Models for stochastic climate prediction

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There has been a recent burst of activity in the atmosphere/ocean sciences community in utilizing stable linear Langevin stochastic models for the unresolved degree of freedom in stochastic climate prediction. Here several idealized models for stochastic climate modeling are introduced and analyzed through unambiguous mathematical theory. This analysis demonstrates the potential need for more sophisticated models beyond stable linear Langevin equations. The new phenomena include the emergence of both unstable linear Langevin stochastic models for the climate mean and the need to incorporate both suitable nonlinear effects and multiplicative noise in stochastic models under appropriate circumstances. The strategy for stochastic climate modeling that emerges from this analysis is illustrated on an idealized example involving truncated barotropic flow on a beta-plane with topography and a mean flow. In this example, the effect of the original 57 degrees of freedom is well represented by a theoretically predicted stochastic model with only 3 degrees of freedom.

An area with great importance for future developments in climate prediction involves simplified stochastic modeling of nonlinear features of the coupled atmosphere/ocean system. The practical reasons for such needs are easy to understand. In the foreseeable future, it will be impossible to resolve the effects of the coupled atmosphere/ocean system through computer models with detailed resolution of the atmosphere on decadal time scales. However, the questions of interest also change. For example, for climate prediction, one is not interested in whether there is a significant deflection of the storm track northward in the Atlantic during a specific week in January of a given year, but rather, whether the mean and variance of the storm track are large during several years of winter seasons and what is the impact of this trend on the overall pole-ward transport of heat in both the atmosphere and ocean. The idea of simplified stochastic modeling for unresolved space-time scales in climate modeling is over 20 years old and emerged from fundamental papers by Hasselman (1) and Leith (2). In the atmosphere/ocean community, there is a recent flourishing of ideas utilizing simple stable linear Langevin stochastic equations to model and predict short-term and decadal climate changes such as El Niño (3, 4), the North Atlantic Oscillation (5, 6), and mid-latitude storm tracks (7–9) with notable positive results but also failure of this simplified stochastic model in some circumstances (10).

Here we introduce and analyze several idealized models for stochastic climate modeling and utilize unambiguous mathematical theory to demonstrate the potential need for more sophisticated stochastic models beyond those developed in earlier works (3–10). In particular, explicit examples demonstrate that simple Langevin models can emerge in stochastic climate modeling with an unstable climate mean and, in appropriate circumstances, stochastic models need to incorporate both suitable nonlinear effects and multiplicative noise beyond standard Langevin equation regression fitting. The mathematical strategy for stochastic climate modeling is illustrated on an idealized example involving truncated barotropic flow on a beta-plane with topography and a mean flow. In this example, the effect of the original 57 degrees of freedom is well represented by a theoretically predicted stochastic model with only 3 degrees of freedom, which yields the model climate behavior with reasonable accuracy after coarse-graining in time.

We summarize the remainder of this paper briefly. First we discuss the general strategy for stochastic climate modeling; then we develop explicit examples with new phenomena for stochastic climate modeling; finally, we illustrate several aspects of the theory on the idealized example mentioned above.

Basic Strategy for Stochastic Climate Modeling

We illustrate the ideas for stochastic climate modeling on an abstract basic model involving quadratically nonlinear dynamics, which is very appropriate for modeling many aspects of atmospheric dynamics. In the abstract model, the unknown variable $\vec{z}$ evolves in time in response to a linear operator, $L\vec{z}$, and a quadratic or bilinear operator, $B(\vec{z}, \vec{z})$, and satisfies

$$\frac{d\vec{z}}{dt} = L\vec{z} + B(\vec{z}, \vec{z}). \qquad [1]$$

In stochastic climate modeling, the variable $\vec{z}$ is decomposed into an orthogonal decomposition through the variables $\vec{x}$, $\vec{y}$ by $\vec{z} = (\vec{x}, \vec{y})$. The variable $\vec{x}$ denotes the climate state of the system; the climate state necessarily evolves slowly in time compared to the $\vec{y}$ variables, which evolve more rapidly in time and are not resolved in detail in the stochastic climate model. Decomposing the dynamic equation in **1** by projecting on the $\vec{x}$ and $\vec{y}$ variables yields the equations

