

Orientation in operator algebras

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ABSTRACT A concept of orientation is relevant for the passage from Jordan structure to associative structure in operator algebras. The research reported in this paper bridges the approach of Connes for von Neumann algebras and ourselves for C*-algebras in a general theory of orientation that is of geometric nature and is related to dynamics.

A problem that dates back to the 1950s is to characterize the ordered linear spaces that are the self-adjoint parts of C*-algebras or of von Neumann algebras. This problem was implicit in Kadison's paper (1), and it was explicitly raised for von Neumann algebras by Sakai (2) and for C*-algebras by Sherman (3). It follows from Kadison's results (1) that the self-adjoint part of a C*-algebra is isometrically isomorphic, as an ordered normed linear space, to the space $A(K)$ of all w^* -continuous affine functions on the state space K . Similarly, the self-adjoint part of a von Neumann algebra is isometrically isomorphic to the space of all bounded affine functions on the normal state space. In view of this, characterizing the self-adjoint part of a C*-algebra (respectively von Neumann algebra) is equivalent to characterizing the state space of a C*-algebra (respectively the normal state space of a von Neumann algebra).

Connes gave a solution of the ordered linear space version of this problem for a σ -finite von Neumann algebra in ref. 4 by first characterizing the associated cone P_{ξ}^{\natural} of Tomita-Takesaki theory. The state spaces of C*-algebras were characterized by Alfsen, Hanche-Olsen, and Shultz (5) and the normal state spaces of von Neumann algebras by Iochum and Shultz (6).

By a theorem of Kadison (7) the ordering and the norm of a C*-algebra (or equivalently, its state space) determine the symmetrized (Jordan) product $\frac{1}{2}(ab + ba)$. However, they do not determine the product itself, because the opposite algebra has the same ordering and norm. Thus some additional structure is needed to determine the associative product.

It was Connes (4) who first realized that a concept of orientation is relevant for this purpose (and his concept was later used in the axiomatic context of JBW-algebras by Bellissard and Iochum in refs. 8 and 9). Alfsen, Hanche-Olsen, and Shultz (5) also introduced a concept with the same name and for the same purpose. However, the definitions had little in common. Connes' notion was algebraic, global in nature, and applied to von Neumann algebras. That of Alfsen, Hanche-Olsen, and Shultz was geometric, local in nature, and applied to state spaces of C*-algebras. One purpose of our current work is to generalize both notions so that they apply to both C* and von Neumann algebras. (There are some significant obstacles to overcome to accomplish this as we will discuss later.) Our second purpose is to relate the two concepts and to explain how they both relate to dynamics. We will build a bridge between these two concepts of orientation by introducing a third concept: that of a dynamical correspondence. This paper is a survey of the results with brief comments on the

proofs. Complete proofs will appear elsewhere (in ref. 10 for the first part of this paper.)

There are two parts to this paper. The first describes the notion of dynamical correspondence and its relationship to Connes' notion of orientation. The context for the first part will be JB and JBW algebras: the Jordan analog of C* and von Neumann algebras. The second part contains a generalization of our geometric notion of orientation of state spaces of C*-algebras to the context of normal state spaces of von Neumann algebras and the connection of this notion with dynamical correspondences. The context for the second part will be von Neumann algebras.

To set the stage for the first part, we first review the connection of Jordan algebras to the original problem of characterizing the spaces of self-adjoint elements of an operator algebra and indicate how dynamics play a role. The problem is motivated by physics, as the self-adjoint elements of such algebras are used to represent bounded observables in algebraic models of quantum mechanics. However, the self-adjoint part A of a C*-algebra is not closed under the given associative product, but only under the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$. This product makes A into a (real) Jordan algebra, and it has been proposed to model quantum mechanics on Jordan algebras rather than associative algebras. This approach is corroborated by the fact that many physically relevant properties of observables are adequately described by Jordan constructs. Knowing an element of A , we can express not only the expectation value of the corresponding observable, but its entire probability law, which is given by spectral functional calculus, and in turn by the squaring operation $a \mapsto a^2$.

The Jordan algebra approach to quantum mechanics was initiated by Jordan, von Neumann, and Wigner (11) where they introduced and studied finite dimensional "formally real" Jordan algebras. The restriction to finite dimensions was removed by von Neumann (12). Jordan operator algebras (linear spaces of self-adjoint operators on a Hilbert space closed under the Jordan product) first were studied by Segal (13), Topping (14), and Størmer (15). In ref. 16 Størmer solved the (spatial) problem of characterizing C*-algebras among such Jordan operator algebras acting on a given Hilbert space. The general (nonspatial) concepts of JB-algebras and JBW-algebras (together with a Gelfand–Naimark type representation theorem) were given by Alfsen, Shultz, and Størmer (17) and by Shultz (18), respectively. These algebras are defined axiomatically as (real) Jordan algebras, which are also Banach spaces, subject to suitable conditions connecting Jordan product and norm. (For the theory of such algebras see ref. 19.) The self-adjoint part of a C*-algebra or a von Neumann algebra is a special case of a JB-algebra or a JBW algebra, respectively. Not all JB-algebras or JBW-algebras arise in this fashion (cf. ref. 19, ch. 3–4), but nevertheless they have enough structure to effectively model quantum mechanical observables.

