Proc. Natl. Acad. Sci. USA
Vol. 95, pp. 13392–13396, November 1998
Mathematics

On representations of finite type

RICHARD V. KADISON

Mathematics Department, University of Pennsylvania, Philadelphia, PA 19104-6395

Contributed by Richard V. Kadison, September 16, 1998

ABSTRACT Representations of the (infinite) canonical anticommutation relations and the associated operator algebra, the fermion algebra, are studied. A "coupling constant" (in (0, 1]) is defined for primary states of "finite type" of that algebra. Primary, faithful states of finite type with arbitrary coupling are constructed and classified. Their physical significance for quantum thermodynamical systems at high temperatures is discussed. The scope of this study is broadened to include a large class of operator algebras sharing some of the structural properties of the fermion algebra.

1. Background

The underlying theme of this note is the study of representations of the (infinite) canonical anticommutation relations (CAR). This study is conducted in the framework of operator algebras. In particular, our definitions and results are stated in terms of "C*-algebras" and "von Neumann algebras." Both are algebras of bounded operators on a Hilbert space (over the complex numbers $\mathbb{C}$) containing the adjoint A^* of A when they contain A ("self-adjoint operator algebras") and closed in the metric topology induced on $\mathcal{B}(\mathcal{H})$, the algebra of all bounded operators on $\mathcal{H}$, by the norm $A \to \|A\|$ on $\mathcal{B}(\mathcal{H})$, where $\|A\|$ is the bound (or, norm) of A. The von Neumann algebras contain the unit operator I on $\mathcal{H}$; for our purposes, we may assume that each C*-algebra does, as well. In addition, a von Neumann algebra $\mathcal{R}$ must satisfy the condition $\mathcal{R}'' = \mathcal{R}$, where $\mathcal{R}'$ is the "commutant" of $\mathcal{R}$, the set of operators in $\mathcal{B}(\mathcal{H})$ that commute with all the operators in $\mathcal{R}$. The commutant is also a von Neumann algebra. These operator algebras provide the basis for mathematical models of infinite quantum systems. We refer to ref. 7 for the basics of the theory of C*-algebras and von Neumann algebras. [A reference of the form (theorem 8.2.8) is to theorem 8 in section 2 of chapter 8 of ref. 7.] The recently published ref. 2 serves as an excellent up-to-date reference for the connections between quantum physics and operator algebras.

In ref. 3, a class of C*-algebras called, variously, "uhf" ("uniformly hyperfinite"), "matricial," and "Glimm" C*-algebras is studied and classified up to algebraic isomorphism. Their "states" and "representations" are examined, as well. Each Glimm algebra $\mathfrak{A}$ is simple, admits a unique tracial state, and is generated (as a C*-algebra) by a countable number (infinite) of elements. [A "state" of $\mathfrak{A}$ is a linear functional ρ on $\mathfrak{A}$ such that $\rho(A^*A) \geq 0$ for each A in $\mathfrak{A}$ and $\rho(I) = 1$; a "tracial" state is a state ρ for which $\rho(AB) = \rho(BA)$ for all A and B in $\mathfrak{A}$.] A countably infinite family $\{\mathfrak{A}_j\}_{j=1,2,\ldots}$ of pairwise-commuting C*-subalgebras $\mathfrak{A}_j$ of $\mathfrak{A}$, each containing the unit I of $\mathfrak{A}$, generates $\mathfrak{A}$ (as a C*-algebra), and each $\mathfrak{A}_j$ is * isomorphic to a full matrix algebra over the complex numbers $\mathbb{C}$ (the orders of the matrix algebras varying with j). Glimm shows that two such algebras $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic if and only if $\mathfrak{A}(p) = \mathfrak{B}(p)$ for each prime p, where $\mathfrak{A}(p)$ is the sum of the powers to which p divides the order of each $\mathfrak{A}_j$ [so that $\mathfrak{A}(p)$ may be 0 or ∞ as well as any natural number]. If the sum of $\mathfrak{A}(p)$ over all primes p is ∞, then there is a Glimm algebra $\mathfrak{A}$ for which this "invariant" occurs.