$$\frac{d\vec{x}}{dt} = L_{11}\vec{x} + L_{12}\vec{y} + B^1_{11}(\vec{x}, \vec{x}) + B^1_{12}(\vec{x}, \vec{y}) + B^1_{22}(\vec{y}, \vec{y}), \qquad [2]$$

$$\frac{d\vec{y}}{dt} = L_{21}\vec{x} + L_{22}\vec{y} + B^2_{11}(\vec{x}, \vec{x}) + B^2_{12}(\vec{x}, \vec{y}) + B^2_{22}(\vec{y}, \vec{y}). \qquad [3]$$

In stochastic climate modeling, the explicit nonlinear self-interaction through $B^2_{22}(\vec{y}, \vec{y})$ of the variables $\vec{y}$, which are not resolved in detail, is represented by a linear stochastic operator

$$B^2_{22}(\vec{y}, \vec{y}) \approx -\frac{\Gamma}{\varepsilon}\vec{y} + \frac{\sigma}{\sqrt{\varepsilon}}\dot{\vec{W}}(t), \qquad [4]$$

where Γ, σ are diagonal matrices (for simplicity in exposition) with positive coefficients and $\dot{\vec{W}}(t)$ is a vector-valued white-noise. We note from **4** that ε measures the ratio of the correlation time of the under-resolved $\vec{y}$-variables to the climate variables $\vec{x}$ and the requirement $\varepsilon \ll 1$ is very natural for stochastic climate models where the climate variables should change more slowly. In fact if we coarse-grain the equations in **2** and **3** with the approximation from **4** on a longer time scale, $t \to \varepsilon t$, to measure the slowly evolving climate variables, we derive the stochastic climate model

$$\begin{aligned} \frac{d\vec{x}}{dt} &= \frac{1}{\varepsilon}\big(L_{11}\vec{x} + L_{12}\vec{y} + B^1_{11}(\vec{x}, \vec{x}) + B^1_{12}(\vec{x}, \vec{y}) + B^1_{22}(\vec{y}, \vec{y})\big), \\ \frac{d\vec{y}}{dt} &= \frac{1}{\varepsilon}\big(L_{21}\vec{x} + L_{22}\vec{y} + B^2_{11}(\vec{x}, \vec{x}) + B^2_{12}(\vec{x}, \vec{y})\big) \\ &\quad - \frac{\Gamma}{\varepsilon^2}\vec{y} + \frac{\sigma}{\varepsilon}\dot{\vec{W}}(t). \end{aligned} \qquad [5]$$

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Abbreviations: SDE, stochastic differential equations; DNS, direct numerical simulation.

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In practice, the climate variables $\vec{x}$ are determined by a variety of procedures including leading order empirical orthogonal functions (EOFs), zonal averaging in space, low pass and high pass time filtering, or a combination of these procedures (3, 7–10). Next we explicitly analyze the equations in **5** in some instructive elementary examples that demonstrate the new phenomena mentioned earlier.

Stochastic Models with Stable and Unstable Langevin Dynamics

We consider the special case of the system in **5** consisting of three modes, x, y_1, y_2, which satisfy the equations

$$\begin{aligned} dx^\varepsilon(t) &= \frac{A_3}{\varepsilon} y_1^\varepsilon(t) y_2^\varepsilon(t)\,dt, \\ dy_1^\varepsilon(t) &= \frac{a}{\varepsilon} y_2^\varepsilon(t)\,dt + \frac{A_1}{\varepsilon} x^\varepsilon(t) y_2^\varepsilon(t)\,dt \\ &\quad - \frac{\gamma_1}{\varepsilon^2} y_1^\varepsilon(t)\,dt + \frac{\sigma_1}{\varepsilon}\,dW_1(t), \\ dy_2^\varepsilon(t) &= \frac{b}{\varepsilon} y_1^\varepsilon(t)\,dt + \frac{A_2}{\varepsilon} x^\varepsilon(t) y_1^\varepsilon(t)\,dt \\ &\quad - \frac{\gamma_2}{\varepsilon^2} y_2^\varepsilon(t)\,dt + \frac{\sigma_2}{\varepsilon}\,dW_2(t), \end{aligned} \qquad [6]$$

