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The capital-asset-pricing model and arbitrage pricing theory: A unification

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ABSTRACT We present a model of a financial market in which naive diversification, based simply on portfolio size and obtained as a consequence of the law of large numbers, is distinguished from efficient diversification, based on mean-variance analysis. This distinction yields a valuation formula involving only the *essential* risk embodied in an asset's return, where the overall risk can be decomposed into a *systematic* and an *unsystematic* part, as in the arbitrage pricing theory; and the systematic component further decomposed into an *essential* and an *inessential* part, as in the capital-asset-pricing model. The two theories are thus unified, and their individual asset-pricing formulas shown to be equivalent to the pervasive economic principle of no arbitrage. The factors in the model are endogenously chosen by a procedure analogous to the Karhunen–Loéve expansion of continuous time stochastic processes; it has an optimality property justifying the use of a relatively small number of them to describe the underlying correlational structures. Our idealized limit model is based on a continuum of assets indexed by a hyperfinite Loeb measure space, and it is asymptotically implementable in a setting with a large but finite number of assets. Because the difficulties in the formulation of the law of large numbers with a standard continuum of random variables are well known, the model uncovers some basic phenomena not amenable to classical methods, and whose approximate counterparts are not already, or even readily, apparent in the asymptotic setting.

Modern asset pricing theories rest on the notion that the expected return of a particular asset depends only on that component of the total risk embodied in it that cannot be diversified away [refs. 1 and 2 (pp. 173–197)]. A market equilibrium, by definition, precludes a price system under which diversification earns a reward, and thus, in a world of costless arbitrage, the fundamental question for asset pricing reduces to the identification and measurement of the relevant component of risk that exercises influence on an asset's expected return.

In the capital-asset-pricing model (CAPM; as in refs. 3 and 4), a particular mean-variance efficient portfolio is singled out and used as a formalization of *essential* risk in the market as a whole, and the expected return of an asset is related to its normalized covariance with this market portfolio—the so-called beta of the asset. The residual component in the total risk of a particular asset, *inessential* risk, does not earn any reward because it can be eliminated by another portfolio with an identical cost and return but with lower level of risk (3–8). On the other hand, in the arbitrage pricing theory (APT; as in ref. 9), a given finite number of factors is used as a formalization of *systematic* risks in the market as a whole, and the expected return of an asset is related to its exposure to each of these factors, and now summarized by a vector of factor loadings. The reward to the residual component in the return to a particular asset, *unsystematic* or *idiosyncratic* risk, can be made arbitrarily small simply by considering portfolios with an arbitrarily large number of assets.

The basic point, however, is that the two theories capture two different sets of risks and address different aspects of the premium-awarding scheme for taking such risks. The CAPM, by its emphasis on efficient diversification in the context of a finite number of assets, neglects unsystematic risks in the sense of the APT; whereas the APT, with its explicit focus on markets with a "large" number of assets, and by its emphasis on naive diversification and on the law of large numbers, neglects essential risks. The two theories seem to be inherently disjoint. It is surprising, however, that a model which unifies their basic ingredients can nevertheless be found; and moreover, that it is one in which the absence of arbitrage opportunities is not only sufficient, but in contrast to the literature, also necessary for the validity of the APT pricing formula. We present this model here.

It is easy to see why such a unification has not been considered so far. A natural way to proceed is to work with a limit model of a financial market with a continuum of assets, to identify the ensemble of systematic risks, and within this, the essential risk emanating from a suitably constructed "market" portfolio. Standard methods, however, do not permit any progress toward a limit model in which nontrivial portfolios with genuinely unsystematic or asset-specific risks can be included; the difficulties associated with a version of the law of large numbers for a standard continuum of random variables—say the Lebesgue unit interval as an index set—are well understood [see refs. 10 (theorem 2.2), and 11–13]. An alternative way is to follow the APT literature and work with an increasing sequence of asset markets, but here one has to overcome at least three obstacles: unsystematic risks are never completely eliminated, exogenous factor structures are not sufficiently refined to yield orthogonal factor loadings, and pricing formulas are approximate with convergent requirements on infinite series (see refs. 14–16). Under these considerations, it is not evident how to introduce a simple explicit formula for essential risk, or more generally, how to relate important portfolios to the associated factor structures.

