$$p_1[M] = 3\sigma(M).$$

(Cf. Hirzebruch, ref. 4, Theorem 8.2.2, p. 85.) The proof of Lemma 1 is based on the fact that $\pi_{n+3}(S^n)$ is cyclic of order 24 for $n \geq 5$.

‡ In the sense of J. H. C. Whitehead.[10] Any two regular neighborhoods of K in M are combinatorially equivalent by reference 10, Theorem 23.

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AN ALGORITHM FOR EQUILIBRIUM POINTS IN BIMATRIX GAMES

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1. Recently N. N. Vorobjev[1] has presented a constructive procedure for computing all equilibrium points for the case of bimatrix (i.e., finite two-person noncooperative non-zero-sum) games. The purpose of the present note is to simplify his algorithm both in theory and application. In the terms of his paper, the classification of extreme equilibrium strategies into two types is eliminated, and the enumeration of all such strategies is reduced to a single routine.

2. For the sake of easy comparison with Vorobjev's work, his notation will be used. If M is any matrix, $M_{i.}$ denotes the ith row of M, $M_{.j}$ denotes the jth column of M, and M^T denotes the transpose of M. Furthermore, J_p denotes the p-dimensional vector with all components equal to one, and O_p denotes the p-dimensional vector with all components equal to zero. Inequalities between vectors are to hold in all components.

A *bimatrix game* Γ is defined by two real m by n payoff matrices, $A = (a_{ij})$ and $B = (b_{ij})$: if player 1 chooses $i \epsilon \{1, \ldots, m\}$ and player 2 chooses $j \epsilon \{1, \ldots, n\}$, then player 1 is paid a_{ij} and 2 is paid b_{ij}. Mixed strategies for 1 and 2 are probability vectors of dimension m and n and are denoted by X and Y respectively. Thus,

$$XJ_m^T = 1, X \geqq O_m \quad \text{and} \quad J_nY^T = 1, \quad Y \geqq O_n.$$

If 1 uses mixed strategy X and 2 uses Y, then their expected payoffs are XAY^T and XBY^T, respectively. An *equilibrium point* is a pair of mixed strategies $(\bar{X}, \bar{Y})$ such that

$$\bar{X}A\bar{Y}^T \geqq XA\bar{Y}^T \quad \text{and} \quad \bar{X}B\bar{Y}^T \geqq \bar{X}BY^T$$

for all X and Y. The set of all equilibrium points for Γ, which is nonempty by the theorem of Nash,[2] will be denoted by $\mathcal{S}_\Gamma$.

Clearly $(\bar{X}, \bar{Y}) \in \mathcal{S}_\Gamma$ if and only if

$$\bar{X}A\bar{Y}^T \geqq A_{i.}\bar{Y}^T \quad \text{and} \quad \bar{X}B\bar{Y}^T \geqq \bar{X}B_{.j}$$

for all i and j.

3. Given any X, set $\mathcal{S}(X) = \{Y | (X, Y) \in \mathcal{S}_\Gamma\}$. Since $\mathcal{S}(X)$ is the solution set of the finite system of linear inequalities,

$$XAY^T \geqq A_{i.}Y^T \quad (i = 1, \ldots, m)$$
$$XBY^T \geqq XB_{.j} \quad (j = 1, \ldots, n)$$
$$Y \geqq O_n, \quad J_nY^T = 1,$$

it is clear that $\mathcal{S}(X)$ is a compact convex set, which is possibly empty (compare the Lemma, pp. 319–320 of ref. 1). For any set $\mathfrak{X}$ of mixed strategies X, set

$$\mathcal{S}(\mathfrak{X}) = \bigcap_{X \in \mathfrak{X}} \mathcal{S}(X).$$

Clearly, $\mathcal{S}(\mathfrak{X})$ is also a compact convex set, which is possibly empty. For any set $\mathcal{S}$, $K(\mathcal{S})$ will denote the set of extreme points of $\mathcal{S}$.

DEFINITION. *The mixed strategy Y is called an extreme equilibrium strategy if*

$$Y \in K\left(\bigcap_{k=1}^{p} \mathcal{S}(X_k)\right) = K(\mathcal{S}(\mathfrak{X}))$$

for some finite set $\mathfrak{X} = \{X_1, \ldots, X_p\}$ of mixed strategies.

It should be clear that analogous definitions can be given with the roles of the players reversed.