where the nonlinear interaction coefficients satisfy $A_1 + A_2 + A_3 = 0$, and $dW_1(t)$, $dW_2(t)$ are independent white-noise processes. Equations like the system in **6** are a prototype for stochastic models for mean flow-wave interaction in barotropic-baroclinic turbulence (8) provided that these general equations are projected on three modes consisting of one mean flow climate variable x arising from zonal averaging and two wave variables y_1, y_2. The nonlinear coupling arises from triad interaction. We are interested in the statistical behavior of the climate variable, $x^\varepsilon(t)$, in the limit as $\varepsilon \to 0$. The equations in **6** are a Markov process and, for any suitable function $f(x)$, the conditional statistics $\langle f(x^\varepsilon(t))\rangle$, where $\langle\cdot\rangle$ denotes ensemble-average, is determined through the backward equation (11)

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{1}{\varepsilon^2}\mathcal{L}_1 u^\varepsilon + \frac{1}{\varepsilon}\mathcal{L}_2 u^\varepsilon, \quad u^\varepsilon\big|_{t=0} = f(x), \qquad [7]$$

with $u^\varepsilon(x, y, t) = \langle f(x^\varepsilon(t))\rangle$. The operators $\mathcal{L}_1$, $\mathcal{L}_2$ in **7** are given explicitly by

$$\begin{aligned} \mathcal{L}_1 &= \sum_{j=1,2} -\gamma_j y_j \frac{\partial}{\partial y_j} + \frac{\sigma_j^2}{2}\frac{\partial}{\partial y_j}, \\ \mathcal{L}_2 &= A_3 y_1 y_2 \frac{\partial}{\partial x} + (a y_2 + A_1 x y_2)\frac{\partial}{\partial y_1} \\ &\quad + (b y_1 + A_2 x y_1)\frac{\partial}{\partial y_2}. \end{aligned} \qquad [8]$$

With the structure in **7** and **8**, we apply a theorem of Kurtz (12) to obtain that, in the limit as $\varepsilon \to 0$, $0 < t \le T$, $T < \infty$, $x^\varepsilon(t) - x(t)$ converges to zero, where $x(t)$ satisfies the linear Langevin stochastic equation

$$dx(t) = -\gamma(x(t) - \bar{x})\,dt + \sigma\,dW(t), \qquad [9]$$

with the same initial data for x from **6** and

$$\begin{aligned} \gamma &= -\frac{A_3}{2(\gamma_1+\gamma_2)}\left(\frac{A_1\sigma_2^2}{\gamma_2} + \frac{A_2\sigma_1^2}{\gamma_1}\right) \\ \gamma\bar{x} &= \frac{A_3}{2(\gamma_1+\gamma_2)}\left(\frac{a\sigma_2^2}{\gamma_2} + \frac{b\sigma_1^2}{\gamma_1}\right), \\ \sigma^2 &= \frac{A_3^2\sigma_1^2\sigma_2^2}{2\gamma_1\gamma_2(\gamma_1+\gamma_2)}. \end{aligned} \qquad [10]$$

Several aspects of the formulas in **9** and **10** merit comments. First, the coarse-grained in time stochastic model in **9** for the climate variable $x(t)$ is a linear Langevin model with a nontrivial climate mean $\bar{x}$ produced through the linear interaction of the wave variables y_1, y_2. On the other hand, this linear stochastic model is stable only for $\gamma > 0$ and, using $A_3 = -(A_1 + A_2)$, this condition is satisfied only if

$$\begin{aligned} &A_2 < -\max\left(A_1, A_1\sigma_2^2\gamma_1/\sigma_1^2\gamma_2\right) \\ \text{or}\quad &A_2 > -\min\left(A_1, A_1\sigma_2^2\gamma_1/\sigma_1^2\gamma_2\right). \end{aligned} \qquad [11]$$

In particular, there is always stability if the nonlinear interaction coefficients satisfy $\mathrm{sgn}(A_1A_2) = 1$ but instability in the climate model occurs for $\mathrm{sgn}(A_1A_2) = -1$ if the conditions in **11** are violated.