Our idealized limit model of asset pricing is based on a hyperfinite continuum of assets (17, 18). In this setting, we can appeal to a hyperfinite analogue of the Karhunen-Loéve expansion of continuous time stochastic processes (18–22), and derive factors endogenously from the process of asset returns. These factors are used to formalize systematic risks and to construct a "market" portfolio for a further specification of essential risk. The valuation formula then shows that the usual claim, based on the APT, that the market only rewards the holding of systematic risks, is simply not sharp

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Abbreviations: CAPM, capital-asset-pricing model; APT, arbitrage pricing theory.

enough; systematic risk can be reduced still further until a portfolio has only essential risk, and it is only this component of systematic risk that earns a premium. Even in the presence of many sources of industry-wide or market-wide factor risks, there is surprisingly only a *unique* source of risk, characterized by one random variable, that is rewarded by the market, and the risk premium assigned to a particular factor risk only reflects the role of that factor in the definition of essential risk. In brief, our study clarifies three different types of risks in financial markets, while previous work only focussed on two of them; each, one at a time, and in a different setting. Our results are asymptotically implementable; asymptotic arguments, besides being pervasive in actuarial situations (ref. 23, especially the final two paragraphs), furnish a valuable check on measurability and other assumptions that are imposed on the limit. Details of proofs will be presented elsewhere.

The work reported here does not simply provide a framework in which epsilons can be rigorously equated to zero, or somewhat less naively, rely on a particular stochastic process for which the law of large numbers holds, as constructed for example in refs. 24–27. It belongs to the genre of ideal limit models that illustrate phenomena obscured in the discrete case; for examples in economic science, see refs. 28 and 29 in general equilibrium theory, see refs. 30 and 31 in cooperative and noncooperative game theory, and ref. 32 in continuous-time finance.

The Underlying Framework

In ref. 18, a hyperfinite Loeb space (17) is used to model probabilistic phenomena involving a large number of random variables in situations where there is no natural topology on the index set. Just as the set $\{1, 2, \cdots, n\}$, endowed with the uniform probability measure, is a natural space of names for a situation with n assets, a hyperfinite Loeb counting measure space provides, by the very terms of the nonstandard extension, a useful framework for the situation of an increasing sequence of asset markets indexed by $\{\{1\}, \{1, 2\}, \cdots, \{1, 2, \cdots, n\}, \cdots\}$. Besides being asymptotically implementable, a Loeb space is a standard measure space, and therefore allows one to invoke mathematical structures not available in the finite, or the large but finite, case. However, the fact that it is constituted by nonstandard entities can be altogether ignored;‡ and attention focussed simply on its special properties not shared by general measure spaces.

Let T be a hyperfinite set, $\mathcal{T}$ the internal power set of T, λ the internal counting probability measure on $(T, \mathcal{T})$, and $(T, L(\mathcal{T}), L(\lambda))$ the standardization of the corresponding internal probability space, the Loeb space. We shall use T as the space of asset names, and another atomless Loeb space $(\Omega, L(\mathcal{A}), L(P))$ as the sample space formalizing all possible uncertain social or natural states relevant to the asset market. There are two ways of formalizing how the index set intertwines with the sample space.§ The completion of the standard product measure space $(T \times \Omega, L(\mathcal{T}) \otimes L(\mathcal{A}), L(\lambda) \otimes L(P))$ can be constructed in the conventional way by taking the product of the two relevant Loeb spaces, while the *Loeb product space* is the standardization $(T \times \Omega, L(\mathcal{T} \otimes \mathcal{A}), L(\lambda \otimes P))$ of the internal product space $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \otimes P)$. It is an interesting fact that the former σ-algebra, denoted here by $\mathcal{U}$, is strictly contained in the Loeb product σ-algebra, except in the trivial case, inapplicable here, when one of the measures $L(\lambda)$ and $L(P)$ is purely atomic [refs. 18 and 38 (example 3.12.13 presents a special case of this fact)]. This observation constitutes the technical point of departure for the approach taken in (18) and also exploited here. If the structure of one-period asset returns is modeled by a real-valued $L(\mathcal{T} \otimes \mathcal{A})$-measurable function x, its mean by μ, and the unexpected return by $f = x - \mu$, the conditional expectation $E(f|_{\mathcal{U}})$ furnishes us with the key "smoothing operation" that we seek for a viable formulation of the ensembles of unsystematic and systematic risks embodied in the market as a whole. To see this, we turn to some specific results in ref. 18.

