abnormal proteins or a new species of active protein. Suppressed mutants derived from non-CRM formers produced relatively little of one of the two components suggesting that some defect in A or B protein formation exists in these strains.

The authors are indebted to Mrs. Mary Hurd and Mrs. Virgina Horn for their able technical assistance.

* This investigation was supported by grants from the National Science Foundation and the U. S. Public Health Service (RG-5652).

† Postdoctoral fellow of the National Foundation.

[1] Suskind, S. R., C. Yanofsky, and D. M. Bonner, these PROCEEDINGS, **41**, 577 (1955).

[2] Lerner, P., and C. Yanofsky, *J. Bact.*, **74**, 494 (1957).

[3] Yanofsky, C., *Biochim. Biophys. Acta*, **31**, 408 (1959).

[4] Crawford, I. P., and C. Yanofsky, these PROCEEDINGS, **44**, 1161 (1958).

[5] Yanofsky, C., and M. Rachmeler, *Biochim. et Biophys. Acta*, **28**, **640** (1958).

[6] Yanofsky, C., and J. Stadler, these PROCEEDINGS, **44**, 245 (1958).

[7] Yanofsky, C., and E. Lennox, manuscript submitted for publication.

[8] Yanofsky, C., *J. Bact.*, **68**, 577 (1954).

[9] Lowry, O. H., N. J. Rosebrough, A. L. Farr, and R. J. Randall, *J. Biol. Chem.*, **193**, 265 (1951).

[10] Suskind, S. R., *J. Bact.*, **74**, 308 (1957).

[11] Eisenstein, R., and C. Yanofsky, manuscript in preparation.

[12] See papers by Gorini and Maas, Vogel, and Magasanik in *The Chemical Basis of Development* (Baltimore: Johns Hopkins Press, 1958), and by Pardee, Jacob and Monod in *Compt. rend.*, **246**, 3125 (1958).

[13] Woodward, D. O., C. W. H. Partridge, and N. H. Giles, these PROCEEDINGS **44**, 1237 (1958)

[14] Lacy, A. M., and D. M. Bonner. *Proceedings of the 10th International Congress of Genetics*, 157, 1958.

[15] Pateman, J. A., and J. R. S. Fincham, *Heredity*, **12**, 317 (1958).

[16] Catcheside, D. G., and A. Overton, *Cold Spring Harbor Symposium on Quantitative Biology*, **23**, 137 (1958).

[17] Evidence has not been presented as yet indicating whether the active protein detected in heterocaryons is identical with the wild type enzyme.

[18] Singer, S. J., and H. A. Itano, these PROCEEDINGS, **45**, 174 (1959).

[19] Crick, F. H. C., *Symposia of the Society for Experimental Biology*, **12**, 138 (1958).

[20] Benzer, S., in *The Chemical Basis of Heredity*, ed. McElroy and Glass (Baltimore: John Hopkins Press, 1957).

[21] Benzer, S., and E. Freese, these PROCEEDINGS, **44**, 112 (1958).

[22] Ingram, V. M., *Nature*, **180**, 326 (1957).

[23] Hunt, J. A., and V. M. Ingram, *Nature*, **181**, 1062 (1958).

ON THE LIMIT OF SOLUTIONS OF DIFFERENTIAL-DIFFERENCE EQUATIONS AS THE RETARDATION APPROACHES ZERO

BY RICHARD BELLMAN AND KENNETH L. COOKE

THE RAND CORPORATION AND POMONA COLLEGE

Communicated by S. Lefschetz, May 20, 1959

1. *Introduction.*—In various parts of mathematical physics, we encounter equations of the form

$$\epsilon L_m(u) + L_n(u) = 0 \tag{1.1}$$

where $L_m(u)$ is a differential operator of order m, $L_n(u)$ is a differential operator of order n, $m > n \geq 1$, and ϵ is a small parameter. Examples of this occur in hydrodynamics where ϵ represents a viscosity, and in the study of the multivibrator (in the form of the Van der Pol equation) where ϵ represents a small inductance. A great deal of effort has been devoted in recent years to the study of the limiting behavior of the solutions of (1.1) as $\epsilon \to 0$.

Equations of this type also arise in parts of the scientific world where time-lags and retardations are present. Given a differential-difference equation of the form

$$\begin{aligned} u'(t) + au(t - \epsilon) &= g(u),\ t > \epsilon, \\ u(t) &= f(t),\ 0 \leq t \leq \epsilon, \end{aligned} \tag{1.2}$$

where $0 < \epsilon \ll 1$, one frequently uses as an approximation an equation of the form

$$u'(t) + au(t) - a\epsilon u'(t) + a\epsilon^2 u''(t)/2 = g(u), \tag{1.3}$$

precisely an equation of the type appearing above.

We shall show that the study of the limiting behavior of the solutions of (1.2) is relatively easy to carry out compared to a similar study for (1.1). A typical result and sketch of proof will be presented below. Complete details, extensions, and generalizations will be presented subsequently.

