of all tiles t so that $t1 < q$ for some $q \in \mathbf{S}_{n11}^*$. These tiles are removed from $\mathbf{S}^r$, and then $\mathbf{S}_{nii}$ is defined similarly for $i = 2, 3$. After the deletion of the tiles $\mathbf{D}_0 = \cup_{i=1}^3 \mathbf{S}_{nii}$, we have $|\langle f_i, \phi_{ti}\rangle| \le 2^{-n/p_i'-1} \sqrt{|I_t|}$ for all tiles $t \in \mathbf{S}^{r'} = \mathbf{S}^r \backslash \mathbf{D}_0$. In the subsequent section we will prove that

$$\sum_{q \in \mathbf{S}_{nii}^*} |I_q| \le C_\eta 2^{n(1+\eta)}, \qquad \eta > 0. \qquad \textbf{[3.7]}$$

We now concentrate on 1-trees $\mathbf{T} \subset \mathbf{S}^{r'}$ for which $\Delta(\mathbf{T}, 2)$ is suitably large. This collection of trees we denote as $\mathbf{S}_{n12}$, and its construction has a particular purpose. Namely, the trees we construct should consist of nearly orthogonal functions, a notion that we encode in the purely combinatorial terms of assertion **3.8**.

Consider tiles q such that there is a 1-tree $\mathbf{T}_q \subset \mathbf{S}^{r'}$ with top q so that $\Delta(\mathbf{T}, 2) \geq 2^{-n/p_2'+1}$. We take $\mathbf{T}_q$ to be the maximal 1-tree with this property. Let $q(1)$ be such a top, which is maximal with respect to $<$, and in addition $\sup\{\xi \mid \xi \in \omega_q\}$ is maximal. Remove the tiles $\mathbf{T}_{q(1)}$, and repeat this procedure to define $\mathbf{T}_{q(2)}$ and so on. $\mathbf{S}_{n12}$ is then $\cup_\ell \mathbf{T}_{q(\ell)}$ and $\mathbf{S}^*_{n12} = \{q(\ell) \mid \ell \geq 1\}$. Observe that for any 1-tree $\mathbf{T} \subset \mathbf{S}^{r'} \backslash \mathbf{S}_{n12}$, we have $\Delta(\mathbf{T}, 2) \leq 2^{-n/p_2'+1}$. These procedures are repeated inductively to define the $\mathbf{S}_{nij}$ for all n, i, j. However, in the case of $i > j$, we choose the top q to be maximal first with respect to '$<$' and then with respect to $\inf\{\xi \mid \xi \in \omega_q\}$. In the subsequent section we will prove that the collections $\mathbf{S}^*_{nij}$ also satisfy inequality **3.7**.

The particulars of the construction of $\mathbf{S}_{nij}$ lead to this combinatorial fact, which for specificity we state in the case of $\mathbf{S}_{n12}$. For $q \in \mathbf{S}^*_{n12}$, denote by $\mathbf{T}^r_q$ the 1-tree $\mathbf{T}_q$ defined above, less those tiles in it that are minimal in the partial order '$<$'. Set $\mathbf{S}^r_{n12} = \cup_{q \in \mathbf{S}^*_{n12}} \mathbf{T}^r_q$. Then

$$s, s' \in \mathbf{S}^r_{n12},\ s \in \mathbf{T}^r_q,\ \omega_{s2} \subset_{\neq} \omega_{s'2} \neq \emptyset \quad \text{implies} \quad I_q \cap I_{s'} = \emptyset, \qquad \mathbf{[3.8]}$$

$$\Delta(\mathbf{T}^r_q, 2) \geq 2^{-n/p_2'}. \qquad \mathbf{[3.9]}$$

The first assertion is a condition concerning the disjointness of the trees in the space–frequency plane that will be a sufficient condition for orthogonality in the subsequent section.

Suppose that condition **3.8** is not true. Thus $I_q \cap I_{s'} \neq \emptyset$ and hence $I_{s'} \subset I_q$. Yet $s' \in \mathbf{T}^r_{q'}$ for some q' and there is an $s'' \in \mathbf{T}_{q'}$ with $s'' < s'$. Hence $\omega_q \subset \omega_{s'2} \subset \omega_{s''1}$ as $\mathbf{T}_{q'}$ is a 1-tree. But $\omega_{q'} \subset \omega_{s'1}$ so that $\sup\{\xi \mid \xi \in \omega_q\} > \sup_{\xi'}\{\xi' \in \omega_{q'}\}$. Hence we have violated the construction of these 1-trees. The second condition, **3.9**, follows immediately from the observation that for any $t \in \mathbf{S}_{n12}$ and $q \in \mathbf{S}^*_{n12}$, we have $4|I_t|^{-1/2}|\langle f_2, \varphi_{t2}\rangle| \leq \Delta(\mathbf{T}_q, 2)$.

Section 4. Counting the Number of Trees

We prove inequality **3.5** first in the case of $\mathbf{S} := \mathbf{S}_{nii}$. The properties that we use are that the tiles in $\mathbf{S}$ are disjoint and $|I_t|^{-1/2}|\langle f_i, \varphi_{ti}\rangle| \geq b$ for $b := 2^{-n/p_i'}$. The proof requires several devices.