We assume the process of asset returns x to have a finite second moment, and work with the Hilbert space $\mathcal{L}^2(L(\lambda \otimes P))$ of real-valued $L(\lambda \otimes P)$-square integrable functions on $T \times \Omega$. For each asset t in $T, x_t \equiv x(t, \cdot) \in \mathcal{L}^2(L(P))$, and for each state ω in Ω, $x_\omega \equiv x(\cdot, \omega) \in \mathcal{L}^2(L(\lambda))$. Consider the infinite-dimensional analogue of the covariance matrix of asset returns, the autocorrelation function $R(t_1, t_2) = \int_\Omega f(t_1, \omega)f(t_2, \omega)dL(P)$, and use it as a kernel to define a compact, self-adjoint and positive semidefinite operator on $\mathcal{L}^2(L(P))$. Similarly, the sample autocorrelation function can be used to define a dual operator on $\mathcal{L}^2(L(\lambda))$. For each operator, let $\{\psi_n\}_{n=1}^\infty$ and $\{\varphi_n\}_{n=1}^\infty$ be the respective complete eigensystems, adjusted to form an orthonormal family, and $\{\lambda_n^2\}_{n=1}^\infty$ their common, nonincreasing sequence of all the positive eigenvalues, each eigenvalue being repeated up to its multiplicity. Theorems 1–3 in ref. 18 show that f can be expressed, for $L(\lambda \otimes \lambda)$-almost all $(t, \omega) \in T \times \Omega$, as

$$f(t,\omega) = x(t,\omega) - \mu(t) = \sum_{n=1}^{\infty} \lambda_n \psi_n(t) \varphi_n(\omega) + e(t,\omega), \quad \textbf{[1]}$$

where $E(f|_{\mathcal{U}})(t, \omega) = \sum_{n=1}^\infty \lambda_n \psi_n(t) \varphi_n(\omega)$,¶ and the residual term e has "low" intercorrelation and satisfies the law of large numbers in a strong sense with $E(e|_{\mathcal{U}}) = 0$. Since Eq. **1** can be seen as a hyperfinite analogue of the usual factor model (20, 39), the φ_n, ψ_n, and λ_n are to be called factors, factor loadings, and scaling constants.‖

Now, for the purpose of this paper, let us refer to a *risk* as any centered random variable defined on the sample space Ω and with a finite variance. As usual, we shall use its variance to measure its *level of risk*. The following is a precise formalization of a basic intuition.

Definition 1: A centered random variable defined on the sample space is an unsystematic (idiosyncratic) risk if it has finite variance and is uncorrelated with x_t for $L(\lambda)$-almost all $t \in T$.

By theorem 2 in ref. 18, one can require that for each $t \in T$, e_t is uncorrelated with x_s for $L(\lambda)$-almost all $s \in T$. Thus e_t is an unsystematic risk for $t \in T$ and e represents the ensemble of all unsystematic risks in the market. Since e satisfies the law of large numbers in a strong sense, it is also easy to see that a risk is unsystematic if and only if it is uncorrelated with all the factors φ_n. It is thus natural to refer to all risks perfectly correlated with some linear combinations of the factors, as systematic risks. Formally,

Definition 2: A centered random variable defined on the sample space exhibits systematic risk if it has finite variance

‡The relevant analogy is to all those situations when the use of a Lebesgue measure space does not depend on the Dedekind set-theoretic construction of real numbers, or on the particular construction of Lebesgue measure. For details on Loeb spaces and on nonstandard analysis, refs. 17, 24–27, 33, and 34.