4. The results of Vorobjev can now be stated as follows:

THEOREM 1. *It is possible to enumerate effectively finite sets $\tilde{\mathfrak{X}}$ and $\tilde{\mathfrak{Y}}$, which contain all extreme equilibrium strategies for the two players.*

THEOREM 2. *For any finite set $\mathfrak{X}$, it is possible to describe effectively the set $\mathcal{S}(\mathfrak{X})$ by computing its extreme points, which are finite in number.*

THEOREM 3.

$$\mathcal{S}_\Gamma = \bigcup_{\mathfrak{X} \subset \tilde{\mathfrak{X}}} [\mathfrak{X}] \times \mathcal{S}(\mathfrak{X}),$$

where $[\mathfrak{X}]$ denotes the convex hull of $\mathfrak{X}$.

Vorobjev has derived Theorems 2 and 3 from Theorem 1 in a very elegant manner. His proofs will be repeated here to make this account self-contained.

Proof of Theorem 2: Since $\mathcal{S}(\mathfrak{X})$ is a compact convex set, to describe $\mathcal{S}(\mathfrak{X})$ we need only find $K(\mathcal{S}(\mathfrak{X}))$. However, $K(\mathcal{S}(\mathfrak{X})) \subset \tilde{\mathfrak{Y}}$, since $\tilde{\mathfrak{Y}}$ contains all extreme equilibrium strategies. Set $\mathfrak{Y} = \mathcal{S}(\mathfrak{X}) \cap \tilde{\mathfrak{Y}}$. To verify whether a point Y of $\tilde{\mathfrak{Y}}$ also lies in $\mathcal{S}(\mathfrak{X})$, we need only test the finite set of inequalities $XAY^T \geqq A_{i.}Y^T$, $XBY^T \geqq$

$XB_{\cdot j}$ for the finite set of points $X \epsilon \mathfrak{X}$. Since this process is finite, $\mathfrak{Y}$ is effectively enumerable.

Since $K(\mathcal{S}(\mathfrak{X})) \subset \mathfrak{Y} \subset \mathcal{S}(\mathfrak{X})$, upon deleting those points of $\mathfrak{Y}$ which are convex combinations of other points of $\mathfrak{Y}$, the set $K(\mathcal{S}(\mathfrak{X}))$ remains. Since $\mathfrak{Y}$ is a finite set, this process of deletion is effective.

Proof of Theorem 3: Let $(X, Y) \epsilon \mathcal{S}_\Gamma$. Then $K(\mathcal{S}(Y)) \subset \tilde{\mathfrak{X}}$, because $\tilde{\mathfrak{X}}$ contains all extreme equilibrium strategies. Let $\mathfrak{X}$ denote the finite set $K(\mathcal{S}(Y))$. If $X \epsilon \mathcal{S}(Y)$, then $X \epsilon [K(\mathcal{S}(Y))] = [\mathfrak{X}]$ because $\mathcal{S}(Y)$ is a compact convex set. Clearly $\mathfrak{X} \subset \mathcal{S}(Y)$ and hence $Y \epsilon \mathcal{S}(X^*)$ for all $X^* \epsilon \mathfrak{X}$. This means $Y \epsilon \mathcal{S}(\mathfrak{X}) = \bigcap_{X^* \epsilon \mathfrak{X}} \mathcal{S}(X^*)$.

Conversely, let $X \epsilon [\mathfrak{X}]$ and $Y \epsilon \mathcal{S}(\mathfrak{X})$ for some finite set $\mathfrak{X} \subset \tilde{\mathfrak{X}}$. Then $\mathfrak{X} \subset \mathcal{S}(Y)$ and $[\mathfrak{X}] \subset \mathcal{S}(Y)$ because $\mathcal{S}(Y)$ is convex. Therefore, $X \epsilon \mathcal{S}(Y)$, which means $(X, Y) \epsilon \mathcal{S}_\Gamma$.

5. *Proof of Theorem 1:* The proof will be based on the following two lemmas.

LEMMA 1. *Let $\bar{Y}$ be an extreme equilibrium strategy. Set $\bar{\alpha} = \max_i A_{i\cdot}\bar{Y}^T$. Then $(\bar{Y}, \bar{\alpha})$ is an extreme solution of the following system:*

$$\alpha J_m^T \geqq AY^T, \quad Y \geqq O_n, \quad J_n Y^T = 1.$$

Proof: For $\bar{Y}$ to be an extreme equilibrium strategy means that there exists a finite set $\mathfrak{X} = \{X_1, \ldots, X_p\}$ such that $\bar{Y}$ is an extreme solution of

$$X_k A Y^T \geqq A_{i\cdot} Y^T \quad (i = 1, \ldots, m; \ k = 1, \ldots, p)$$
$$X_k B Y^T \geqq X_k B_{\cdot j} \quad (j = 1, \ldots, n; \ k = 1, \ldots, p)$$
$$Y \geqq O_n, \qquad J_n Y^T = 1.$$

Clearly, $X_k A \bar{Y}^T = \bar{\alpha}$ for $k = 1, \ldots, p$.

