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On the geometry of solutions of the quasi-geostrophic and Euler equations

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ABSTRACT We study solutions of the two-dimensional quasi-geostrophic thermal active scalar equation involving simple hyperbolic saddles. There is a naturally associated notion of simple hyperbolic saddle breakdown. It is proved that such breakdown cannot occur in finite time. At large time, these solutions may grow at most at a quadruple-exponential rate. Analogous results hold for the incompressible three-dimensional Euler equation.

The question of whether singularities form in finite time in incompressible fluid flows is an important open problem in theoretical fluid mechanics. In other words, if one starts with smooth initial data, will the solution remain smooth for all time?

For the two-dimensional (2D) incompressible Euler equations, it is well known that there is no breakdown of solutions. For three-dimensional (3D) Euler equations the question is still open, and the existence of such singularities would have important consequences for the understanding of turbulence.

The aim of this work is to study the following 2D quasi-geostrophic equation:

$$(\partial_t + u\cdot\nabla)\theta = 0 \qquad \textbf{[1]}$$

$$u = \nabla^{\perp}\psi \quad \text{where} \quad \theta = -(-\Delta)^{1/2}\psi$$

$\theta = \theta(x, t)$ with $x \in \mathbf{R}^2$, $t \in \mathbf{R}^+$ is a scalar temperature, u is the velocity, and ψ is the stream function. Majda and Tabak (1) studied the similarities and differences of Eq. **1** with the 2D incompressible Euler equation in vorticity form. This equation, a version of a quasi-geostrophic model, was proposed as a 2D model for 3D vorticity intensification by Constantin (2). In another paper, by Constantin, Majda, and Tabak (3), they show that there is a geometric and analytic analogy with 3D Euler:

3D Euler:

$$\left(\frac{\partial}{\partial t} + u\cdot\nabla\right)\omega = \omega\cdot\nabla u.$$

$$u(\varkappa, t) = \frac{1}{4\pi}\int_{R^3}\frac{y\times\omega(x+y, t)}{|y|^3}\,dy$$

2D quasi-geostrophic:

$$\left(\frac{\partial}{\partial t} + u\cdot\nabla\right)\nabla^{\perp}\theta = \nabla^{\perp}\theta\cdot\nabla u.$$

$$u(\varkappa, t) = -\int_{R^2}\frac{\nabla^{\perp}\theta(x+y, t)}{|y|}\,dy.$$

In both cases $\nabla\cdot u = 0$.

Therefore, for the 2D quasi-geostrophic active scalar, the level sets of θ are analogous to the vortex lines for 3D Euler. It was also proved in ref. 3 that if the direction field ξ

$$\xi(x) = \frac{\nabla^{\perp}\theta}{|\nabla^{\perp}\theta|}$$

is smooth in regions of high $|\nabla^{\perp}\theta|$, then blow-up does not occur. A similar result was obtained by Constantin, Fefferman, and Majda (4) for 3D Euler with

$$\xi(x) = \frac{\omega(x)}{|\omega(x)|}$$

and for Navier–Stokes by Constantin and Fefferman (5).

The results mentioned above and the one presented in this paper are based on a theorem proved by Beale, Kato, and Majda (6); a necessary condition for having a singularity at time T is that

$$\int_0^T ||\omega(t)||_\infty\,dt = +\infty.$$

An analogous theorem is proved in ref. 3 for the quasi-geostrophic equation (replace $|\omega|$ by $|\nabla^{\perp}\theta|$).

In ref. 3, Constantin, Majda, and Tabak studied numerically a particular example where the geometry of the level sets of θ contains a hyperbolic saddle with the vertex angle α of the saddle going to zero. This numerical experiment shows evidence of singular behavior. There are more recent numerical studies by Ohkitani and Yamada (7), which suggest that $|\nabla^{\perp}\theta|$ does not go to infinity in finite time, but rather goes to infinity at a double-exponential rate.

A number of blow-up scenarios were discussed in ref. 8 by Constantin.

RESULTS

The theorem in this paper shows that the angle of the saddle cannot reach zero in finite time. That will rule out blow-up by a simple hyperbolic saddle.

The main hypothesis in our definition of such points is to assume that there is a nonlinear, time-dependent coordinate change, so that in the new coordinates (y_1, y_2) the level curves of θ are given by the equation $\rho = \text{const}$, where $\rho = y_1y_2 - \cot\alpha y_2^2$. Here α is the angle of opening of the simple hyperbolic saddle.