Stochastic Models Requiring Nonlinear Effects and Multiplicative Noise

We consider the special case of the system in **5** involving two climates x_1, x_2 and a single stochastic variable, y, which solve the nonlinear stochastic triad interaction equations

$$\begin{aligned} dx_1^\varepsilon(t) &= \frac{A_1}{\varepsilon} x_2^\varepsilon(t) y^\varepsilon(t)\,dt, \\ dx_2^\varepsilon(t) &= \frac{A_2}{\varepsilon} x_1^\varepsilon(t) y^\varepsilon(t)\,dt, \\ dy^\varepsilon(t) &= \frac{A_3}{\varepsilon} x_1^\varepsilon(t) x_2^\varepsilon(t)\,dt - \frac{\gamma}{\varepsilon^2} y^\varepsilon(t)\,dt \\ &\quad + \frac{\sigma}{\varepsilon}\,dW(t), \end{aligned} \qquad [12]$$

where $A_1 + A_2 + A_3 = 0$ and $dW(t)$ is a white-noise. We claim that as $\varepsilon \to 0$, $x_1^\varepsilon(t) - x_1(t) \to 0$, $x_2^\varepsilon(t) - x_2(t) \to 0$ where $x_1(t)$, $x_2(t)$ satisfy

$$\begin{aligned} dx_1(t) &= \frac{A_1}{\gamma}\left(A_3 x_2^2(t) + \frac{\sigma^2}{2\gamma}A_2\right) x_1(t)\,dt \\ &\quad + \frac{\sigma}{\gamma} A_1 x_2(t)\,dW(t), \\ dx_2(t) &= \frac{A_2}{\gamma}\left(A_3 x_1^2(t) + \frac{\sigma^2}{2\gamma}A_1\right) x_2(t)\,dt \\ &\quad + \frac{\sigma}{\gamma} A_2 x_1(t)\,dW(t), \end{aligned} \qquad [13]$$

with the same initial data for x_1, x_2 from **12**. We note that the stochastic equations for the climate variables x_1, x_2 in **13** in the coarse-grained limit involve both nonlinear interaction between these climate variables and also multiplicative rather than additive noises. Such features are usually not incorporated in contemporary stochastic climate models (3, 7–10) and are potentially significant.

The method from ref. 12 we have utilized in analyzing the previous model in **6** can also be applied here; however, it is also instructive to establish the result in **13** by direct calculation. For each $(x_1(t), x_2(t))$, the solution of the third equation in **12** is

$$y(t) = e^{-\gamma t/\varepsilon^2} y + \frac{A_3}{\varepsilon}\int_0^t e^{-\gamma(t-s)/\varepsilon^2} x_1(s) x_2(s)\,ds + g(t), \qquad [14]$$

where

$$g(t) = \frac{\sigma}{\varepsilon}\int_0^t e^{-\gamma(t-s)/\varepsilon^2}\,dW(s),$$

and y is the initial data. Inserting this expression (possibly supplemented by the statistics of y) in the first two equations in **12** yields an exact, non-Markovian system of equations for $x_1(t)$,

$x_2(t)$. The noise entering (multiplicatively) these equations is Gaussian, with zero-mean and covariance:

$$\langle g(t)g(t')\rangle = \frac{\sigma^2\varepsilon^2}{2\gamma}\left(e^{-\gamma|t-t'|/\varepsilon^2} - e^{-\gamma(t+t')/\varepsilon^2}\right).$$

We analyze the long-time form of the resulting equations for $x_1(t)$, $x_2(t)$ by considering the limit as $\varepsilon \to 0$, $0 < t \le T$, $T < \infty$ of $1/\varepsilon$ times the three terms in the right-hand side of **14**. First, we have

$$\frac{1}{\varepsilon}e^{-\gamma t/\varepsilon^2}y \to 0.$$

Second, we have

$$\frac{A_3}{\varepsilon^2}\int_0^t e^{-\gamma(t-s)/\varepsilon^2}x_1(s)x_2(s)\,ds \to \frac{A_3}{\gamma}x_1(t)x_2(t).$$

