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# Regular and stochastic dynamics in the real quadratic family

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**ABSTRACT** **We prove the Regular or Stochastic Conjecture for the real quadratic family which asserts that almost every real quadratic map $P_c$, $c \in [-2, 1/4]$, has either an attracting cycle or an absolutely continuous invariant measure.**

## 1. Statement of the Results

The goal of this note is to outline a proof of the Regular or Stochastic Conjecture for the real quadratic family. A quadratic map $P_c$: $x \mapsto x^2 + c$ is called *regular* if it has an attracting cycle. In this case, the attracting cycle is unique and attracts almost all orbits. It is called *stochastic* if it has a finite, absolutely continuous invariant measure (acim). In this case, the measure is unique and weakly Bernoulli, and almost all orbits are asymptotically equidistributed with respect to it.

MAIN THEOREM [REGULAR OR STOCHASTIC]. *Almost every real quadratic polynomial, $P_c(z) = z^2 + c$, $c \in [-2, 1/4]$, is either regular or stochastic.*

Regular quadratic maps are also called (*uniformly*) *hyperbolic*, because they are uniformly expanding outside the basin of the attracting cycle. Stochastic maps can also be called (*nonuniformly*) *hyperbolic* in the sense of the Pesin theory. Thus one can say that *almost any real quadratic map is hyperbolic.*

Previously, it was known that the set of stochastic maps has positive measure (1, 2), while the set of regular maps is open and dense (see ref. 3 for the proof of this result and further reference comments). Our *Regular or Stochastic Theorem* completes the measure-theoretical picture of dynamics in the real quadratic family.

Let us remind the reader of the following topological decomposition of the parameter interval: $[-2, 1/4] = \mathcal{R} \cup \mathcal{N} \cup \mathcal{I}$, where $\mathcal{R}$ stands for the regular parameter values, $\mathcal{N}$ stands for nonregular at most finitely renormalizable parameter values, and $\mathcal{I}$ stands for infinitely renormalizable parameter values. The set $\mathcal{S}$ of stochastic parameter values is contained in $\mathcal{N}$. Thus the *Main Theorem* will follow from the following two results:

THEOREM 1.1 (4, 5). *Almost every nonregular real quadratic that is at most finitely renormalizable is stochastic:* $\text{meas}(\mathcal{N} \setminus \mathcal{S}) = 0$.

Namely, in our joint project, Martens and Nowicki gave a geometric condition for existence of an absolutely continuous invariant measure (5), and the author showed that this condition is satisfied almost everywhere in $\mathcal{N}$ (4).

THEOREM 1.2. *The set of infinitely renormalizable real quadratics has zero Lebesgue measure:* $\text{meas}(\mathcal{I}) = 0$.

We derive this result from the following Renormalization Theorem for all real combinatorial types. Let us consider the following objects (see Section 2 for the definitions or references): $\mathcal{QL}$ is the space of quadratic-like germs considered up to affine conjugacy, and $\mathcal{C} = \{f \in \mathcal{QL} : J(f) \text{ is connected}\}$ is the connectedness locus in $\mathcal{QL}$; $\mathcal{H}(f)$ is the hybrid class of $f \in \mathcal{C}$; $z \mapsto z^2 + \chi(f)$ is the straightening of $f \in \mathcal{C}$; $\mathcal{M}$ is the family of maximal real Mandelbrot copies; $p(M)$ is the renormalization period of a copy $M \in \mathcal{M}$; $R$: $\bigcup_{M \in \mathcal{M}} \mathcal{T}_M \to \mathcal{QL}$ is the renormalization operator defined on the disjoint union of the renormalization strips $\mathcal{T}_M$ labeled by the Mandelbrot copies; $\Sigma$ is the space of two-sided sequences of natural numbers; and $\omega$ is the shift on this symbolic space.