The Glimm algebra for which $\mathfrak{A}(2) = \infty$ and $\mathfrak{A}(p) = 0$ for all other primes p is of special significance in quantum physics. If each $\mathfrak{A}_j$ is (isomorphic to) the algebra of complex 2×2 matrices, then $\mathfrak{A}$ has that invariant. In that case, $\mathfrak{A}$ provides us with a mathematical model for the kinematical structure of an infinite system of identical fermion particles. Representations of groups by * automorphisms of $\mathfrak{A}$ model the symmetries of such systems. One-parameter groups of automorphisms model the dynamics generated by certain hamiltonians. This special Glimm algebra is called the "CAR algebra" (also, the "fermion algebra"). Although we shall study representations of broader classes of C*-algebras, sharing with the CAR algebra the properties of being countably generated and admitting a unique tracial state, our principal interest is in the CAR algebra. (A representation of $\mathfrak{A}$ is a homomorphism φ of $\mathfrak{A}$ into $\mathcal{B}(\mathcal{K})$, for some Hilbert space $\mathcal{K}$, such that $\varphi(A^*) = \varphi(A)^*$ for each A in $\mathfrak{A}$.) A family of operators $C_1, C_2, \ldots$ acting on a Hilbert space $\mathcal{H}$ is said to be a representation of the CAR when

$$
(*) \qquad \begin{aligned}
C_jC_k + C_kC_j &= 0 \qquad (j, k = 1, 2, \ldots),\\
C_jC_k^* + C_k^*C_j &= 0 \qquad (j \neq k),\\
C_jC_j^* + C_j^*C_j &= I \qquad (j = 1, 2, \ldots).
\end{aligned}
$$

[The system of equations $(*)$ is referred to as the "canonical anticommutation relations" and the set of elements $\{C_j\}$ is said to "satisfy the CAR."] An important part of the mathematical analysis of (infinite) fermion systems involves the study of the representations of the CAR. In each $\mathfrak{A}_j$, regarded as the algebra of 2×2 matrices, let $\sigma_x^{(j)}$, $\sigma_y^{(j)}$, and $\sigma_z^{(j)}$ be $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}$, and $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, respectively. (These are the "Pauli spin matrices." They generate $\mathfrak{A}_j$ as an algebra and $\{\sigma_x^{(j)}, \sigma_y^{(j)}, \sigma_z^{(j)} : j = 1, 2, \ldots\}$ generates $\mathfrak{A}$ as a C*-algebra.)

Let C_j be $\sigma_z^{(1)} \cdots \sigma_z^{(j-1)}(\sigma_x^{(j)} - i\sigma_y^{(j)})/2$. Then $\{C_j\}$ satisfies the CAR. In addition,

$$
\begin{aligned}
\sigma_z^{(j)} &= 2C_j^*C_j - I,\\
\sigma_x^{(j)} &= \sigma_z^{(1)} \cdots \sigma_z^{(j-1)}(C_j + C_j^*),\\
\sigma_y^{(j)} &= i\sigma_z^{(1)} \cdots \sigma_z^{(j-1)}(C_j - C_j^*).
\end{aligned}
$$

Thus $\{C_1, \ldots, C_j\}$ generates the same (finite-dimensional) C*-algebra as $\{\mathfrak{A}_1, \ldots, \mathfrak{A}_j\}$ does, and $\{C_1, C_2, \ldots\}$ generates $\mathfrak{A}$ as a C*-algebra. With some further calculation, it follows that each representation of the CAR algebra $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ gives rise to a representation of the CAR. Conversely, each representation of the CAR gives rise to a representation of the CAR algebra (that leads, again, to the given representation of the CAR). This identification of the representations of the CAR with the representations of the CAR algebra makes a very powerful mechanism available to us for studying the representations of the CAR; there is a deep and highly developed theory of representations of C*-algebras in the mathematical literature.