However, it is an important feature of quantum mechanics that the physical variables play a dual role, as observables and as generators of transformation groups. The observables are random variables with a specified probability law in each state of the quantum system, whereas the generators determine one-parameter groups of transformations of observables (Heisenberg picture) or states (Schrödinger picture).

Both aspects can be adequately dealt with in the C^* -algebra or von Neumann algebra formulation of quantum mechanics. An element a in the self-adjoint part A of such algebras represents an observable whose probability law is determined by spectral theory as indicated above, whereas an element h in A determines the one-parameter group $\alpha_\tau : x \mapsto e^{i\tau h} x e^{-i\tau h}$ (equivalently $d\alpha_\tau(x)/d\tau|_{\tau=0} = i[h, x]$), which represents the time evolution of the observable x . The spectral functional calculus is a Jordan construct, but the generation of one-parameter groups cannot be expressed in terms of the symmetrized product. Instead it is determined by the antisymmetrized product in A , which we will write as follows $a \star b = \frac{i}{2}[a, b] = \frac{i}{2}(ab - ba)$. Thus the decomposition of the associative product into its Jordan part and its Lie part

$$ab = a \circ b - i(a \star b)$$

separates the two aspects of a physical variable.

To solve the characterization problem, we must find appropriate conditions for an ordered normed linear space A , under which it is possible to define an associative product on $A + iA$ making this space a C^* -algebra or a von Neumann algebra. By the discussion above, this problem can be divided in two parts: first to construct the Jordan part of the associative product, then the Lie part when the Jordan part is known. As was discussed above, the key to doing this is the concept of orientation and the related notion of a dynamical correspondence that we will now discuss.

Dynamical Correspondences

A bounded linear operator δ on a JB-algebra A is called an order derivation if $\exp(\tau\delta)(A^+) \subset A^+$ for all $\tau \in \mathbf{R}$, or what is equivalent, if $\exp(\tau\delta)$ is a one-parameter group of order automorphisms. (This concept was first used by Connes in the context of ref. 4.) An important class of order derivations are the Jordan multipliers $\delta_b(a) = b \circ a$. An order derivation δ on a unital JB-algebra A is self-adjoint if $\delta = \delta_a$ for some $a \in A$ and skew-adjoint (or just skew) if $\delta(1) = 0$. Each order derivation δ on a unital JB-algebra A has a unique decomposition $\delta = \delta_1 + \delta_2$ where δ_1 is self-adjoint and δ_2 is skew, and the adjoint of δ is defined by $\delta^* = \delta_1 - \delta_2$. The skew order derivations are precisely the Jordan derivations (satisfying the Leibniz rule for the Jordan product). They are also the ones for which the operators $\exp(\tau\delta)$ fix 1, so the duals of these operators leave the state space invariant. Note that an order derivation on a JBW-algebra is automatically σ -weakly continuous, so it determines a one-parameter group $\exp(\tau\delta^*)$ of affine automorphisms of the normal state space as well as the state space.

One can show that the set $D(A)$ of all order derivations of a unital JB-algebra A is a real linear space closed under Lie brackets $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1$. One also can show (using the Kadison-Sakai theorem on inner derivations) that if A is a self-adjoint part of a von Neumann-algebra \mathcal{M} , then the order derivations are the operators δ_m defined by $\delta_m(x) = \frac{1}{2}(mx + xm^*)$ for $m \in \mathcal{M}$. In particular, a skew order derivation is of the form $\delta_{ia}(x) = \frac{i}{2}[a, x]$ where $a \in A$. In this case, the associated one-parameter group is

$$\exp(\tau\delta_{ia})(x) = e^{i\tau a/2} x e^{-i\tau a/2}.$$

Connes' notion of orientation can be transferred from the cone P_ξ^h of a JBW-algebra A to the algebra itself, and we will call the resulting notion a "Connes orientation on A ." Like the original concept, such an orientation is a complex structure compatible with involution on the Lie algebra of order derivations modulo its center $Z(D(A))$. To simplify the notation, we write $\tilde{D}(A)$ in place of $D(A)/Z(D(A))$, and we denote the equivalence class of an element δ of $D(A)$ modulo $Z(D(A))$ by $\tilde{\delta}$. Note that the involution $(\tilde{\delta})^* = (\delta^*)$ is well defined. Thus a Connes orientation on a JBW-algebra A is a complex structure on $\tilde{D}(A)$, which is compatible with Lie brackets and involution, i.e., a linear operator I on $\tilde{D}(A)$ which satisfies the requirements:

- (i) $I^2 = -1$ (the identity map),
- (ii) $[I\tilde{\delta}_1, \tilde{\delta}_2] = [\tilde{\delta}_1, I\tilde{\delta}_2] = I[\tilde{\delta}_1, \tilde{\delta}_2]$,
- (iii) $I(\tilde{\delta}^*) = -(I\tilde{\delta})^*$.