THEOREM 1.3. *There is a set $\mathcal{A}$ of real quadratic-like germs such that*:

- *$\mathcal{A}$ is $R$-invariant and $R|\mathcal{A}$ is topologically conjugate to the two-sided shift $\omega$*;
- *The restriction $R|\mathcal{A}$ is uniformly hyperbolic*;
- *Any stable leaf $W^s(f)$, $f \in \mathcal{A}$, is a complex manifold of codimension one within $\mathcal{QL}$ coinciding with the hybrid class $\mathcal{H}(f)$*;
- *Any unstable leaf $W^u(f)$ is an analytic curve that passes transversally through all real hybrid classes except possibly the cusp class $\mathcal{H}(P_{1/4})$*;
- *For any $\delta > 0$, the renormalization operator has uniformly bounded nonlinearity on the curves $Y^u(f) = W^u(f) \cap \chi^{-1}[-2, 1/4 - \delta]$*;
- *The straightenings $\chi$: $Y^u(f) \to [-2, 1/4 - \delta]$ are uniformly quasi-symmetric.*

*Notes: 1.* The hyperbolicity and nonlinearity above are understood with respect to a suitable Banach metric (compare the Remark in Section 2.11).

*2.* The Renormalization Conjecture stated by Feigenbaum and independently by Coullet and Tresser in 1978 has a rich history (see ref. 10 for references).

*3.* This work completes a program of study of the real quadratic family by complex methods carried in the series of papers (refs. 6, 7, 3, 8, 4, 5, 9, and 10) and preprint IMS at Stony Brook # 1997/8, http://www.math.sunysb.edu/~mlyubich/horseshoe.ps.gz.

## 2. Outline of the Proof

**2.1. Background.** We assume familiarity with a basic holomorphic dynamics, including the theory of quadratic-like maps, complex renormalization, and little Mandelbrot copies (see refs. 3 and 11).

The Mandelbrot set will be denoted by $M_*$. A Mandelbrot copy is called *maximal* if it is not contained in any other copy except $M_*$ itself. It is called *real* if it is centered on the real line.

The critical point of quadratic-like maps will be assumed to be at 0. A *quadratic-like germ* is roughly a class of quadratic-like maps coinciding near the common Julia set (see ref. 10 for the precise definition). We normalize the germs by the requirement that $f'(0) = 0$ and $f''(0) = 1$. Given a quadratic-like germ $f$, let $\text{mod}(f)$ denote the supremum of the $\text{mod}(A)$ where $A$ runs over all fundamental annuli of $f$.

Two quadratic-like germs are called *hybrid equivalent* if they are quasi-conformally conjugate by a map $h$ with $\bar\partial h = 0$ a.e. on the filled Julia set $K(f)$. By the Straightening Theorem (11), if $K(f)$ is connected, then the hybrid class $\mathcal{H}(f)$ has a

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single intersection point $c = \chi(f)$ with the quadratic family $\{P_c,\ c \in M_*\}$.

If $M \in \mathcal{M}$ and $\chi(f) \in M$, we say that $f$ is renormalizable with real combinatorics $M$. Its renormalization will be denoted as $Rf \equiv R_M f$. If $f$ is infinitely renormalizable, then its combinatorial type $\tau(f)$ is defined as the string $(M_0, M_1, \dots)$, where $M_n \in \mathcal{M}$ and $\chi(R^n f) \in M_n$, $n = 0, 1, \dots$.

Given a quasi-conformal map $h$, $\mathrm{Dil}(h) > 1$ will stand for its dilatation.

**2.2. Geometry of the Yoccoz Puzzle.** Let $O$ denote the "Misiurewicz wake" rooted at the real tip of the doubling Mandelbrot copy (bounded by the two rays of angle $\pm 5/12$ landing at this root). Let $f$ be a quadratic map whose hybrid class belongs to $O$. In ref. 3 we construct a nest $V^0 \supset V^1 \supset \cdots$ of Yoccoz puzzle pieces about 0 called the *principal nest*. For any $n \geq 1$, there is a quadratic-like map $g_n\colon V^n \to V^{n-1}$ corresponding to the first return map of 0 to $V^{n-1}$. A level $n$ is called *central* if $g_{n+1}0 \in V^{n+1}$. Let $\{n_k\}$ stand for the sequence of noncentral levels.