§For the standard mathematical concepts in this and the next two paragraphs, see ref. 35 (chapter 8), ref. 19 (chapter 8), and ref. 36 (chapter 2). For the relevant version of Fubini's theorem, see refs. 37 and 38.

¶This is the analogue of the Karhunen–Loéve biorthogonal representation but for a setting without a topology on the universe T of asset indices. The representation has had applications in many fields, though not in financial economics to our knowledge; see refs. 20–22 and references therein.

‖The infinite sum is replaced by a sum with m terms if there are only m nontrivial factors in the market.

and is in the linear space $\mathcal{F}$ spanned by all of the factors φ_n, $n \geq 1$.

Eq. **1** now tells us that $E(f|_{\mathcal{U}})$ represents the ensemble of systematic risks in the market. In the context an individual asset t, its total risk f_t is decomposed into two components, a systematic risk component $\sum_{n=1}^{\infty} \lambda_n \psi_n(t) \varphi_n$ and an unsystematic risk component e_t.

Note that the factors φ_n are constructed endogenously from the given process of asset returns, and thereby respond to the criticism of the arbitrariness of the choice of factors in the APT (40, 41). From another point of view, a principal motivation behind factor analysis is to find a small enough set of latent variables so that the systematic behavior of the directly observed random variables can be adequately identified and explained (20, 39). If one is allowed to use only m sources of risk to measure the ensemble of systematic risks of the market, then the m random variables ought to be chosen in a way that any other set of m random variables makes a smaller contribution towards explaining the correlational structure of the process of asset returns. It can be shown that this kind of best approximation is indeed achieved by using the first m elements of the set of factors $\{\varphi_n\}$.

The specification of our underlying framework is now complete, and we turn to the formulation of APT and CAPM, and to a further decomposition of systematic risks.

Results

A portfolio p constituted from a process x is a square integrable function on T, its cost $C(p)$ is given by $\int_T p(t)dL(\lambda)$, its expected return by $E(p) = \int_T p(t)\mu(t)dL(\lambda)$, and its random return $\mathcal{R}_p(\omega)$ by $\int_T p(t)x(t, \omega)dL(\lambda)$. The ensemble of unsystematic risks identified above in the market as a whole satisfy a consistent version of the law of large numbers in the sense that the sample averages of various variations of e are $L(P)$-almost surely equal to zero (ref. 18, theorems 1 and 3). Such a statement can be informally expressed as "aggregation removes individual uncertainty," or in another context, "no betting system can *ever* break the house" (18, 23). This fact allows us to compute, from Eq. **1** and $\mathcal{R}_p(\omega)$, the variance $V(p) = \sum_{n=1}^{\infty} \lambda_n^2 (\int_T p(t)\psi_n(t)dL(\lambda))^2$ of any portfolio p by ignoring the unsystematic risk component. This formula shows that a riskless portfolio, namely one with zero variance, is orthogonal to all of the factor loadings ψ_n, and that every portfolio is completely diversified in the sense that it embodies only factor variance and no unsystematic variance.

We can now present the basic theorem of APT, but without the assumption of an exogenously given, exact or approximate, factor structure. It needs to be emphasized that the no arbitrage condition is not only sufficient but also necessary for the validity of the asset pricing formula. Such a necessity condition is surprisingly absent in the APT literature.

Definition 3: The market permits no arbitrage opportunities if and only if for any portfolio p, $V(p) = C(p) = 0$ implies $E(p) = 0$.