Finally, for an arbitrary test function $\psi(t, t')$, we have

$$\frac{1}{\varepsilon^2}\int_0^T\int_0^T \psi(t,t')\langle g(t)g(t')\rangle\,dt\,dt' \to \frac{\sigma^2}{\gamma^2}\int_0^T \psi(t,t)\,dt.$$

Thus $g(t)$ is itself approximately a white-noise as $\varepsilon \to 0$, i.e.,

$$\frac{1}{\varepsilon}g(t)\,dt \to \frac{\sigma}{\gamma}\,dW(t).$$

Note, however, that, as an approximation of a process with finite correlation time, $dW(t)$ has to be interpreted in the Stratonovich sense (11).

Combining these formulae in the first two equations in **12**, we obtain the following Stratonovich system of SDEs:

$$\begin{aligned} dx_1(t) &= \frac{A_1A_3}{\gamma}x_2^2(t)x_1(t)\,dt + \frac{\sigma}{\gamma}A_1x_2(t)\circ dW(t),\\ dx_2(t) &= \frac{A_2A_3}{\gamma}x_1^2(t)x_2(t)\,dt + \frac{\sigma}{\gamma}A_2x_1(t)\circ dW(t). \end{aligned} \quad [15]$$

This system is equivalent to the Itô system in **13** (see ref. 11, theorem 10.2.5 pp. 169–170).

Using the Stratonovich form of the SDEs in **15** can be useful because usual rules of calculus (in contrast to Itô rules) apply. Zero is an equilibrium state for **13** or **15** and the linearized equations from **15** about this equilibrium reduce to

$$\begin{aligned} dx_1(t) &= \frac{\sigma}{\gamma}A_1x_2(t)\circ dW(t),\\ dx_2(t) &= \frac{\sigma}{\gamma}A_2x_1(t)\circ dW(t). \end{aligned} \quad [16]$$

The solution of this system is easily obtained:

$$\begin{aligned} x_1(t) &= Cx_1 + \sqrt{\frac{A_1}{A_2}}Sx_2,\\ x_2(t) &= Cx_2 + \sqrt{\frac{A_2}{A_1}}Sx_1, \end{aligned} \quad [17]$$

where

$$\begin{aligned} C &= \cosh\left(\frac{\sigma\sqrt{A_1A_2}}{\gamma}\int_0^t dW(s)\right),\\ S &= \sinh\left(\frac{\sigma\sqrt{A_1A_2}}{\gamma}\int_0^t dW(s)\right). \end{aligned}$$

We note that the solutions of the linearized equations in **17** grow if and only if $\mathrm{sgn}(A_1A_2) = 1$. Thus, strong stochastic forcing of one mode of the real triad interaction equations in **12** yields substantial growth of energy in the other two modes if and only if these two modes have interaction coefficients with the same sign. These results solve the interesting recent conjecture of Smith and Waleffe (13) on stochastic forcing of triad equations in an extreme limiting regime for real-valued amplitudes. Similar results will be reported elsewhere for the full complex valued triad interaction equations as well as suitable multi-mode generalizations. Our explicit analysis of **12** is also useful for understanding higher order non-Markovian corrections; this will be developed elsewhere.

A Priori Stochastic Climate Modeling in an Idealized Model

There is an idealized model for geophysical flows where both the stochastic modeling assumptions in **4** and **5** as well as the new stochastic phenomena elucidated in **6–17** above can be checked *a priori* in an unambiguous fashion. This model consists of truncated barotropic fluid equations on a periodic beta-plane with mean flow and topography:

$$\begin{aligned} \frac{\partial q}{\partial t} + \nabla^{\perp}\psi\cdot\nabla q + U\frac{\partial q}{\partial x} + \beta\frac{\partial \psi}{\partial x} &= 0,\\ q &= \Delta\psi + h,\\ \frac{dU}{dt} &= \fint h\frac{\partial\psi}{\partial x}. \end{aligned} \quad [18]$$

There are nontrivial Gaussian measures (14) defining canonical Gibbs distributions depending on two parameters, α, $\mu > 0$ for the statistics of the truncated dynamics with

$$\begin{aligned} &\text{mean } U = -\frac{\beta}{\mu}, \quad \text{var } U = \frac{1}{\alpha\mu},\\ &\text{mean } \psi_l = \frac{h_l}{\mu + |\vec{l}|^2}, \quad \text{var } \psi_l = \frac{1}{\alpha|\vec{l}|^2(\mu + |\vec{l}|^2)}. \end{aligned} \quad [19]$$

In **19** and below, f_l denotes the $\vec{l}$-th Fourier coefficient of the 2π-periodic function, f. In spherical geometry such models capture a number of large-scale features of the atmosphere (15).