THEOREM 2.1 (3). $\mathrm{mod}(V^{n_k+1} \setminus V^{n_k+2}) \geq Ck$ *where* $C > 0$ *is an absolute constant.*

*Remark:* A related result for *real* quadratics was independently proven in ref. 12. Note that the further argument needs in a crucial way the above *Theorem 2.1* for complex parameter values.

There is an important special type of unbounded combinatorics when a big renormalization period is created by means of the saddle-node phenomenon. The combinatorial parameter that controls such combinatorics is called the *essential period* $p_e \equiv p_e(M)$, $M \in \mathcal{M}$ (see refs. 3 and 8).

THEOREM 2.2 (3). *Let $f$ be renormalizable by $R_M$ and let* $\mathrm{mod}(f) \geq \mu > 0$. *Then* $\mathrm{mod}(R_M f) \geq \nu_M(\mu) \geq \nu(\mu) > 0$. *Moreover,* $\nu_M(\mu) \to \infty$ *as* $p_e(M) \to \infty$.

**2.3. A Priori Bounds.** We say that a real map $f$ is *close to the cusp* if it has an attracting fixed point with multiplier greater than 1/2.

THEOREM 2.3 (8, 13). *Let $f$ be an $n$ times renormalizable real quadratic-like map with* $\mathrm{mod}(f) \geq \mu > 0$. *Then* $\mathrm{mod}(R^n f) \geq \nu_n(\mu) \geq \nu(\mu) > 0$, *unless the last renormalization is of doubling type and $R^n f$ is close to the cusp. Moreover,* $\liminf \nu_n(\mu) \geq \nu > 0$, *where $\nu$ is an absolute constant.*

**2.4. Combinatorial Rigidity Theorem.**

THEOREM 2.4 (3). *Let $f$ and $g$ be two infinitely renormalizable quadratic-like maps (not necessarily real) with the same real combinatorial type $\tau = \{M_0, M_1, \dots\}$, $M_n \in \mathcal{M}$, and with a priori bounds. Then $f$ and $g$ are hybrid equivalent.*

**2.5. Criterion for Existence of the Acim.** Let $I^n = V^n \cap \mathbb{R}$, $\lambda_n = |I^n|/|I^{n-1}|$.

THEOREM 2.5 (Martens and Nowicki, ref. 5). *If* $\sum \sqrt{|\lambda_n|} < \infty$ *then $f$ has an acim.*

This result was derived in ref. 5 from the criterion of Nowicki and van Strien (15): $\sum |Df^n(f(0))|^{-1/2} < \infty$ implies existence of acim.

COROLLARY 2.6. *If all but finitely many levels of the principal nest are noncentral, then $f$ has an acim.*

Indeed, by ref. 7 (or by *Theorem 2.1*), the scaling factors $\lambda_n$ exponentially decay under the assumption of the *Corollary*.

**2.6. Parapuzzle Geometry.** In ref. 4 we have constructed a special nest of tilings of the parameter plane by *parapuzzle pieces* $\Delta^l(c)$. The generalized renormalization $g_{l,\lambda}$ of level $l$ form a full holomorphic family over $\Delta^l(c)$. Let $\Pi^l(c) \subset \Delta^l(c)$ be the set of parameter values with central return on level $l$.

THEOREM 2.7 (4). *For any* $c \in M_* \cap O$, $\mathrm{mod}(\Delta^l(c) \setminus \Pi^l(c)) \geq Cl$ *with an absolute constant* $C > 0$.

Let us consider a holomorphic quadratic-like family $\mathbf{f}$, $f_\lambda\colon \tilde{U}_\lambda \to \tilde{U}'_\lambda$, over a topological disk $\tilde{\Lambda} \subset \mathbb{C}$. Let us restrict it to smaller disks $\Lambda \Subset \tilde{\Lambda}$, $U'_\lambda \Subset \tilde{U}'_\lambda$, $\lambda \in \tilde{\Lambda}$, in such a way that $f_\lambda\colon U_\lambda \equiv f_\lambda^{-1} U'_\lambda \to U'_\lambda$ is also a quadratic-like family over $\Lambda$. This family is called *proper* over $\Lambda$ if $f_\lambda(0) \in \partial U'_\lambda$ for $\lambda \in \partial\Lambda$.