Suppose $\bar{Y} = {}^1/_2(Y' + Y'')$ and $\bar{\alpha} = {}^1/_2(\alpha' + \alpha'')$, where (Y', α') and (Y'', α'') are distinct solutions to the system

$$\alpha \geqq A_{i\cdot} Y^T \quad (i = 1, \ldots, m)$$
$$X_k B Y^T \geqq X_k B_{\cdot j} \quad (j = 1, \ldots, n; \ k = 1, \ldots, p)$$
$$Y \geqq O_n, \ J_n Y^T = 1.$$

Then,

$$\bar{\alpha} = \max_i A_{i\cdot}\bar{Y}^T \leqq {}^1/_2(\max_i A_{i\cdot}Y'^T + \max_i A_{i\cdot}Y''^T) \leqq {}^1/_2(\alpha' + \alpha'') = \bar{\alpha},$$

and hence,

$$\max_i A_{i\cdot}Y'^T = \alpha', \ \max_i A_{i\cdot}Y''^T = \alpha''.$$

On the other hand,

$$\bar{\alpha} = X_k A \bar{Y}^T = {}^1/_2(X_k A Y'^T + X_k A Y''^T) \leqq {}^1/_2(\alpha' + \alpha'') = \bar{\alpha},$$

and hence,

$$X_k A Y'^T = \alpha', \ X_k A Y''^T = \alpha''$$

for $k = 1, \ldots, p$. However, this proves $Y' \epsilon \mathcal{S}(\mathfrak{X})$ and $Y'' \epsilon \mathcal{S}(\mathfrak{X})$. Since $\bar{Y} =$

$^1/_2(Y' + Y'')$ is extreme in $\mathcal{S}(\mathfrak{X})$, this implies $Y' = Y'' = \bar{Y}$ and hence $\alpha' = \alpha'' = \bar{\alpha}$. However, this contradicts $(Y', \alpha') \neq (Y'', \alpha'')$.

Therefore, $\bar{Y}$ is an extreme solution to the system in which the inequalities

$$X_k B Y^T \geqq X_k B_{.j} \quad (j = 1, \ldots, n; \ \ k = 1, \ldots, p)$$

are adjoined to the system of the lemma. However, these merely require that $y_j = 0$ for those j for which $X_k B_{.j} < \max_l X_k B_{.l}$ for some k. Call this set N. Suppose $\bar{Y} = {}^1/_2(Y' + Y'')$, where Y' and Y'' are distinct solutions to the system of the lemma. Since $Y' \geqq O_n$, $Y'' \geqq O_n$, and $\bar{y}_j = 0$ for $j \in N$, we have $y_j' = y_j'' = 0$ for $j \in N$ and hence Y' and Y'' also solve the enlarged system. This contradiction proves the lemma.

LEMMA 2. *Let $(\bar{Y}, \bar{\alpha})$ be an extreme solution of the system*

$$\alpha J_m^T \geqq AY^T, \quad Y \geqq O_n, \quad J_n Y^T = 1.$$

Then there exists an s by s submatrix D of A such that

$$\bar{D} = \begin{pmatrix} D & -J_s^T \\ J_s & 0 \end{pmatrix}$$

is nonsingular. Furthermore, renumbering rows and columns, if necessary, to place D in the upper left corner of A,

$$\bar{y}_j = \sum_{i=1}^{s} D_{ij} \Big/ |\bar{D}| = \sum_{i=1}^{s} D_{ij} \Big/ \sum_{i,j=1}^{s} D_{ij} \quad (j = 1, \ldots, s)$$

$$\bar{y}_j = 0 \quad (j = s + 1, \ldots, n)$$

$$\bar{\alpha} = |D| \Big/ |\bar{D}| = |D| \Big/ \sum_{i,j=1}^{s} D_{ij},$$

(D_{ij} denotes the cofactor of a_{ij} in D; $|D|$ and $|\bar{D}|$ denote the determinants of D and $\bar{D}$).