At the heart of the representation theory for C*-algebras is the GNS (Gelfand–Neumark–Segal) construction associating with each state ρ of a C*-algebra $\mathfrak{A}$ a representation π_ρ of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}_\rho$ such that, for some unit vector x_ρ in $\mathcal{H}_\rho$, $\pi_\rho(\mathfrak{A})x_\rho$ is dense in $\mathcal{H}_\rho$ and $\rho(A) = \langle \pi_\rho(A)x_\rho, x_\rho \rangle$, for each A in $\mathfrak{A}$ (see theorem 4.5.2). In effect, the Hilbert space $\mathcal{H}_\rho$ is $\mathfrak{A}$ provided with the inner product $\langle A, B \rangle = \rho(B^*A)$. In general, we must form the quotient $\mathfrak{A}/\mathcal{L}_\rho$ of $\mathfrak{A}$ by the left ideal $\mathcal{L}_\rho$ $(= \{A \in \mathfrak{A}\colon \rho(A^*A) = 0\}$, the *left kernel* of ρ) in $\mathfrak{A}$ of "null vectors" for this inner product, and "complete" the quotient to construct $\mathcal{H}_\rho$. Then $\pi_\rho(A)$ stems from left multiplication by A on $\mathfrak{A}$ and x_ρ corresponds to the unit I of $\mathfrak{A}$. When $\mathcal{L}_\rho = (0)$, no quotient is needed. We say that ρ is a *faithful* state of $\mathfrak{A}$ in that case. If $\pi_\rho(\mathfrak{A})x$ is dense in $\mathcal{H}_\rho$ for each non-zero vector x in $\mathcal{H}_\rho$, we say that π_ρ is an *irreducible* representation of $\mathfrak{A}$ and that ρ is a *pure* state of $\mathfrak{A}$. That the state ρ is pure is equivalent to each of the following conditions: $\mathcal{L}_\rho$ is a maximal left ideal in $\mathfrak{A}$ (theorem 10.2.10); the null space of ρ is $\mathcal{L}_\rho + \mathcal{L}_\rho^*$ (theorem 10.2.8); if $\rho = \frac{1}{2}(\rho_1 + \rho_2)$, with ρ_1 and ρ_2 states of $\mathfrak{A}$, then $\rho = \rho_1 = \rho_2$ (theorem 10.2.3). The process leading to π_ρ, $\mathcal{H}_\rho$, and x_ρ, is called the "GNS construction," and π_ρ is the "GNS representation" for ρ.

In ref. 9, the class of von Neumann algebras (there called "rings of operators") whose centers consist just of the scalar multiples of I, the *factors*, is studied in detail. These factors are separated into three large subclasses: those of type I contain a minimal projection; those of type II do not have a minimal projection and either have a tracial state, the factors of "type II_1," or can be viewed as infinite matrices with entries from a factor of type II_1, the factors of "type II_∞"; the remaining factors are said to be of "type III." Each factor of type I is isomorphic to some $\mathcal{B}(\mathcal{H})$. If $\mathcal{H}$ has dimension n (a finite or infinite cardinal), the factor is said to be of "type I_n."

The tracial state on a factor $\mathcal{M}$ of type II_1 is unique. Its restriction to the (lattice of) projections in $\mathcal{M}$ serves as a "dimension function" on the lattice of projections and provides us with a realistic concept of "continuous dimensionality" (relative to $\mathcal{M}$). It makes sense to speak of spaces (the ranges of projections in $\mathcal{M}$) of dimension, say, $\sqrt{2}/2$. More than 60 years of intensive research, and a vast mathematical literature on the subject, have made it clear that the class of factors of type II_1 are pivotal to our understanding of the structure of all factors (and all von Neumann algebras).