THEOREM 1. *Let* $\theta(x_1, x_2, t)$ *be a smooth solution of* Eq. **1** *defined for* $0 \le t < T_*$, $(x_1, x_2) \in R^2$. *Assume for* $0 \le t < T_*$ *that*

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Abbreviations: 2D and 3D, two- and three-dimensional.
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θ is constant along the curves $\rho = y_1y_2 - \cot \alpha \cdot y_2^2$ for all (x_1, x_2) in a neighborhood U of the origin. Here,

$$y_1 = F_1(x_1, x_2, t)$$

$$y_2 = F_2(x_1, x_2, t)$$

*and $\alpha(t) \in \mathscr{C}^\infty([0, T_*])$, $F_i \in \mathscr{C}^\infty(\bar{U} \times [0, T_*])$, $det|\partial F_i/\partial x_j| \geq c > 0$ whenever $x \in U$, $t \in [0, T_*]$. Outside U assume that the $|\nabla^\perp \theta|$ is bounded and θ decays rapidly at infinity. Then $\lim_{t \to T_*} \alpha(t) \neq 0$, and θ continues to a smooth solution of Eq. **1** for $0 \leq t < T_* + \varepsilon$, $(x_1, x_2) \in R^2$ for some ε.*

THEOREM 2. *Let $\theta(x_1, x_2, t)$, α, U, and F_j be as in* Theorem 1, *but with $T_* = \infty$. Assume that the $\mathscr{C}^\infty$ seminorms of F_j are bounded for all time $t \in [0, \infty)$. Then*

$$\left| \log \log \frac{1}{\alpha(t)} \right| \leq (constant) \cdot t$$

for all t.

Remarks:

(*i*) The saddle is allowed to rotate and dilate with respect to time. The center of the saddle can move in U in time.

(*ii*) The inequality not only shows that α cannot be zero in finite time, it also tells us that it can go to zero at most as a double exponential. That result implies that $|\nabla^\perp \theta|$ can tend to infinity at most as a quadruple exponential of time.

(*iii*) The same techniques give analogous results for the incompressible 3D Euler equation. For example:

THEOREM 3. *Let $u(x, t)$ be a smooth solution of 3D Euler incompressible equation defined for $0 \leq t < T_*$, $x \in R^3$ with $\omega = curl(u) = (|\omega|/r)(-\partial\rho/\partial x_2, \partial\rho/\partial x_1, 0)$ where $r = |\nabla\rho|$, u is bounded up to $t = T_*$, and ρ is defined as in* Theorem 1 *with the same nonlinear time-dependent coordinate change and the same assumptions. Outside U assume that the $|\omega|$ is bounded and decays rapidly at infinity. Then $\lim_{t \to T_*} \alpha(t) \neq 0$, and u continues to a smooth solution for $0 \leq t < T_* + \varepsilon$ for some ε.*

The proof of *Theorem 1* can be divided into two steps. First, we make a change of variables $(y_1, y_2) \to (\rho, \sigma)$, where ρ are the level curves of θ and σ moves along a fixed ρ. We get an expression of the stream function

$$\psi(\rho, \sigma, t) = G_1(\rho, t) \cdot \sigma + \int^\sigma \frac{\partial \rho}{\partial t} \, d\sigma + G_2(\rho, t).$$

In the second step we subtract the value of ψ at $(\rho = 0, \sigma_1)$ from $\psi(\rho = 0, \sigma_2)$, and we use the other set of variables,

$$\psi(y_1, 0) - \psi(y_1, y_1 \cdot \tan \alpha),$$

to control $\partial\alpha/\partial t$.

Next we use the expression

$$\psi(y_1, y_2) = -\int_{R^2} \frac{\theta(x)}{|x - y|} \, dx$$

and the properties of θ to get the following estimate:

$$\left| \frac{\partial \alpha}{\partial t} \right| \leq |\alpha \cdot \ln \alpha| (\text{const}).$$

The representation of the 2D incompressible Euler equation in vorticity form is

$$(\partial_t + u \cdot \nabla)\omega = 0$$

$$u = \nabla^\perp \psi \quad \text{where} \quad \omega = \Delta\psi.$$

The two active scalars θ and ω are similar, but they differ on the characterization of the stream function. Using the same scheme as before, we assume ω is constant along hyperbolas and α is the angle of the saddle. We can show

$$|\log \alpha(t)| \leq (\text{constant}) \cdot t.$$

That means α can go to zero at most as an exponential.

The proof is identical to the one above, but in this case ψ is defined by

$$\psi = \frac{1}{2\pi} \int_{R^2} \omega(x + y) \log |y| dy.$$

COMMENTS

One way to understand the 3D Euler incompressible equation is by studying models in lower dimension. Constantin, Lax, and Majda (9) developed and studied a one-dimensional mathematical model for 3D Euler where they showed that the equation can produce singularities and the solutions exhibit some of the phenomena observed in numerical simulations for breakdown of the 3D Euler equation. Eq. **1** is a system that comes from a geophysical context, where θ is the potential temperature and u is the velocity of the geophysical flow. As was explained before, Eq. **1** is a 2D mathematical model for 3D Euler. It is not known at this moment if this equation can produce breakdown.

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