For our purpose here, we regard the mean flow U as the climate variable to be determined through a stochastic model. We pick the parameters μ and α in the Gibbs distribution described in **19** so that the correlation time of U is longer than those of all the other fluid variables ψ_l. We also pick an extreme case where the topography consists of only one single mode, $h_k \neq 0$, with ψ_k the stream function coefficient.

The Stochastic Model

Under the assumptions above, following the stochastic climate modeling strategy in **4** and **5** above, we will predict the stochastic behavior of the climate variable, U, through the coupled stochastic equations

$$\begin{aligned} d\psi_k^{\varepsilon} &= \frac{i}{\varepsilon}(\Omega_k - k_xU^{\varepsilon})\psi_k^{\varepsilon}\,dt + \frac{i}{\varepsilon}H_kU^{\varepsilon}\,dt\\ &\quad - \frac{\gamma_k}{\varepsilon^2}(\psi_k^{\varepsilon} - \bar{\psi}_k)\,dt + \frac{\sigma_k}{\varepsilon}\,dW(t),\\ dU^{\varepsilon} &= -\frac{2}{\varepsilon}k_x\,\mathrm{Im}(h_k^*\psi_k^{\varepsilon})\,dt, \end{aligned} \quad [20]$$

where $dW(t)$ is complex valued white-noise and $\Omega_k = k_x|\vec{k}|^{-2}\beta$, $H_k = k_x|\vec{k}|^{-2}h_k$, γ_k is a complex damping coefficient. We will provide a severe test of stochastic modeling by choosing the parameters of the stochastic model in **20** consistent with **19** *a priori*, i.e., we set

$$\bar{\psi}_k = \frac{h_k}{\mu + |\vec{k}|^2}, \qquad \frac{\sigma_k^2}{\mathrm{Re}\,\gamma_k} = \frac{1}{\alpha|\vec{k}|^2(\mu + |\vec{k}|^2)}, \quad [21]$$

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Table 1. Statistics for $\alpha = 1$, $\mu = 2$, $\beta = 0.5$

	Mean $\psi_{(1,0)}$	Var $\psi_{(1,0)}$	Mean U	Var U
DNS **21**	$(5.66 + i6.44)10^{-2}$	0.33	−0.215	0.409
Stat Mech **19**	$(5.51 + i6.28)10^{-2}$	0.333	−0.25	0.5
SDE **20**	$(5.5 + i6.27)10^{-2}$	0.32	−0.249	0.466

in **20** so that the correlation time, γ_k, is the only parameter supplied to the theory. The same procedure outlined in **7** and **8** above yields the theoretical prediction that as $\varepsilon \to 0$, $U^\varepsilon - U \to 0$, where the climate variable U satisfies the stable Langevin equation

$$dU = -\gamma_U(U - \bar{U})\,dt + \sigma_U\,dW(t), \qquad [22]$$

with

$$\gamma_U = \frac{2k_x^2\mu|h_k|^2\,\mathrm{Re}\,\gamma_k}{|\gamma_k|^2|\vec{k}|^2(\mu + |\vec{k}|^2)},$$
$$\sigma_U = \frac{2|k_x|\sigma_k|h_k|}{|\gamma_k|}, \quad \bar{U} = -\frac{\beta}{\mu}. \qquad [23]$$

It is simple matter to check that the mean and variance of U predicted through **22** agrees with the values in **19** from equilibrium statistical theory. Thus the simple stochastic model in **20** is consistent with the mean and variance of the climate variable U.