The idea behind this definition is to axiomatize the transition from δ_a to δ_{ia} in the von Neumann algebra case. Here it is necessary to work with equivalence classes because there is no well-defined map $\delta_a \mapsto \delta_{ia}$ on A itself. (The element a is not determined by the operator δ_a if a is not known to be self-adjoint.)

An alternative approach is to axiomatize the map $a \mapsto \delta_{ia}$. Then we arrive at the following notion, which makes sense both in the JB and the JBW context (and in particular in the C^* and the von Neumann context).

A dynamical correspondence on a unital JB-algebra A is a linear map $\psi : a \mapsto \psi_a$ from A into the set of skew order derivations on A , which satisfies the requirements:

- (i) $[\psi_a, \psi_b] = -[\delta_a, \delta_b]$ for $a, b \in A$,
- (ii) $\psi_a(a) = 0$ for all $a \in A$.

The skew order derivations determine one-parameter groups of affine automorphisms of the state space of A (and also of the normal state space in the JBW case). Thus a dynamical correspondence gives the elements of A a double identity, which reflects the dual role of physical variables as observables and as generators of a one-parameter group of motions; hence the name "dynamical correspondence."

Because the Jordan product is abelian, there is no useful concept of "commutator" for elements in a JB-algebra, but the commutators of the associated Jordan multipliers can be used as a substitute in view of the identity $[\delta_a, \delta_b] = \frac{1}{2}\delta_{(a,b)}$ in C^* -algebras. [As before, $\delta_m(x) = \frac{1}{2}(mx + xm^*)$ for a non-self-adjoint element m .] Thus the condition (i) above is a kind of quantization requirement, relating commutators of elements to the commutators of the associated generators. Note also that the equation $\psi_a(a) = 0$ is equivalent to $\exp(\tau\psi_a)(a) = a$ for all $\tau \in \mathbf{R}$. Thus condition (ii) says that the time evolution associated with an observable fixes that observable.

To relate the definition of a dynamical correspondence to the geometric notion of orientation that we will discuss later, we will explain the geometric meaning of this definition in the case of the 2×2 matrix algebra M_2 (which models a two-level quantum system, cf. e.g., ref. 20, ch. 15). Here the state space is the Euclidean 3-ball B^3 , and a self-adjoint element $a \in \mathcal{M}$ acts as an affine function on the ball. This function attains its maximum and its minimum at two antipodal points (the North Pole and the South Pole in Fig. 1), and the corresponding one-parameter group consists of rotations of the ball about the diameter between these two points (in either one of the two possible directions depending on orientation).

The orbits of $\exp(\tau(\delta_{ia})^*)$ are the "parallel circles" in Fig. 1. The orbits of $\exp(\tau(\delta_a)^*)$ will take us out of the state space, but

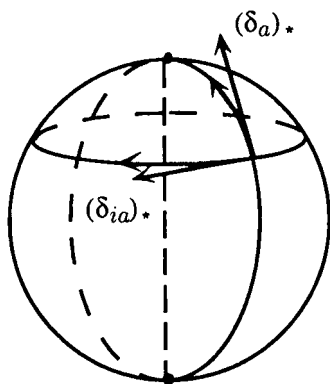


FIG. 1. State space of M_2 .

this can be remedied by a normalization. Then the resulting orbits will be the “longitudinal semicircles” traced out from South Pole to North Pole. In Fig. 1 the generators $(\delta_{ia})^*$ and $(\delta_a)^*$ are visualized as tangent vectors (velocity vectors) at an arbitrary point on the surface of the ball.

Note that a Connes orientation is defined in the context of JBW-algebras, whereas a dynamical correspondence is defined in the general context of unital JB-algebras (which include JBW-algebras as a special case), so the two concepts cannot be equivalent. However, for a JBW-algebra, each Connes orientation I determines a unique dynamical correspondence ψ determined by $\psi_a \in I(\delta_a)$, and each dynamical correspondence arises in this way from a unique Connes orientation, so the two concepts are in fact equivalent in the JBW case.

If A is a unital JB-algebra (JBW algebra) and $(x, y) \mapsto xy$ is an associative product on the complex linear space $A + iA$, which induces the given Jordan product on A and organizes $A + iA$ to a C^* -algebra (von Neumann algebra) with the involution $(a + ib)^* = a - ib$ and the norm $\|x\| = \|x^*x\|^{1/2}$, then this product is said to be a Jordan compatible C^* -product (respectively, von Neumann-product).

1. THEOREM. A unital JB-algebra is (isomorphic to) the self-adjoint part of a C^* -algebra iff there exists a dynamical correspondence on A . In this case each dynamical correspondence ψ on A determines a unique Jordan compatible C^* -product such that $\psi_a b = \frac{i}{2}(ab - ba)$ for each pair $a, b \in A$, and each Jordan compatible C^* -product arises in this way from a unique dynamical correspondence ψ on A . The same conclusions hold with JBW in place of JB and “von Neumann” in place of C^* .

In the proof of this theorem one constructs the associative product from the dynamical correspondence via the equation $ab = a \circ b - i(a \star b)$ where $a \star b = \psi_b(a)$.