Step 1. We set $N_{\mathbf{S}'}(x) := \sum_{t \in \mathbf{S}'} \chi_{I_t}(x)$. We are to estimate the L^1 norm of $N_{\mathbf{S}}$, but it is an integer-valued function, so it suffices to prove a weak type $1 + \varepsilon$ estimate for some $\varepsilon > 0$. That is, we prove

$$|\{N_{\mathbf{S}} \geq \lambda\}| \leq K_{\delta,\varepsilon} b^{-p_i'-\delta} \lambda^{-1-\varepsilon}, \qquad \mathbf{[4.1]}$$

for certain arbitrarily small $\delta, \varepsilon > 0$. Fix such a $\lambda \geq 1$. Because the intervals I_q are dyadic, there is a subcollection $\mathbf{S}' \subset \mathbf{S}$ so that $\{N_{\mathbf{S}} \geq \lambda\} = \{N_{\mathbf{S}'} = \lambda\}$ and $\|N_{\mathbf{S}'}\|_\infty = \lambda$. We work with the collection $\mathbf{S}'$.

Step 2. For $A > 1$ to be specified we can write $\mathbf{S}' = \mathbf{S}^\flat \cup \cup_{m=1}^{A^{10}} \mathbf{S}_m$ for which the tiles in each $\mathbf{S}_m$ are widely separated in the space variable, namely

$$\{AI_t \times \omega_{t1} | t \in \mathbf{S}_m\} \text{ are pairwise disjoint for } 1 \leq m \leq A^{10}.$$

While $\mathbf{S}^\flat$ is small in that

$$\sum_{t \in \mathbf{S}^\flat} |I_t| \leq Ce^{-A}\|N_{\mathbf{S}_1}\|_1.$$

For a proof of this inequality see the separation lemma of ref. 3. It is then clear that

$$|\{N_{\mathbf{S}'} = \lambda\}| \leq C\lambda^{-1}e^{-A}\|N_{\mathbf{S}_1}\|_1 + CA^{10}\lambda^{-1-\varepsilon} \sum_{m=1}^{A^{10}} \|N_{\mathbf{S}_m}\|^{1+\varepsilon}_{1+\varepsilon}.$$

We will show that for appropriate A and $0 < \varepsilon, \delta$ to be chosen, but arbitrarily small, that we have a uniform estimate on the $1 + \varepsilon$ norm of the counting functions of the $\mathbf{S}_m$. Namely,

$$\|N_{\mathbf{S}_m}\|_{1+\varepsilon} \leq Cb^{-p_i'-\delta}\lambda^{\varepsilon/2}. \qquad \mathbf{[4.2]}$$

This will prove inequality **4.1**.

Step 3. Orthogonality decisively enters the argument. As a consequence of the orthogonality lemmas of ref. 1, we have for any $g \in L^2$ the inequality

$$\sum_{t \in \mathbf{S}_m} |\langle g, \varphi_{ti}\rangle|^2 \leq C(1 + A^{-1/\varepsilon}\lambda)\|g\|_2^2.$$

On the other hand, the following is trivially bounded by the maximal function,

$$\sup_{t \in \mathbf{S}_m} \frac{|\langle g, \varphi_{ti}\rangle|}{\sqrt{|I_t|}} \chi_{I_t}(x).$$

Hence it maps L^p into itself for all $p > 1$. Interpolation with the better L^2 bound then provides the bound below for all $1 < p < 2$ and $\delta > 0$.

$$\left\| \left[\sum_{t \in \mathbf{S}_m} \left| \frac{\langle g, \varphi_{ti}\rangle}{\sqrt{|I_t|}} \right|^{p'+\delta} \chi_{I_t}(x) \right]^{\frac{1}{p'+\delta}} \right\|_p \leq C_{p\delta}(1 + A^{-1/\varepsilon}\lambda)\|g\|_p.$$

This estimate can be localized, yielding a better inequality for our purposes. For a dyadic interval J set $\mathbf{S}_{m,J} := \{t \in \mathbf{S}_m \mid I_t \subset J\}$. Then

$$\left\| \left[\sum_{t \in \mathbf{S}_{m,J}} \left| \frac{\langle g, \varphi_{ti}\rangle}{\sqrt{|I_t|}} \right|^{p'+\delta} \chi_{I_t}(x) \right]^{\frac{1}{p'+\delta}} \right\|_p \leq C_{p\delta}\lambda^{\varepsilon}(1 + A^{-1/\varepsilon}\lambda) \inf_{x \in J} M_p(Mg)(x).$$

This last inequality holds for all p and g. We specialize to the case of f_i and $p = p_i$. Moreover, we have the information supplied from the exclusion of tiles t with $I_t \subset E$. Thus,

$$(N^{p_i/(p_i'+\delta)}_{\mathbf{S}_m})^\sharp(x) \leq Cb^{-p_i}\lambda^{\varepsilon}(1 + A^{-1/\varepsilon}\lambda)^{p_i} \min\{1, M_{p_i}(Mf_i)(x)\}^{p_i},$$

where $g^\sharp$ denotes the sharp maximal function. This inequality proves inequality **4.2** by raising to the $r = (p_i' + 2\delta)/p_i$ power, integrating and using $\|g\|_r \leq C_r\|g^\sharp\|_r$.

Finally, there is the case of providing the counting function estimate for $\mathbf{S}_{nij}$ for $i \neq j$. But we have taken care to construct trees that are disjoint in the sense of condition **3.8** and so they too satisfy an orthogonality principle; see *Lemma 3* of ref. 1. The argument is then much like the one just presented.

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