A representation π of a C*-algebra $\mathfrak{A}$ is said to be a *primary* (or *factor*) representation of $\mathfrak{A}$ when $\pi(\mathfrak{A})''$, the von Neumann algebra generated by $\pi(\mathfrak{A})$, is a factor. We say that π is a primary representation of type I_n, II_1, II_∞, or III, when $\pi(\mathfrak{A})''$ is a factor of the corresponding type. The *finite factors* are those of type I_n, with n finite, or of type II_1. A primary representation π is of finite type when $\pi(\mathfrak{A})''$ is a factor of finite type. A state ρ of $\mathfrak{A}$ is a *primary* (or *factor*) state of $\mathfrak{A}$ (of type I_n, II_1, II_∞, III, of finite type) when its GNS representation π_ρ is of the corresponding type.

The finite canonical anticommutation relations were introduced in 1928 and studied by Jordan and Wigner. The relations $(*)$ (for $C_1, \ldots, C_n$) are satisfied by the algebraic combinations of spin matrices described before (an assignment known as the "Jordan–Wigner transform"). These C_j generate the same algebra as $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ (as noted). This algebra is the full matrix algebra of complex $2^n \times 2^n$ matrices. It has just one irreducible representation (up to unitary equivalence)—its usual action on 2^n-dimensional Hilbert space. It was felt, for a number of years, that the same would be true of the (infinite) CAR; aside from the technical, analytic details encountered in passing from the finite to the infinite, there ought not be "qualitative" differences. In ref. 4, Gårding and Wightman displayed an infinite number of inequivalent, irreducible representations of the CAR.

The powerful techniques of the theory of operator algebras make it routine, now, to produce inequivalent, irreducible representations of the CAR. (In example 10.4.19, an uncountable infinity of inequivalent irreducible representations of $\mathfrak{A}$ are constructed.) These methods produce representations of $\mathfrak{A}$ that are not of type I. In effect, Murray and von Neumann (10) exhibit a type II_1 representation of the CAR by constructing a factor $\mathcal{M}$ of type II_1 with $\mathfrak{A}$ as a C*-subalgebra such that $\mathfrak{A}''$ is $\mathcal{M}$. In ref. 11, R. T. Powers finds a family of states $\{\rho_t\}$ of $\mathfrak{A}$ of type III, with t in $(0, \frac{1}{2})$, for which the GNS representations are not only unitarily inequivalent, but are "algebraically" inequivalent as well [that is, $\pi_t(\mathfrak{A})''$ and $\pi_{t'}(\mathfrak{A})''$ are not isomorphic when $t \neq t'$—compare, section 12.3]. That deep work remains one of the cornerstones of the impressive body of results clarifying the structure of type III factors, developed over the past 30 years. A definitive classification of the "quasi-free" states of the CAR algebra (defined in terms of "annihilation" and "creation" operators), by Powers and Størmer, appears in ref. 13.

The type II_1 representations of the CAR are to a large extent, the subject of the sections of this article that follow. Their algebraic equivalence is a difficult result established in ref. 10. The algebraic equivalence of the type II_∞ representations remained a major challenge until its (brilliant!) proof by A. Connes (1).

If ρ is a primary state of finite type of the CAR algebra $\mathfrak{A}$, then $\pi_\rho(\mathfrak{A})''$ is a finite factor acting on $\mathcal{H}_\rho$ with a (unit) cyclic vector x_ρ. It follows (from proposition 9.1.2 and theorem 9.1.3) that $\pi_\rho(\mathfrak{A})'$, the commutant of $\pi_\rho(\mathfrak{A})''$, is a finite factor (of type II_1 when $\pi_\rho(\mathfrak{A})''$ is of type II_1). Let $\mathcal{M}$ and its commutant $\mathcal{M}'$, acting on a Hilbert space $\mathcal{H}$ be finite factors and τ and τ' their respective tracial states. If x is a non-zero vector in $\mathcal{H}$ and E and E' are the projections whose ranges are the closures of $\mathcal{M}'x$ and $\mathcal{M}x$, respectively, then $E \in \mathcal{M}$ and $E' \in \mathcal{M}'$. A difficult result of Murray and von Neumann (9) tells us that the ratio $\tau(E)/\tau'(E')$ does not depend on the vector x we choose. This ratio is called *the coupling constant* for $\mathcal{M}$ and $\mathcal{M}'$. It is the starting point for the celebrated work of Jones (6) on the "index of subfactors."