Proof: If $(\bar{Y}, \bar{\alpha})$ is an extreme solution to

$$\alpha J_m^T \geqq AY^T, \ Y \geqq O_n, \ J_n Y^T = 1,$$

then $A_{i.}\bar{Y}^T = \bar{\alpha}$ for some i. Reindex rows, if necessary, so that $A_{i.}\bar{Y}^T = \bar{\alpha}$ for $i = 1, \ldots, r$ and $A_{i.}\bar{Y}^T < \bar{\alpha}$ for $i = r + 1, \ldots, m$. Since $\bar{Y} \geqq O_n$ and $J_n\bar{Y}^T = 1$, $\bar{y}_j > 0$ for some j. Reindex columns, if necessary, so that $\bar{y}_j > 0$ for $j = 1, \ldots, s$ and $\bar{y}_j = 0$ for $j = s + 1, \ldots, n$.

I contend that the system of equations

$$\sum_{j=1}^{s} a_{ij} y_j = \alpha \quad (i = 1, \ldots, r)$$

$$\sum_{j=1}^{s} y_j = 1$$

has the unique solution $y_j = \bar{y}_j$ for $j = 1, \ldots, s$ and $\alpha = \bar{\alpha}$. Suppose, to the contrary, that there are distinct solutions $(y_1', \ldots, y_s', \alpha')$ and $(y_1'', \ldots, y_s'', \alpha'')$ and define $\bar{Y}'$ and $\bar{Y}''$ by

$$\bar{y}_j' = \bar{y}_j + \epsilon(y_j' - y_j'') \quad \text{and} \quad \bar{y}_j'' = \bar{y}_j + \epsilon(y_j'' - y_j') \qquad (j = 1, \ldots, s)$$
$$\bar{y}_j' = \bar{y}_j'' = 0 \quad (j = s + 1, \ldots, n),$$

where $\epsilon > 0$ is to be chosen. Since $\bar{y}_j > 0$ for $j = 1, \ldots, s$ it is clear that $\bar{Y}' \geqq O_n$ and $\bar{Y}'' \geqq O_n$ for ϵ sufficiently small. Furthermore, since $\sum_{j=1}^{s} y_j' = \sum_{j=1}^{s} y_j'' = 1$, we have $J_n\bar{Y}'^T = 1$ and $J_n\bar{Y}''^T = 1$.

For these mixed strategies, we have

$$A_{i.}\bar{Y}'^T = \bar{\alpha} + \epsilon(\alpha' - \alpha''), \quad A_{i.}\bar{Y}''^T = \bar{\alpha} + \epsilon(\alpha'' - \alpha') \quad (i = 1, \ldots, r)$$

and

$$A_{i.}\bar{Y}'^T = A_{i.}\bar{Y}^T + \epsilon \sum_{j=1}^{s} a_{ij}(y_j' - y_j''), \quad A_{i.}\bar{Y}''^T = A_{i.}\bar{Y}^T + \epsilon \sum_{j=1}^{s} a_{ij}(y_j'' - y_j')$$
$$(i = r + 1, \ldots, m).$$

Since $A_{i.}\bar{Y}^T < \bar{\alpha}$ for $i = r + 1, \ldots, m$, it is possible to choose a fixed δ with the same sign as $\alpha' - \alpha''$ such that

$$(\bar{\alpha} + \delta)J_m^T \geqq A\bar{Y}'^T, \quad (\bar{\alpha} - \delta)J_m^T \geqq A\bar{Y}''^T$$

for some $\epsilon > 0$ sufficiently small. Hence, $(\bar{Y}', \bar{\alpha} + \delta)$ and $(\bar{Y}'', \bar{\alpha} - \delta)$ solve the system of the lemma. However, $(\bar{Y}, \bar{\alpha}) = {}^1/_2\{(\bar{Y}', \bar{\alpha} + \delta) + (\bar{Y}'', \bar{\alpha} - \delta)\}$ is an extreme solution. Hence, $\delta = 0$, which implies $\alpha' = \alpha''$, and $\bar{Y}' = \bar{Y}''$, which implies $(y_1', \ldots, y_s') = (y_1'', \ldots, y_s'')$. This proves the contention made above.