Comparison of the Stochastic Model and Direct Numerical Simulations

We provide a severe *a priori* test for stochastic climate model in **20–22** by using the single mode topography, $h(x) = 0.5\sin(x + 2.93)$ with $\beta = 0.5$, and values $\alpha = 1$, $\mu = 2$ for the canonical Gibbs measure in **19** with the truncation $|\vec{l}|^2 \leq 17$ so that there are 57 active modes. We take a single random initial data with micro-canonical ensemble values of energy and enstrophy consistent with **19** and compute the statistics of the solution by time averaging a direct numerical simulation after omitting a short transient period. The comparison of the mean and variance of the direct numerical simulation (DNS) and the predictions of **20** are presented in Table 1. The theoretical values of the mean and variance of $\psi_{(1,0)}$ agree almost exactly, while those for the climate variable, U, have about 15% error for mean U and 20% error for var U. The DNS spectrum of ψ_l for $1 \leq |\vec{l}|^2 \leq 17$ is compared with the canonical spectrum using **19** in Fig. 1 and the agreement is excellent.

As discussed in **3–5** above, a tacit assumption of climate modeling is that the correlation time of the unresolved variables is much shorter than those of the climate variables. In Fig. 2 we plot the correlation functions determined by the DNS for the modes, $1 \leq |\vec{l}|^2 \leq 5$, and in Fig. 3 we plot the correlation function of U. In Fig. 2 the higher modes have shorter correlation times. These DNS results indicate that replacing the nonlinear interaction in **18** for the $(1, 0)$ mode by a white-noise model is an accurate approximation. From a least square fit of the DNS data we obtain the empirical correlation times for the mode $\psi_{(1,0)}$ and U to be given by $\mathrm{Re}\,\gamma_{(1,0)} = 0.61$ and $\gamma_U = 0.0566$, respectively, with the ratio $\gamma_U/\,\mathrm{Re}\,\gamma_{(1,0)} = 0.094$ so that the empirical parameter ε in the theory from **20–23** is quite small.

In Fig. 3 we compare the correlation predicted from the theory in **20–23** with the single input parameter $\gamma_{(1,0)}$ with the DNS and also with the numerical simulation of the stochastic equation in **20** (see also Tables 1 and 2). From a least square fit of the DNS correlation function of $\mathrm{Re}\,\psi_{(1,0)}$ we obtain the value of the parameter $\gamma_{(1,0)} = 0.61 + i0.74$. Using theoretical predictions **23** we obtain values $\gamma_U^T = 0.0555$ and $\sigma_U^T = 0.2356$. We compute the numerical solution of **20** with the mean values from **21** and $\varepsilon = 0.1$ in order to compare asymptotic predictions in the limit as $\varepsilon \to 0$ from **22** with the finite ε effect in **20**. We obtain values of $\gamma_{U,SDE} = 0.0536$, $\sigma_{U,SDE} = 0.2234$. This is excellent overall agreement of the stochastic modeling procedure given the crudeness of the approximation.