THEOREM 1. *The market permits no arbitrage opportunities* $\Leftrightarrow$ *there is a sequence* $\{\tau_n\}_{n=0}^{\infty}$ *of real numbers such that for* $L(\lambda)$*-almost all* $t \in T$, $\mu(t) = \tau_0 + \sum_{n=1}^{\infty} \tau_n\psi_n(t)$.

Next, we turn to our results on the equivalence between the APT and the CAPM.

THEOREM 2. *The following equivalence holds: There is a portfolio* M *and a real number* ρ *such that for* $L(\lambda)$*-almost all* $t \in T$, $\mu(t) = \rho + cov(x_t, M) \Leftrightarrow$ *there is a sequence* $\{\tau_n\}_{n=0}^{\infty}$ *of real numbers such that* $\sum_{n=1}^{\infty} (\tau_n^2/\lambda_n^4) < \infty$, *and for* $L(\lambda)$*-almost all* $t \in T$, $\mu(t) = \tau_0 + \sum_{n=1}^{\infty} \tau_n\psi_n(t)$.

Theorem 2 does not furnish a precise specification of the portfolio M. We can construct a portfolio I_0 based on parameters extracted from the given market process of asset returns x and use it to identify the essential risk embodied in the realized return of a particular asset. Let s_n, μ_n be the inner products $(1, \psi_n)$, (μ, ψ_n), h a risk-free portfolio defined by $h = (1 - \sum_{n=1}^{\infty} s_n\psi_n)$, and $\mu_0 = (\mu, h)/(h, h)$, if $h \not\equiv 0$ and $\mu_0 = 0$ if $h = 0$. We can now define the index portfolio I_0 to be $\sum_{n=1}^{\infty} ((\mu_n - \mu_0 s_n)/\lambda_n^2)\psi_n$, with a net random return $X_0 = \sum_{n=1}^{\infty} ((\mu_n - \mu_0 s_n)/\lambda_n)\varphi_n$, cost $C(I_0) = \sum_{n=1}^{\infty} s_n(\mu_n - \mu_0 s_n)/\lambda_n^2$, mean $E(I_0) = \sum_{n=1}^{\infty} \mu_n(\mu_n - \mu_0 s_n)/\lambda_n^2$ and variance $V(I_0) = \sum_{n=1}^{\infty} (\mu_n - \mu_0 s_n)^2/\lambda_n^2$. We can use I_0 to present

Definition 4: A centered random variable defined on the sample space exhibits essential risk if it is in the linear space generated by the net random return X_0 of the index portfolio I_0, and exhibits inessential risk if it is in the orthogonal complement of X_0 in the space $\mathcal{F}$ of systematic risks.

We can now present a further refinement of systematic risks, and thereby a tri-partite decomposition of the total risk of an asset. If $V(I_0) \neq 0$, let the normalized covariance of any asset t with the index portfolio be given by $\beta_t = \text{cov}(x_t, I_0)/V(I_0)$. It can be checked that $Y_t(\omega) = \sum_{n=1}^{\infty} (\lambda_n\psi_n(t) - ((\mu_n - \mu_0 s_n)/\lambda_n)\beta_t)\varphi_n(\omega)$ is orthogonal to X_0 and its sum with $\beta_t X_0(\omega)$ is the portion of systematic risk $\sum_{n=1}^{\infty} \lambda_n\psi_n(t)\varphi_n$ in asset t. Then, by Eq. **1**, the total risk of asset t can be written as

$$x_t(\omega) - \mu(t) = \beta_t X_0(\omega) + Y_t(\omega) + e_t(\omega). \qquad \textbf{[2]}$$

The importance of this decomposition lies in the fact that the risk premium of almost all assets is equal to the beta of the asset multiplied by the risk premium of the index portfolio.