If ρ is a primary state of finite type of a C*-algebra $\mathfrak{A}$, we say that ρ has coupling a when the finite factors $\pi_\rho(\mathfrak{A})''$ and $\pi_\rho(\mathfrak{A})'$ have coupling constant a. In case x_ρ is a separating vector for $\pi_\rho(\mathfrak{A})''$, it is generating for $\pi_\rho(\mathfrak{A})'$, and the coupling is 1. When ρ is a faithful state of $\mathfrak{A}$, x_ρ is a separating vector for $\pi_\rho(\mathfrak{A})$, which is dense in $\pi_\rho(\mathfrak{A})''$ in the strong-operator topology (corresponding to convergence on vectors in $\mathcal{H}_\rho$). With the added structure of finite factors and the matricial structure of the CAR algebra, does this force x_ρ to be separating for $\pi_\rho(\mathfrak{A})''$ as well? In other words, must a faithful, primary, state of finite type of a C*-algebra, in general, or the CAR algebra, in particular, have coupling 1? We shall show (Theorem 4), by construction, that there are such states of the CAR algebra with coupling a for each a in $(0, 1]$ and that the parameter a determines the GNS representations for these states up to unitary equivalence.

The physical significance of the finite-type representations of operator algebras can be seen best in the context of quantum thermodynamical systems. For such a system with a large (though, finite) number of degrees of freedom, Gibbs describes an equilibrium state ω_T for temperature T. The kinematical model (algebra of bounded observables) is a full matrix algebra over $\mathbb{C}$ whose order depends on the number of degrees of freedom. The "expectation functional" corresponding to ω_T is given in terms of the (unique) tracial state on this matrix algebra and a "density matrix" $\exp(-H/kT)$ (up to choices of units and normalizations), where H is the hamiltonian of the system and k is Boltzmann's constant. The expectation $\omega_T(A)$ of an observable A in the equilibrium state is $\tau(A\exp(-H/kT))$. As the temperature T tends to infinity $\exp(-H/kT)$, tends to I, the unit matrix; the equilibrium state at "infinite" temperature is τ, the tracial state.

Many of the phenomena we seem to observe in such systems (e.g., phase transitions) occur, in a mathematically precise sense, only when the system is infinite. Viewing large systems (with $\sim 10^{23}$ molecules) makes us feel that we are observing them as if they were infinite. To study these phenomena by mathematical methods, we deal with the infinite system. This is achieved, traditionally, by "taking the thermodynamical limit"—allowing the finite system (the system "in a box") to expand to infinity (maintaining control of "density"). Calculations are made in the finite system and the ("thermodynamical") limit taken as the systems expand. There are, of course, difficulties in passing to these limits.