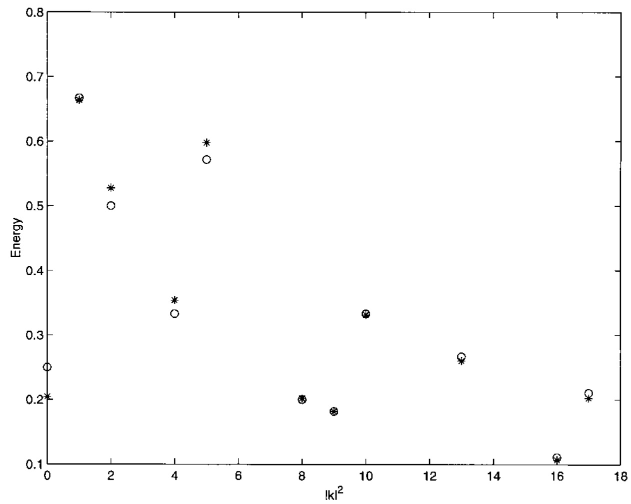

Fig. 1. Energy spectrum. $*$, DNS; $\circ$, canonical spectrum

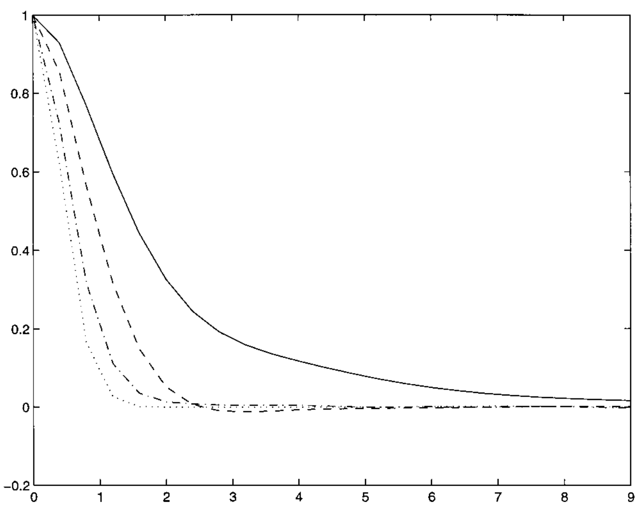

Fig. 2. Averaged correlation functions of ψ_k vs. time.

Table 2. Numerical and analytical estimates for γ_U, σ_U

	γ_U	σ_U
DNS **21**	0.0566	0.205
Theory **23**	0.0555	0.2356
SDE **20**	0.0536	0.2234

Concluding Discussion

Here we have utilized mathematical models and theory to elucidate several new and potentially important phenomena in stochastic climate models. These phenomena include unstable Langevin equations for the climate variables and the necessity for both incorporating nonlinear effects and multiplicative noise in stochastic equations for the climate variables under appropriate circumstances. We have utilized truncated models for the barotropic equations on a beta-plane with topography

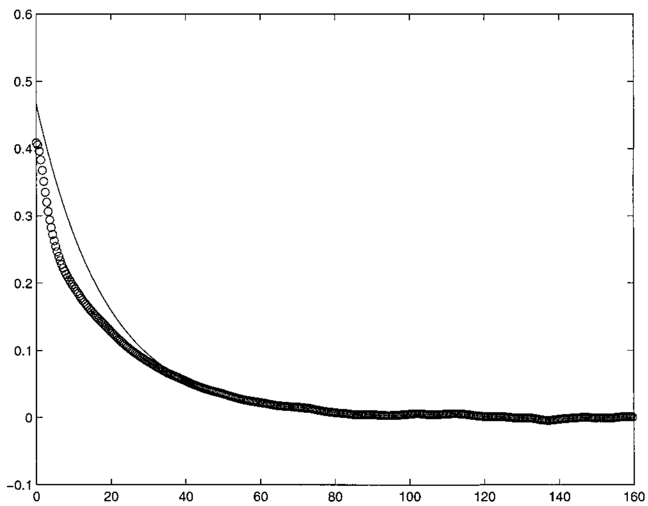

Fig. 3. Correlation function of *U* vs. time. Circles, DNS; solid line, theoretical predictions, SDE.

and mean flow as an idealized climate model where one can check the predictions of stochastic climate modeling *a priori*; these results include both the tacit assumptions of the theory and the excellent agreement of direct simulations with an explicit stochastic theory with only a single empirical parameter defined by a natural correlation time with all other parameters theoretically determined.

The theory developed here has been extended by the authors to several more complex stochastic models with wave-mean flow interactions, non-trivial topography, and more general mean states. Also by varying the statistical parameters μ and α from **19** as well as the mean flow, two of the authors have demonstrated the need for utilizing nonlinear effects and multiplicative stochastic noise in the truncated models from **18** representing idealized climate models. All of these results will be reported elsewhere in the near future in additional publications.

Finally, we mention that the work in **18–23** provides an example of a highly inhomogeneous conservative Hamiltonian system with nonlinear stability that inherits a crude version of effective stochastic dynamics in a single large scale variable U which interacts with a "heat bath" of modes for $1 \leq |\vec{l}|^2 \leq 17$ with much shorter correlation times. Thus, this work provides further empirical evidence beyond the interesting observation of Cai, McLaughlin, and Shatah (16) in idealized Schrödinger models that even a single mode of instability can lead to effective stochastic dynamics.

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