2. COROLLARY. A JBW-algebra A is the self-adjoint part of a von Neumann algebra iff there exists a Connes orientation on A . In this case each Connes orientation I on A determines a unique Jordan compatible von Neumann product such that $I(\delta_d) = \delta_{id}$ for each $d \in A + iA$, and each Jordan compatible von Neumann product arises in this way from a unique Connes orientation on A .

Geometric Orientations

The characterization of C^* -algebra state spaces by Alfsen, Hanche-Olsen, and Shultz (5) is based on two earlier papers of Alfsen and Shultz characterizing the state space K of a JB-algebra (21, 22). To move from JB-state spaces to C^* -state spaces, two conditions are imposed: the 3-ball axiom and orientability. The former condition states that the face generated by any two extreme points (“pure states”) is either a line segment or else is affinely isomorphic to the standard 3-ball B^3 in \mathbf{R}^3 . (In a general JB-algebra, these faces are also balls, i.e., affinely isomorphic to the unit ball of a Hilbert space, but these balls can be of arbitrary finite or infinite dimension.) The 3-ball

axiom guarantees that the order-unit space $A(K)$ admits a faithful representation Φ onto a norm closed Jordan subalgebra of $\mathcal{B}(H)_{sa}$ for a complex Hilbert space H . The orientability axiom guarantees that there is one such representation Φ for which $\Phi(A)$ is the self-adjoint part of the C^* -algebra it generates in $\mathcal{B}(H)$.

The concept of “orientability” is defined in terms of facial 3-balls, i.e., faces that are (affinely isomorphic to) the standard 3-ball B^3 . An orientation of a facial 3-ball F is an equivalence class of affine isomorphisms $\phi : B^3 \rightarrow F$ where $\phi_1 \sim \phi_2$ if $\det(\phi_2^{-1}\phi_1) = 1$. Thus an orientation of a 3-ball is determined by an (orthogonal) frame, i.e., an ordered triple of orthogonal directed axes, with two frames determining the same orientation if one can be rotated into the other. An orientation of all of K is a “continuous choice” of orienting for each facial 3-ball. We make this precise by topologizing the set of all oriented facial 3-balls as well as the set of all (nonoriented) facial 3-balls in a natural way involving the w^* -topology of K (see ref. 5) for the details). This makes the former set a locally trivial Z_2 -bundle over the latter. If this bundle is trivial, then K is said to be orientable, and each continuous cross-section of the bundle is called a global orientation of K (or just an orientation of K). The main result of ref. 5 says that a JB-algebra A with state space K is the self-adjoint part of a C^* -algebra iff K has the 3-ball property and is orientable (ref. 5, th. 8.4). Moreover, if these conditions are satisfied, then there is a 1–1 correspondence between C^* -structures on $A + iA$ and global orientations of K (ref. 5, cor. 8.5).

Iochum and Shultz (6) first characterize the normal state space of a JBW-algebra. Then they turn to von Neumann algebras. Because the normal state space of a von Neumann algebra may be devoid of extreme points, and hence also of facial 3-balls, one must replace the 3-balls by “blown up 3-balls” (which were introduced under the name “global 3-balls” in ref. 6, Def. 2.1). We will say more about blown-up 3-balls later on. Here we will only point out that the normal state space of a von Neumann algebra is characterized among the normal state spaces of JBW-algebras by an axiom similar to the original 3-ball axiom (ref. 6, th. 2.9). No orientability axiom is needed in this case. Nevertheless, here, too, one may ask whether there is a notion of “global orientation” in 1–1 correspondence with associative products in the same way as for C^* -algebras.

We will present our general definition of orientation for von Neumann algebras in two versions, one for the algebras themselves and one for their normal state spaces. But we will concentrate on the normal state space version, which is closer to the existing definition of orientation in the C^* -case (5).

Henceforth \mathcal{M} shall be a von Neumann algebra with normal state space K . There is a 1–1 map $p \mapsto F_p = \{\omega \in K \mid \omega(p) = 1\}$ from projections in \mathcal{M} to norm closed faces in K , and this map is a homeomorphism from the norm topology on \mathcal{M} to the Hausdorff metric on closed subsets of K . Generally we refer to $p' = 1 - p$ as the complementary projection to p and to $F' = F_{1-p}$ as the complementary face to $F = F_p$. In the M_2 case (where $K = B^3$) the complementary faces are exactly the pairs of antipodal boundary points. In the general case an ordered pair (F, F') of complementary faces will be called a generalized axis.

There is a natural 1–1 correspondence between generalized axes in K and symmetries (self-adjoint unitaries) in \mathcal{M} because symmetries are of the form $s = p - p'$. For each generalized axis (F, F') with corresponding symmetry s there is a unique affine automorphism with period 2 of K whose fixed point set is exactly $\text{co}(F \cup F')$, namely the dual of the conjugation $U_s : x \mapsto sxs$. We call this automorphism the reflection of K about the generalized axis (F, F') . A triple of generalized axes is called a 3-frame if the associated reflections have the following properties generalizing elementary properties of an ordinary (orthogonal) frame in B^3 :

(i) Each reflection exchanges the faces F, F' of each of the other two axes.