A proper family is called *unfolded* if the curve $\lambda \mapsto f_\lambda(0)$, $\lambda \in \partial\Lambda$, has winding number 1 around 0.

A quadratic-like family is called *equipped* if there is a holomorphic motion $h_\lambda\colon (U'_*, U_*) \to (U'_\lambda, U_\lambda)$ over $\Lambda$ respecting the boundary dynamics, i.e., $h_\lambda(f_* z) = f_\lambda(h_\lambda z)$ for $z \in \partial U_*$ (where $* \in \Lambda$ is a base point). For $\mu > 0$, let $\mathcal{G}_\mu$ stand for the collection of equipped unfolded quadratic-like families with $\mathrm{mod}(f_\lambda) \geq \mu$, $\mathrm{mod}(\Lambda \setminus M(\mathbf{f})) \geq \mu$ (where $M(\mathbf{f})$ is the Mandelbrot set of $\mathbf{f}$), and $\mathrm{Dil}(h_\lambda) \leq \mu^{-1}$.

THEOREM 2.8 (4). *Let $\mathbf{f} \in \mathcal{G}_\mu$ be an equipped quadratic-like family over $\Lambda$, and let $M \in \mathcal{M}$, $p(M) > 2$. Then $R_M\mathbf{f}$ is an equipped quadratic-like family of class $\mathcal{G}_{\nu(\mu)}$. Moreover, $\nu(M, \mu) \to \infty$ as $p_e(M) \to \infty$.*

This result is crucial for the transverse control of the renormalization operator.

**2.7. Proof of *Theorem 1.1*.** *Theorem 2.7* shows that $\Pi^l(c)$ has an exponentially small size relative to $\Delta^l(c)$. Thus central returns have exponentially decaying probabilities (in the sense of the Lebesgue measure). Hence the probability to have infinitely many central returns is equal to 0. By *Corollary 2.6*, the probability not to have an acim is equal to 0 as well.

**2.8. McMullen Towers.** *McMullen tower* $\bar{f}$ is a sequence $\{f_k\}_{k=l}^n$ of quadratic-like germs with connected Julia sets such that $f_k = Rf_{k+1}$. The combinatorial type $\bar{\tau} = \tau(\bar{f})$ of such a tower is the sequence of Mandelbrot copies $M_k \in \mathcal{M}$ such that $\chi(f_k) \in M_k$. One says that the tower has a bounded combinatorics if $\sup p(f_k)$ is finite. One says that a tower has *a priori* bounds if $\mathrm{mod}(\bar{f}) \equiv \inf \mathrm{mod}(f_k) > 0$. Combining *Theorem 2.4* and the Rigidity Theorem of ref. 14, we obtain the following theorem:

THEOREM 2.9. *Two bi-infinite towers with the same bounded combinatorics and a priori bounds are affinely equivalent.*

**2.9. Parabolic Towers.** Parabolic towers are geometric limits of McMullen towers with uniformly bounded essential period (see ref. 9). One can naturally define combinatorics and the modulus of such a tower.

THEOREM 2.10 (9). *If two parabolic towers with a priori bounds are combinatorially equivalent then they are affinely equivalent.*

**2.10. Space of Quadratic-Like Germs and the Hybrid Foliation.** In ref. 10 we supplied the space $\mathcal{QL}$ of normalized quadratic-like germs with a complex analytic structure modeled on a family of Banach spaces. (Note: this structure does not turn $\mathcal{QL}$ into a Banach manifold but turn it roughly speaking into an "inductive limit of Banach manifolds.") We then showed that the hybrid classes form a foliation of the connectedness locus $\mathcal{C}$ with complex analytic leaves that have codimension one in $\mathcal{QL}$. This foliation is transversally quasiconformal. This, in particular, implies that the maximal real Mandelbrot copies are uniformly quasi-conformally equivalent to the whole Mandelbrot set $M_*$ (except for the doubling copy near its root $c = -3/4$).