By using operator algebra techniques, we can start with the operator algebra model of the infinite system, define the thermodynamical functions and features of interest, intrinsically, and eliminate the passage to the thermodynamical limit. A case in point, is the prescription for equilibrium states, in the C*-algebra setting, given by Haag, Hugenholtz, and Winnink (5). Without details, these states are boundary values of a function analytic in a strip in the complex plane of width $1/kT$. Moving from one point on the boundary (corresponding to a particular time in the dynamical evolution of the system), orthogonally, to the point on the other boundary of the strip, the expectation value undergoes a trace-like interchange of variables. Such states are called "KMS states" in ref. 5 to note the starting point for the development in ref. 5 in the work of Kubo, Martin, and Schwinger. The presence of several KMS states for a given system at a given temperature indicates the presence of phase transition at that temperature. Again, at infinite temperature, the strip has 0 width and a KMS state will be a tracial state. If our kinematical model is the CAR algebra, there is just one tracial state [and this is the case for other C*-algebras such as those stemming from the free groups on 2 or more generators (12)]. In these cases, the GNS representations for the tracial state is primary and of finite type, as we shall note in the next section. The states associated with this representation (the *normal* states for that representation) are "local" (or "quasi-local") perturbations of the infinite temperature equilibrium state. Among them are the states we construct with coupling in the parameter interval $(0, 1]$. Is this parameter physically detectable in systems at very high temperatures?

2. States of Finite Type

We begin with a technical concept, *separating projection*, that we shall need for our construction of the coupling parameter.

Definition 1: With $\mathfrak{A}$ a C*-subalgebra of a von Neumann algebra $\mathcal{R}$, a projection E in $\mathcal{R}$ is said to be *separating* for $\mathfrak{A}$ if $A = 0$ when $A \in \mathfrak{A}$ and $AE = 0$. A separating projection E for $\mathfrak{A}$ is said to be *a-separating* for $\mathfrak{A}$, with a in the interval $(0, 1]$, when $a = \sup\{b\colon \|AE\| > b,\ A \in [\mathfrak{A}]_1\}$, where $[\mathfrak{A}]_1 = \{A \in \mathfrak{A}\colon \|A\| = 1\}$.

If E is a-separating for $\mathfrak{A}$, then $\|AE\| \geq a$ for each A in $[\mathfrak{A}]_1$. Since $\|AE\| \leq \|A\| = 1$ for A in $[\mathfrak{A}]_1$, if E is a-separating, then $a \leq 1$. Of course, I is 1-separating for each C*-subalgebra. Moreover, if E is a-separating for $\mathfrak{A}$ and $E \leq F$, with F in $\mathcal{R}$, then F is a'-separating for $\mathfrak{A}$ and $a \leq a'$. To see this, note that if $\|AE\| > b$ for A in $[\mathfrak{A}]_1$, then

$$b < \|AE\| = \|AFE\| \leq \|AF\|\|E\| = \|AF\|.$$

For the most part, our aim is to find "small" a-separating projections with a as large as possible.

LEMMA 2. *If $\mathfrak{A}$ is a countably generated C*-subalgebra of a factor $\mathcal{M}$ of type II_1, then for each positive ε, there is a 1-separating projection E in $\mathcal{M}$ such that $\tau(E) < \varepsilon$, where τ is the* (*unique*) *tracial state on $\mathcal{M}$.*

Proof: Since $\mathfrak{A}$ is countably generated, it is norm separable. With $T_1, T_2, \ldots$ a countable, dense subset of (nonzero) elements of $\mathfrak{A}$, we have that $\|T_1\|^{-1}T_1, \|T_2\|^{-1}T_2, \ldots$ is a countable, dense subset of the unit sphere, $\{A \in \mathfrak{A}\colon \|A\| = 1\}$, of $\mathfrak{A}$. (If $A_j \to A$ and $\|A\| = 1$, then $\|A_j\|^{-1}A_j \to A$.) We write $A_1, A_2, \ldots$ in place of $\|T_1\|^{-1}T_1, \|T_2\|^{-1}T_2, \ldots$. If F is a projection that commutes with A^*A, then

$$\|AF\|^2 = \|FA^*AF\| = \|A^*AF\|.$$

Thus $\|AF\| = 1$ if and only if $\|A^*AF\| = 1$. At the same time, if $\|AF\| = 1 = \|A\|$, and F is a subprojection of G, then

$$1 = \|AF\| = \|AGF\| \leq \|AG\|\|F\| = \|AG\| \leq \|A\| = 1,$$

whence $\|AG\| = 1$.