(ii) The product of all three reflections in any order is the identity map.

The following properties of the corresponding symmetries s_1, s_2, s_3 are equivalent to those above:

- (i)' $s_i \circ s_j = 0$ for $i \neq j$.
- (ii)' $U_{s_1}U_{s_2}U_{s_3} = 1$ (the identity map).

If the triple (s_1, s_2, s_3) satisfies (i)' and (ii)' above, then we call it a Cartesian triple of symmetries (or just a Cartesian triple). An example of a Cartesian triple of symmetries are the Dirac spin matrices in M_2 .

A symmetry $r = p - p'$ is called balanced if the projections p and p' are equivalent, and then (by orthogonality) also exchangeable by a symmetry s (i.e., $p' = sps$). If r is a balanced symmetry, then there exists another balanced symmetry s such that $r \circ s = 0$ (the same s as above) and then in turn a third balanced symmetry t such that (r, s, t) is a Cartesian triple. The possible choices for t are $t = izrs$ where z is a central symmetry. Thus there exists a Cartesian triple in \mathcal{M} (and a 3-frame in K) iff there exists a balanced symmetry in \mathcal{M} , or equivalently iff the unit 1 is halvable (i.e., the sum of two equivalent projections).

If K has a 3-frame, then we call it a blown-up 3-ball. (This definition applies also in the JBW-case. It is equivalent to the original definition of a "global 3-ball" in ref. 6, Def. 2.1, which provides further justification for the name "blown-up 3-ball.") By the above, K is a blown-up 3-ball iff \mathcal{M} has a halvable unit. This is the case iff \mathcal{M} has 2×2 matrix units, or equivalently iff $\mathcal{M} \cong M_2 \otimes \mathcal{N}$ for a von Neumann algebra \mathcal{N} . Hence K is a blown-up 3-ball iff K is the normal state space of an algebra of this type.

We shall see that the Cartesian triples in \mathcal{M} determine *-derivations (=skew order derivations) with properties analogous to those of *-derivations on M_2 .

3. THEOREM. *If δ is a *-derivation on a von Neumann algebra \mathcal{M} , then the following are equivalent:*

- (i) $\delta = \delta_{is}$ for a symmetry $s \in \mathcal{M}$.
- (ii) $\delta^3 = -\delta$.
- (iii) Each $\omega \in K$ is either a fixed point for the one-parameter group $\exp(\tau\delta^*)$ or a periodic point with minimal period 2π .
- (iv) $Sp(\delta) \subset \{-i, 0, i\}$.

In the proof of this theorem (i) \Rightarrow (ii) is easy. (ii) \Rightarrow (iii) follows by a direct calculation that shows orbits are circles with period 2π . (iii) \Rightarrow (iv) applies known results on Banach algebras to the operator δ^* on the predual \mathcal{M}^* of \mathcal{M} (which satisfies the equation $f(\delta^*) = 0$ where f is the entire function $f(\lambda) = e^{2\pi\lambda} - 1$ with simple zeros), and (iv) \Rightarrow (i) makes use of arguments similar to those used in the theory of Arveson spectrum and spectral subspaces without actually quoting results from this theory.

A *-derivation δ of a von Neumann algebra that satisfies the requirement $\delta^3 = -\delta$ will be called a rotational derivation. If $\delta = \delta_{is}$ is a rotational derivation and $s = p - p'$ where p is a projection with associated norm closed face $F \subset K$, then the fixed point set of $\exp(\tau\delta^*)$ is $\text{co}(F \cup F')$ and the orbit of each $\omega \notin \text{co}(F \cup F')$ is (affinely isomorphic to) a circle with center in $\text{co}(F \cup F')$. Thus the one-parameter group $\exp(\tau\delta^*)$ of a rotational derivation $\delta = \delta_{is}$ may be thought of as rotational motion of K with period 2π about the generalized axis (F, F') .

Assume that \mathcal{M} has a halvable unit. For every Cartesian triple (r, s, t) the operator $\psi = [\delta_s, \delta_t]$ is a rotational derivation such that $\ker \psi = \{r\}^c$ where $\{r\}^c$ denotes the relative

commutant of r in \mathcal{M} (in particular $\psi(r) = 0$), and also such that $\psi(s) = t$. (Actually $\psi = [\delta_s, \delta_t]$ is the unique rotational derivation with these properties.) We call ψ the rotational derivation associated with the given Cartesian triple $\alpha = (r, s, t)$, and we denote it by ψ_α . If α^* is the 3-frame determined by α , then the one-parameter group $\exp(\tau\psi^*)$ fixes the first generalized axis of α^* , and for $\tau = \pi/2$ it carries the second generalized axis into the third. We will call it the rotational action associated with α^* .

Assume now that \mathcal{M} is a general von Neumann algebra. For each projection $e \in \mathcal{M}$ the associated face $F_e \subset K$ is the normal state space of the subalgebra $e\mathcal{M}e$. Thus F_e is a blown-up 3-ball iff e is a halvable projection. We will use 3-frames to define orientation for these (facial) blown-up 3-balls F_e , and then later "globalize" to K .