The renormalization operator $R$ is defined on the union of the renormalization strips $\mathcal{T}_M = \chi^{-1}M$, $M \in \mathcal{M}$. Each restriction $R_M = R|\mathcal{T}_M$ admits an analytic continuation to an appropriate Banach neighborhood of the strip.

**2.11. Schwarz Lemma and Exponential Contraction.**

THEOREM 2.11. *Let $f$ and $g$ be two hybrid equivalent quadratic-like maps with modulus at least $\mu$. Assume that $f$ and $g$ are $n+1$ times renormalizable. Then*

$$\mathrm{dist}(R^n f, R^n g) \leq C\rho^n, \qquad [2.1]$$

*where* $\rho \in (0, 1)$ *is an absolute constant, and* $C > 0$ *depends only on* $\mu$.

*Remark:* The above distance is induced by the uniform norm in the Banach space $\mathcal{B}_\epsilon$ of bounded holomorphic functions on $\mathbb{D}_\epsilon = \{z\colon |z| < \epsilon\}$. Here $\epsilon = \epsilon(\mu) > 0$ should be selected in such a way that $R^n f$ and $R^n g$ are well-defined on $\mathbb{D}_\epsilon$ (which is possible by *Theorem 2.3*).

For maps with essentially bounded combinatorics, Eq. **2.1** follows from the rigidity of parabolic towers (*Theorem 2.10*) and the Schwarz lemma in Banach spaces. The renormalizations of maps with big essential periods are close to the quadratic family (by *Theorem 2.2*), and the Schwarz lemma yields strong contraction.

*Remark:* For bounded combinatorics, *Theorem 2.11* was originally proven by Sullivan (16) and McMullen (14) (in a different way).

**2.12. Full Renormalization Horseshoe.** Let $\mathcal{A} \subset \mathcal{C}$ stand for the set of $f \in \mathcal{C}$ such that there is sequence $\{f_n\}_{n=-\infty}^{\infty}$ with $f_0 = f$, $Rf_n = f_{n+1}$ and $\text{mod}(f_n) \geqslant \mu = \mu(f) > 0$, $n \in \mathbb{Z}$.

THEOREM 2.12. *There exists a homeomorphism* $\eta$: $\Sigma \to \mathcal{A}$ *conjugating* $\omega$ *and* $R|\mathcal{A}$.

*Theorem 2.3* implies that any combinatorial type $\bar{\tau} \in \Sigma$ is realizable by a nonescaping map $f \in \mathcal{A}$. *Theorem 2.11* yields uniqueness of $f$.

**2.13. Shadowing Lemma and Hyperbolicity (Proof of Theorem 1.3).** To complete the proof of *Theorem 1.3*, we show that lack of hyperbolicity of $R|\mathcal{A}$ implies existence of the slow shadowing orbits, i.e., there exists an $f \in \mathcal{A}$ and $g \in \mathcal{QL} \setminus \mathcal{H}(f)$ such that $\text{dist}(R^n f, R^n g) < \epsilon$, $n = 0, 1, \ldots$. On the other hand, this situation is ruled out by *Theorem 2.4*.

**2.14. Proof of Theorem 1.2.** *Theorem 1.3* implies that the set $\mathcal{A} \cap Y^u(f)$, $f \in \mathcal{A}$, has definite gaps in arbitrary small scales on the curves $Y^u(f)$. Since the straightenings from $Y^u(f)$ to the quadratic family are uniformly quasi-symmetric, the set $\mathcal{I} \subset \mathbb{R}$ has the same property. Now the Lebesgue density points theorem yields the statement.

The final result of *Theorem 1.3* was obtained in the fall of 1996 during author's visit to Institut des Hautes Études Scientific. It was partially supported by the National Science Foundation Grant DMS-9505833.


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