With A in $[\mathfrak{A}]_1$ and a positive ε' given, let $\{E_\lambda\}$ be the spectral resolution for A^*A (in the form defined on pp. 310–312). Then $\bigwedge_{\lambda<1} I - E_\lambda = F_1$, where F_1 is the projection on the eigenspace [possibly (0)] for A^*A for the eigenvalue 1. Since $\lim_{\lambda\to 1-}(I - E_\lambda)$ is F_1 in the strong-operator topology, and τ is ultraweakly continuous on $\mathcal{M}$ (compare, theorem 8.2.8), we have that $\lim_{\lambda\to 1-} \tau(I - E_\lambda) = \tau(F_1)$. In addition, $I - E_\lambda \neq 0$ when $\lambda < 1$, since $\|A^*A\| = 1$. Thus $0 < \tau(I - E_\lambda) < \varepsilon'$ if $F_1 = 0$ and λ is near 1. If $F_1 \neq 0$, then there is a subprojection F_0 of F_1 in $\mathcal{M}$ such that $0 < \tau(F_0) < \varepsilon'$, since $\mathcal{M}$ is a factor of type II_1. As $A^*AF_1 = F_1$, $A^*AF_0 = F_0$, and F_0 commutes with A^*A. In any event, there is a projection F in $\mathcal{M}$ commuting with A^*A such that $\|A^*AF\| = 1$ and $0 < \tau(F) < \varepsilon'$.

Applying this result to each A_j, with $\varepsilon/2^j$ in place of ε', we find a projection E_j in $\mathcal{M}$, commuting with $A_j^*A_j$, such that $0 < \tau(E_j) < \varepsilon/2^j$ and $\|A_j^*A_jE_j\| = 1$. As noted, this implies that $\|A_jE_j\| = 1$. From the Kaplansky formula (ref. 8; cf. theorem 6.1.7), $F \vee G - G \sim F - F \wedge G$, where F and G are projections in a von Neumann algebra. Thus $\tau(F \vee G) + \tau(F \wedge G) = \tau(F) + \tau(G)$, when F and G are in $\mathcal{M}$, and $\tau(F \vee G) \leq \tau(F) + \tau(G)$. Applying this, successively, we conclude that $\tau(E_1 \vee E_2 \vee \cdots \vee E_n) \leq \tau(E_1) + \cdots + \tau(E_n)$. Since $\{E_1 \vee E_2 \vee \cdots \vee E_n\}$ is strong-operator convergent to $\bigvee_{j=1}^\infty E_j\ (= E)$,

$$\tau(E) \leq \sum_{j=1}^{\infty} \tau(E_j) < \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon.$$

The function, $T \to \|TE\|$ is norm continuous on $\mathfrak{A}$ and takes the constant value 1 on $\{A_j\}$, hence on the norm closure of $\{A_j\}$, the unit sphere $[\mathfrak{A}]_1$ of $\mathfrak{A}$. Thus E is a 1-separating projection for $\mathfrak{A}$ such that $\tau(E) < \varepsilon$. ■

We certainly don't expect that the hypothesis of Lemma 2 can be weakened to encompass all C*-subalgebras of $\mathcal{M}$; for $\mathcal{M}$, itself, $(I - E)E = 0$ and $I - E \neq 0$ when $\tau(E) < 1$. There is no separating projection for $\mathcal{M}$ (in $\mathcal{M}$) with small trace. Are there proper C*-subalgebras of $\mathcal{M}$ without separating projections? Are there any without 1-separating projections? Each proper von Neumann subalgebra has a separating projection different from I. "Separation" [related to the Jones Index (6)] will be defined and discussed elsewhere.

With very little change, the proof of Lemma 2 applies to finite von Neumann algebras, in general (in place of factors). In this case the tracial state is replaced with the center-valued trace on the von Neumann algebra. What we have proved in Lemma 2 is what we need to construct faithful, primary states of the fermion algebra with arbitrary coupling. This is effected in Theorem 4. We make use of some other results in its proof.