We topologize the closed subsets of K by the Hausdorff metric. Then we topologize the 3-frames of blown-up 3-balls of K by the product topology (regarding 3-frames as 6-tuples of norm closed faces).

If B is a global 3-ball of K , we define two 3-frames of B to be equivalent if there is a continuous path from one to the other. An orientation of B is then an equivalence class of 3-frames of B . We will refer to a blown-up 3-ball B of K together with a choice of orientation of B as an oriented blown-up 3-ball. It is easily seen that the definition above reduces to the previous definition of orientation for ordinary 3-balls.

We now will define the concept of global orientation for K . Let \mathcal{B} be the space of all facial blown-up 3-balls with the Hausdorff metric, and let \mathcal{OB} be the space of all oriented blown-up 3-balls B topologized by the quotient topology from the set of 3-frames.

4. THEOREM. *The natural surjection $\pi : \mathcal{OB} \rightarrow \mathcal{B}$ is continuous and open, and it organizes \mathcal{OB} to a locally trivial Z_{sym} bundle over \mathcal{B} , where Z_{sym} is the group of central symmetries in \mathcal{M} .*

Our proof of this theorem makes use of the fact that two projections that are near to each other can be exchanged by a unitary that is near to 1. More precisely: two projections $p, q \in \mathcal{M}$ with $\|p - q\| < 1$ admit a unitary u in the norm closed subalgebra generated by p and q such that u is the product of two symmetries in \mathcal{M} and satisfies

$$u^*pu = q \quad \text{and} \quad \|1 - u\| < \sqrt{2}\|p - q\|.$$

The bundle $\mathcal{OB} \rightarrow \mathcal{B}$ is called the bundle of oriented blown-up 3-balls.

We will define a global orientation of K as a continuous cross-section $\phi : \mathcal{B} \rightarrow \mathcal{OB}$ with an additional consistency condition by which "cutting down a blown-up 3-ball to a smaller blown-up 3-ball" preserves orientation. (There is no such requirement in the C^* case because ordinary 3-balls are minimal.) Let $B_1 \supset B_2$ be blown-up 3-balls of K . Then we say that an orientation of B_2 is the restriction of an orientation of B_1 if the two orientations can be represented by 3-frames such that the first generalized axis in B_2 is the first generalized axis in B_1 intersected by B_2 and such that the two associated rotational actions coincide on B_2 . A technical lemma guarantees the existence of restricted orientations: if B_1 and B_2 are blown-up 3-balls with $B_1 \supset B_2$ and B_1 is oriented, then there is a unique orientation of B_2 , which is the restriction of the given orientation of B_1 .

We will say that a cross-section ϕ of the bundle of oriented facial 3-balls is consistent if $B_1 \supset B_2$ implies that $\phi(B_2)$ is the restriction of $\phi(B_1)$. Then we define a global orientation of K (or just an orientation of K) as a consistent continuous cross-section of the bundle of oriented blown-up 3-balls.

Alternatively we can use Cartesian triples to define an algebraic notion of orientation for the corresponding local subalgebras $e\mathcal{M}e$ where e is a halvable projection, and then define a global orientation for \mathcal{M} . Two Cartesian triples α_1 and

α_2 in the same local subalgebra eMe are said to be equivalent if they can be connected by a continuous path from α_1 to α_2 in eMe . Here one can show that α_1 and α_2 are equivalent iff there is a unitary in eMe , which carries α_1 into α_2 , and also iff there is a unitary in eMe , which carries the element r_1 into the element r_2 and the operator ψ_{α_1} into ψ_{α_2} . Thus one may alternatively interpret an orientation of eMe as a (unitary) equivalence class of pairs (r, ψ) where r is a symmetry and ψ is a rotational derivation on eMe such that $\ker \psi = \{r\}^c$. Then an orientation of e_1Me_1 is the restriction of an orientation of e_2Me_2 if $e_1 \leq e_2$, and the two orientations can be represented by pairs (r_1, ψ_1) and (r_2, ψ_2) such that $r_1 = e_1r_2 = r_2e_1$ and ψ_2 restricts to ψ_1 on e_1Me_1 . (Note that the technical lemma on the existence of restricted orientations that was mentioned before is proved in this setting.)

Now the natural surjection of the local subalgebras eMe onto the halvable projections e defines a Z_{sym} bundle, which is locally trivial for the norm topology. We call it the bundle of oriented local algebras. Then the concept of consistency for a cross-section of this bundle and the concept of a global orientation of \mathcal{M} is defined as in the dual context.

Representing orientations of local subalgebras by pairs (r, ψ) as explained above, we can prove a theorem in which the definition of a global orientation of \mathcal{M} is restated in a form closer to the definition of a dynamical correspondence. Recall that a partial symmetry in \mathcal{M} is a self-adjoint element r whose square $e = r^2$ is a projection, so r is a symmetry in the subalgebra eMe and $r = p - q$, where p and q are projections such that $p + q = e$. We will write $r_1 \ll r_2$ if the partial symmetries $r_1 = p_1 - q_1$ and $r_2 = p_2 - q_2$ satisfy $p_1 \leq p_2$ and $q_1 \leq q_2$.