THEOREM 3. *Assume that* $\sum_{n=1}^{\infty} (\mu_n - \mu_0 s_n)^2/\lambda_n^4 < \infty$ *and the market is nontrivial in the sense that the expected return function* μ *is not the constant function* μ_0. *Then there is no arbitrage* $\Leftrightarrow$ *there is a sequence* $\{\tau_n\}_{n=0}^{\infty}$ *of real numbers such that for* $L(\lambda)$*-almost all* $t \in T$, $\mu_t = \tau_0 + \sum_{n=1}^{\infty} \tau_n\psi_n(t) \Leftrightarrow$ *for* $L(\lambda)$*-almost all* $t \in T$, $\mu_t = \mu_0 + \text{cov}(x_t, I_0) = \mu_0 + \beta_t(E(I_0) - \mu_0 C(I_0))$.

Once we move to the asymptotic setting of an increasing sequence of finite asset markets, the insights of the limit model can only be extracted in an approximate form. The structure of asset returns is now formalized by a triangular array of random variables $\{x_n\}_{n=1}^{\infty}$ defined on the product space $(T_n \times \Omega, \mathcal{T}_n \otimes \mathcal{A}, \lambda_n \otimes P)$, $T_n = \{1, 2, \cdots, n\}$ endowed with a uniform probability measure λ_n on its power set $\mathcal{T}_n$, and $(\Omega, \mathcal{A}, P)$, a fixed common probability space. For the nth market, the realized return of asset t in T_n is given by x_{nt}, and its mean by $\mu_{nt} = \int_\Omega x_{nt}(\omega)dP$. The square integrability restriction on portfolios in the idealized case translates to a limitation to sequences of portfolios $\{p_n\}_{n=1}^{\infty}$, for which $\sup_{n\geq 1}\{\int_{T_n} p_{nt}^2 d\lambda_n = (1/n)\sum_{t \in T_n} p_{nt}^2\}$ is finite, and that on the processes of asset returns by the requirement that there is a positive number M such that $\int \int_{T_n \times \Omega} x_{nt}^2 dL(\lambda_n \otimes P) \leq M$, for all $n \geq 1$. In particular, in the context of each finite market in this sequence, one can draw on the raw intuition developed for the ideal case, and identify the component of unsystematic risks, the residual error terms $e_n(t, \omega)$, and the factors, factor loadings and scaling constants, $\{\varphi_{ni}, \psi_{ni}, \lambda_{ni}\}_{i=1}^{s_n}$, s_n the number of factors, all pertaining to the nth market. Furthermore, s_n can be chosen to be much smaller than the number of assets in the precise sense that $\lim_{n\to\infty} s_n/n = 0$, and the e_n's satisfy the law of large numbers in an approximate way. Space considerations force us to defer a fuller elaboration of the asymptotic interpretation of the model; we only present the asymptotic analogue of Theorem 2 for illustrative purposes.

THEOREM 4. *The following equivalence holds: There is a sequence* $\{M_n\}_{n=0}^{\infty}$ *of portfolios, and a sequence* $\{\rho_n\}_{n=1}^{\infty}$ *of real numbers such that* $\lim_{n\to\infty} \|\mu_{nt} - (\rho_n + \text{cov}(x_{nt}, M_n))\|_2 = 0 \Leftrightarrow$ *there is a sequence* $\{\tau_{ni}\}_{i=0}^{s_n}$ *of real numbers and a positive number*

B *such that for all* $n \geq 1$, $\sum_{i=1}^{s_n} (\tau_{ni}^2/\lambda_{ni}^4) \leq B$ *and* $\lim_{n\to\infty} \|\mu_n - (\tau_{n0} + \sum_{i=1}^{s_n} \tau_{ni}\psi_{ni})\|_2 = 0$.

Concluding Remarks

The concrete nature of the idealized limit model that we report above allows us to explore conditions for the existence of various important portfolios—risk-free, factor, mean, cost, and mean-variance efficient—and to develop explicit formulas for them. The hyperfinite model, besides being *asymptotically implementable*, also exhibits a *universality* property in the sense that the distributions of the individual random variables in the ensemble of unsystematic risks may be allowed any variety of distributions (18).

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