It is interesting to observe that we have a situation in which the more accurate description yields an easier problem, and the obvious approximation yields a more difficult problem.

2. *A Typical Result.*—Representative of the results that can be obtained is the following.

THEOREM. *Consider the equation*

$$\begin{aligned} u'(t) + au(t - \epsilon) &= g(u),\ t > \epsilon, \\ u(t) &= f(t),\ 0 \leq t \leq \epsilon, \end{aligned} \tag{2.1}$$

where $\epsilon > 0$, *and*

$$\begin{aligned} &(a)\ \ g(u)\ \text{satisfies a Lipschitz condition for}\ |u| \leq c_1, \\ &(b)\ \ f(t)\ \text{is continuous at}\ t = 0, \\ &(c)\ \ |f(0)| = c_2 < c_1. \end{aligned} \tag{2.2}$$

Then, for $0 \leq t \leq t_0$, *where* t_0 *is dependent upon* c_1 *and* $g(u)$,

$$\lim_{\epsilon \to 0} u(t) = v(t), \tag{2.3}$$

where $v(t)$ *is the solution of*

$$v'(t) + av(t) = g(v),\ v(0) = f(0). \tag{2.4}$$

The convergence is uniform for $0 \leq t \leq t_0$.

Similar results can be obtained for systems of equations with any finite number of lags.

3. *Sketch of Proof.*—The proof depends, as might be expected, upon the properties of the solution of the linear equation

$$\begin{aligned} u'(t) + au(t - \epsilon) &= g(t),\ t > \epsilon, \\ u(t) &= f(t),\ 0 \leq t \leq \epsilon, \end{aligned} \tag{3.1}$$

which, in turn, depends upon the properties of the kernel

$$K(t, \epsilon) = \frac{1}{2\pi i}\int_C \frac{e^{st}ds}{s + ae^{-\epsilon s}}. \tag{3.2}$$

Here C represents any line of the form $s = s_0 + it$, $-\infty < t < \infty$, with $s_0 > |a|$. It is easy to show that as $\epsilon \rightarrow 0$,

$$K(t, \epsilon) \rightarrow K(t) = e^{-at} \tag{3.3}$$

uniformly for $0 \leq t \leq t_0 < \infty$, where t_0 is any fixed finite quantity.

The solution of (1.2), under the assumptions that we have made, can be obtained as the solution of the integral equation

$$u(t) = u_0(t) + \int_\epsilon^t g(u(t_1))K(t - t_1, \epsilon)dt_1,\ t > \epsilon, \tag{3.4}$$

where $u_0(t)$ is the solution of (1) corresponding to $g(u) \equiv 0$.

This we solve by the method of successive approximations. Set

$$\begin{aligned} u_{n+1}(t) &= f(t),\ 0 \leq t \leq \epsilon,\ n \geq 0, \\ u_{n+1}(t) &= u_0(t) + \int_\epsilon^t g(u_n(t_1))K(t - t_1, \epsilon)dt_1,\ t > \epsilon,\ n \geq 0. \end{aligned} \tag{3.5}$$

Under the foregoing assumptions, it is easy to show that $u_n(t) \rightarrow u(t)$ uniformly for $0 \leq \epsilon \leq \epsilon_0$, $0 \leq t \leq t_0$, for $n = 0, 1, \ldots$, and that as $\epsilon \rightarrow 0$, $u_n(t) - v_n(t) \rightarrow 0$, uniformly for $0 \leq t \leq t_0$, where the sequence $\{v_n(t)\}$ is determined by the relations

$$\begin{aligned} v_0(t) &= f(0), \\ v_{n+1}(t) &= v_0(t) + \int_0^t g(v_n(t_1))e^{-a(t - t_1)}\,dt_1, \\ &\qquad t \geqq 0,\ n \geqq 0. \end{aligned} \tag{3.6}$$

Since the same assumptions ensure that $v_n(t) \rightarrow v(t)$ uniformly in $0 \leq t \leq t_0$, where $v(t)$ is the solution of

$$v(t) = f(0) + \int_0^t g(v(t_1))e^{-a(t - t_1)}dt_1, \tag{3.7}$$

we have the desired uniform convergence of $u(t)$ to $v(t)$ in $0 \leq t \leq t_0$ as $\epsilon \rightarrow 0$.

The same method can be applied to demonstrate the corresponding result for more general differential-difference equations.

THE RICCI PRINCIPAL DIRECTION VECTORS

BY LUTHER P. EISENHART

PRINCETON UNIVERSITY

Communicated May 11,1959

1. In a space V_n of n dimensions the Ricci principal direction vectors denoted $\lambda_{h|}{}^i$, as h takes the values 1 to n, are defined by the equations[1]

$$(R_{ij} + \rho_h g_{ij})\,\lambda_{h|}{}^i = 0, \tag{1}$$

where R_{ij} is the Ricci tensor, g_{ij} is the metric tensor, and ρ_h as h takes the values 1 to n, are scalars. Both R_{ij} and g_{ij} are symmetric in their indices i and j.