5. THEOREM. *There is a natural 1-1 correspondence between global orientations of \mathcal{M} and maps $r \mapsto \psi_r$, which assign to each partial symmetry r a rotational derivation ψ_r of eMe , where $e = r^2$, such that*

- (i) $\ker \psi_r = \{r\}^c$
- (ii) *Each unitary that carries r_1 into r_2 carries ψ_{r_1} into ψ_{r_2} .*
- (iii) *If $r_1 \ll r_2$, then $\psi_{r_2} = \psi_{r_1}$ on e_1Me_1 where $e_1 = r_1^2$.*

We now will explain how one can use the associative product in \mathcal{M} to construct a global orientation of K from a central symmetry $z \in \mathcal{M}$. If $B \in \mathcal{B}$, then $B = F_e$ for a halvable projection $e \in \mathcal{M}$. Choose a symmetry $r \in eMe$, then another symmetry $s \in eMe$ such that $r \circ s = 0$, and let $t \in eMe$ be the symmetry defined by $t = izrs$. Then (r, s, t) is a Cartesian triple in eMe , and it can be shown that different choices of r and s give equivalent Cartesian triples. Thus (r, s, t) determines an orientation of B that depends only on z . This gives a cross-section of the bundle $\mathcal{O}\mathcal{B} \rightarrow \mathcal{B}$, and it can be shown that this cross-section is consistent and continuous. With this we have associated a global orientation to each central symmetry $z \in \mathcal{M}$.

We also can associate a Jordan compatible von Neumann product to each central symmetry $z \in \mathcal{M}$. It is expressed by the central projection $c = \frac{1}{2}(1 + z)$ through the formula

$$a \times b = cab + (1 - c)ba.$$

6. THEOREM. *If \mathcal{M} is a von Neumann algebra with normal state space K , then there is 1-1 correspondence between (i) global orientations of K , (ii) Jordan compatible von Neumann products in \mathcal{M} , (iii) central symmetries in \mathcal{M} , such that each central symmetry $z \in \mathcal{M}$ corresponds to the associated orientation and product (as defined above).*

In the proof of this theorem one must show that each global orientation of K is associated to a central symmetry $z \in \mathcal{M}$. This z is obtained by “pasting together” central symmetries from

local subalgebras eMe in a construction that involves a strong version of the comparison theorem with unitary equivalence in place of Murray von Neumann equivalence. (This version of the comparison theorem follows from the general JBW-theorem in ref. 19, 5.2.13, but the proof can be simplified for von Neumann algebras.)

The given product in \mathcal{M} determines a natural orientation of K (the one for which $z = 1$), and it follows easily from theorem 6 that a Jordan automorphism $\Phi : \mathcal{M} \rightarrow \mathcal{M}$ is a *-automorphism iff the affine automorphism $\Phi^* : K \rightarrow K$ is orientation preserving. As a consequence, a von Neumann algebra is determined by the combination of the normal state space and an orientation. Another consequence is the following:

7. COROLLARY (Kadison, ref. 7). *If Φ is a Jordan automorphism of a von Neumann algebra \mathcal{M} , then there is a central projection $c \in \mathcal{M}$ such that Φ restricted to $c\mathcal{M}$ is a *-automorphism and Φ restricted to $(1 - c)\mathcal{M}$ is a *-anti-automorphism.*

Note that this corollary does not require the topological bundle properties of theorem 4, as one can rely on the purely algebraic characterization of a global orientation given in theorem 5.

Note that the bijection between (i) and (ii) of theorem 6 is canonical in that it is uniquely determined by the Jordan structure of \mathcal{M} , whereas the bijections between these and (iii) are noncanonical, as the associative product \times assigned to the central symmetry z in the displayed equation before theorem 6 depends on the given product in the von Neumann algebra \mathcal{M} . Thus it is not true that a Jordan compatible von Neumann product can be specified by the combination of the normal state space and a choice of central symmetry. Unless one is given a multiplication or orientation to start with, a central symmetry is not enough to determine an orientation. What a central symmetry does determine is a change from one orientation to another.

The central symmetry $z = -1$ changes an orientation $\phi : \mathcal{B} \rightarrow \mathcal{O}\mathcal{B}$ to its opposite orientation for which the three generalized axes of any 3-frame in any $\phi(B)$ are reversed (or just one of them is reversed, or their order is changed by an odd permutation.) If \mathcal{M} is a factor, then the only central symmetries are $z = 1$ and $z = -1$, so in this case there are just two orientations, the one being the opposite of the other. If \mathcal{M} is the direct sum of $n < \infty$ factors (in particular if \mathcal{M} is finite dimensional), then there are 2^n orientations.

By theorems 1 and 6, there is also a (canonical) bijection between global orientations and dynamical correspondences. It is easily seen that if $a \mapsto \psi_a$ is the dynamical correspondence associated to a given orientation, then $a \mapsto -\psi_a$ is the dynamical correspondence associated to the opposite orientation. Thus a passage to the opposite orientation induces a change from the one-parameter group $\exp(\tau\psi_a)$ to $\exp(-\tau\psi_a)$, i.e., a change of sign for the time parameter τ .