The remainder of the proof is standard.[3] The uniqueness of the solution to the equation system implies that the columns of the matrix

$$\begin{pmatrix} a_{11} & \ldots & a_{1s} & -1 \\ \vdots & & \vdots & \vdots \\ a_{r1} & \ldots & a_{rs} & -1 \\ 1 & \ldots & 1 & 0 \end{pmatrix}$$

are linearly independent. Hence we can choose s rows (including the last row) so that

$$\bar{D} = \begin{pmatrix} a_{11} & \ldots & a_{1s} & -1 \\ \vdots & & \vdots & \vdots \\ a_{s1} & \ldots & a_{ss} & -1 \\ 1 & \ldots & 1 & 0 \end{pmatrix}$$

is nonsingular (the rows may have to be renumbered again). We shall call the matrix

$$D = \begin{pmatrix} a_{11} & \ldots & a_{1s} \\ \vdots & & \vdots \\ a_{s1} & \ldots & a_{ss} \end{pmatrix},$$

which is a square submatrix of A, a *kernel* for the extreme solution $\bar{Y}$ (not "*the*" kernel since the steps in its construction are not unique). The formulas for $\bar{Y}$ and $\bar{\alpha}$ then follow by Cramer's rule.

Since there are only a finite number of square submatrices of A, the proof of Theorem 1 follows by combining Lemmas 1 and 2. Note that not every kernel which provides an extreme solution to the system of Lemma 2 need provide an extreme equilibrium strategy. However, the finite set $\tilde{\mathcal{Y}}$ consisting of all mixed strategies computed from the kernels certainly contains all extreme equilibrium strategies.

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*ORBIT SPACES OF FINITE ABELIAN TRANSFORMATION GROUPS**

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1.—We shall consider transformation groups—or actions—(π, X) where X is a locally compact Hausdorff space and $\pi = \beta_1 \times \ldots \times \beta_r$, each β_i being cyclic of prime order p. An action (π, X) is *free* if the fixedpoint set $F(\gamma)$ is empty for each cyclic subgroup γ. The *free part* $\mathfrak{F}$ of an action (π, X) is the open set $X - \cup F(\gamma)$; the induced action $(\pi, \mathfrak{F})$ is free. We shall consider the problem of determining the cohomology of the orbit spaces of free actions, in particular, the orbit spaces $\mathfrak{F}/\pi$. In the case of cohomology spheres, we simplify and complete the computation given in an earlier note.[1]

2. *Some Cohomology Groups.*[2]—An action (π, X) being given, let $A'(X)$ be the $Z_p(\pi)$-module of Alexander-Spanier cochains on X, values in Z_p, modulo those of empty support, and let $A(X)$ be the submodule of compactly supported elements. $A(X)$ is identical with the module of compactly supported sections of the Alexander-Spanier sheaf on X, values in Z_p. Let $H(X) = H(A(X))$ where H stands for cohomology, and for $\xi \in Z_p(\pi)$, let $H_\xi(X) = H(\xi A(X))$. Multiplication by an element η in $Z_p(\pi)$ induces $\eta^* : H_\xi(X) \rightarrow H_{\xi\eta}(X)$. We have also $i^* : H_{\xi\eta}(X) \rightarrow H_\xi(X)$ induced by the inclusion $\xi\eta A(X) \rightarrow \xi A(X)$.

For $g \in \pi$, $g \neq 1$, let $\sigma(g) = 1 + g + \ldots + g^{p-1}$, $\tau(g) = 1 - g$ and let $\rho(g)$ be either one of $\sigma(g)$, $\tau(g)$ and $\bar{\rho}(g)$ the other. Evidently, $\rho(g)\bar{\rho}(g) = 0$.

Let γ be the cyclic subgroup generated by g. The support of each element of $\rho(g)A(X)$ lies in $X - F(\gamma)$. In fact, if $x \in F(\gamma)$, $c \in A(X)$, we have $(gc)(x) = c(g(x)) = c(x)$. Hence, $(\sigma c)(x) = pc(x) = 0$, $(\tau c)(x) = c(x) - c(x) = 0$; so ρc $(\rho = \rho(g))$ vanishes on $F(\gamma)$. *If γ acts trivially on X, then ρ^* annuls $H(X)$*, since in this case $F(\gamma) = X$.

Let $\mathcal{G}_0 = X$ and for $s = 1, \ldots, r$ let $\mathcal{G}_s = X - \cup_1^s F(\beta_i)$. We have inclusions $A(\mathcal{G}_s) \subset A(X)$, $\rho_1 \ldots \rho_s A(\mathcal{G}_s) \rightarrow \rho_1 \ldots \rho_s A(X)$. The second of these is bijective. For let $c \in A(X)$. The support of $\rho_1 \ldots \rho_s c$ is in $X - F(\beta_i)$, $i = 1, \ldots s$, hence is in $\mathcal{G}_s$. Therefore, $\rho_1 \ldots \rho_s c$ can be identified with an element of $\rho_1 \ldots \rho_s A(\mathcal{G}_s)$. It is