Finally, we will relate the concept of orientation for the normal state space of a von Neumann algebra to our previous concept of orientation for a C^* -algebra \mathcal{A} with state space K , which is also the normal state space of the enveloping von Neumann algebra $\mathcal{M} \cong \mathcal{A}^{**}$. Consider the restriction of the bundle $\mathcal{O}\mathcal{B} \rightarrow \mathcal{B}$ of oriented blown-up 3-balls to the set of 3-balls. Set theoretically this bundle can be identified with the bundle of facial 3-balls defined in ref. 5. (In both cases an orientation of a 3-ball is determined by a frame, with two frames determining the same orientation if one can be rotated into the other.) Topologically the two bundles are different. The topology of the bundle considered here is derived from the norm (via the corresponding Hausdorff metric for closed subsets of K), whereas the topology of the previous bundle is derived from the w^* -topology of K (as a subset of \mathcal{A}^*). To keep these two topological bundles apart, we now will denote the former by $\mathcal{O}\mathcal{B}_3 \rightarrow \mathcal{B}_3$ (norm) and the latter by $\mathcal{O}\mathcal{B}_3 \rightarrow \mathcal{B}_3$ (weak*). (A different notation is used in ref. 5.)

It is not hard to show that the natural identification gives a continuous bundle isomorphism from $\mathbb{C}\mathcal{B}_3 \rightarrow \mathcal{B}_3$ (norm) to $\mathbb{C}\mathcal{B}_3 \rightarrow \mathcal{B}_3$ (weak*), and that each continuous cross-section of the latter (i.e., each C^* -orientation of K) induces a continuous cross-section of the former. In fact, each continuous cross-section of $\mathbb{C}\mathcal{B}_3 \rightarrow \mathcal{B}_3$ (weak*) induces a consistent continuous cross-section of the entire bundle $\mathbb{C}\mathcal{B} \rightarrow \mathcal{B}$ (i.e., a von Neumann orientation of K), but this is less trivial. Here one goes from a C^* -orientation to a C^* -multiplicative structure in \mathcal{A} (ref. 5, Th. 8.4), then via the bidual to a von Neumann multiplicative structure in \mathcal{M} , and then back to a von Neumann orientation of K .

1. Kadison, R. V. (1951) *A Representation Theory for Commutative Topological Algebra: Memoirs of the American Mathematical Society* 7 (Am. Math. Soc., Providence, RI).
2. Sakai, S. (1956) *Tohoku Math. J.* **8**, 70–85.
3. Sherman, S. (1956) *Ann. Math.* **64**, 593–601.
4. Connes, A. (1974) *Ann. Inst. Fourier (Grenoble)* **24**, 121–155.
5. Alfsen, E. M., Hanche-Olsen, H. & Shultz, F. W. (1980) *Acta Math.* **144**, 267–305.
6. Iochum, B. & Shultz, F. W. (1983) *J. Funct. Anal.* **50**, 317–328.
7. Kadison, R. V. (1951) *Ann. Math.* **54**, 325–338.
8. Bellissard, J. & Iochum, B. (1978) *Ann. Inst. Fourier (Grenoble)* **28**, 26–67.
9. Iochum, B. (1984) *Cônes autopolaires et algèbres de Jordan*, Lecture Notes in Math, (Springer, Heidelberg).
10. Alfsen, E. M. & Shultz, F. W. (1998) *Comm. Math. Phys.*, in press.
11. Jordan, P., von Neumann, J. & Wigner, E. (1934) *Ann. Math.* **35**, 29–64.
12. von Neumann, J. (1936) *Mat. Sbornik* **1**, 415–482.
13. Segal, I. E. (1947) *Ann. Math.* **48**, 930–948.
14. Topping, D. (1965) *Jordan Algebras of Self-Adjoint Operators: Memoirs of the American Mathematical Society* 53 (Am. Math. Soc., Providence, RI).
15. Størmer, E. (1966) *Acta Math.* **115**, 165–184.
16. Størmer, E. (1965) *Trans. Am. Math. Soc.* **120**, 438–447.
17. Alfsen, E. M., Shultz, F. W. & Størmer, E. (1978) *Adv. Math.* **28**, 11–56.
18. Shultz, F. W. (1979) *J. Funct. Anal.* **31**, 360–376.
19. Hanche-Olsen, H. & Størmer, E. (1984) *Jordan Operator Algebras* (Pitman, London).
20. Mandel, L. & Wolf, E. (1995) *Optical Coherence and Quantum Optics* (Cambridge Univ. Press, New York).
21. Alfsen, E. M. & Shultz, F. W. (1976) *Noncommutative Spectral Theory for Affine Function Spaces on Convex Sets: Memoirs of the American Mathematical Society* 172 (Am. Math. Soc., Providence, RI).
22. Alfsen, E. M. & Shultz, F. W. (1978) *Acta Math.* **140**, 155–190.