PROPOSITION 3. *If $\mathfrak{A}$ is a self-adjoint operator algebra acting essentially on a Hilbert space $\mathcal{H}$, $\mathcal{R}$ is the strong-operator closure of $\mathfrak{A}$, u is a trace vector for $\mathfrak{A}$, and P is the projection with range $[\mathcal{R}'u]$, then P is a central projection in $\mathcal{R}$. If u is also generating for $\mathfrak{A}$, then u is separating for $\mathcal{R}$.*

Proof: Since $\mathfrak{A}$ acts essentially on $\mathcal{H}$ [that is, $\mathfrak{A}(\mathcal{H})$ is dense in $\mathcal{H}$], I is in $\mathcal{R}$, $\mathcal{R}$ is a von Neumann algebra, and P is in $\mathcal{R}$. If $Gu = 0$, with G a projection in $\mathcal{R}$, then

$0 = \mathcal{R}'Gu = G\mathcal{R}'u$, and $GP = 0$. Thus $G \leq I - P$. If U is a unitary operator in $\mathcal{R}$, then $U^*(I - P)U$ is a projection in $\mathcal{R}$ and since u is a trace vector for $\mathcal{R}$,

$$\langle U^*(I-P)Uu, u\rangle = \langle UU^*(I-P)u, u\rangle = \langle (I-P)u, u\rangle = 0.$$

Thus $U^*(I - P)Uu = 0$ and $U^*(I - P)U \leq I - P$. With U^* in place of U, we have that $U(I - P)U^* \leq I - P$, and $I - P \leq U^*(I - P)U$. It follows that $I - P = U^*(I - P)U$ for each unitary operator U in $\mathcal{R}$. Thus $I - P$ and P are central projections in $\mathcal{R}$.

If u is generating for $\mathfrak{A}$, then u is separating for $\mathcal{R}'$. (See corollary 5.5.12.) Since $I - P$ is in $\mathcal{R}'$ (as well as $\mathcal{R}$) and $(I - P)u = 0$, $I - P = 0$. Thus u is generating for $\mathcal{R}'$, hence, separating for $\mathcal{R}$. ■

THEOREM 4. *If $\mathfrak{A}$ is an infinite-dimensional, countably generated C*-algebra with a unique tracial state τ that is faithful, and a in $(0, 1]$ is given, then there is a faithful, type II_1, primary state of $\mathfrak{A}$ with coupling a that is quasi-equivalent to τ.*

Each faithful, type II_1, primary, cyclic representation of a countably generated C-algebra is the GNS representation for some faithful state of the algebra.*

Two faithful, primary, cyclic representations of finite type of a C-algebra with a unique tracial state are unitarily equivalent if and only if they have the same coupling.*

Proof: Considering $\pi_\tau(\mathfrak{A})$ acting on $\mathcal{H}_\tau$, where π_τ is the GNS representation of $\mathfrak{A}$ corresponding to τ (see pp. 278–279 of ref. 7), we may assume that $\mathfrak{A}$ acts on $\mathcal{H}$ with a generating (unit) trace vector u. Thus u is a generating trace vector for $\mathcal{M}$, the strong-operator closure of $\mathfrak{A}$. From Proposition 3, u is separating for $\mathcal{M}$. If P is a central projection in $\mathcal{M}$, and neither P nor $I - P$ is 0, then $A \to \|Pu\|^{-2}\langle APu, u\rangle$ and $A \to \|(I-P)u\|^{-2}\langle A(I-P)u, u\rangle$ are tracial states of $\mathfrak{A}$. By assumption, these tracial states coincide. Since $\mathfrak{A}$ is strong-operator dense in $\mathcal{M}$, there is an A in $\mathfrak{A}$ such that APu is close to PPu $(= Pu)$ and $A(I - P)u$ is close to $P(I - P)u$ $(= 0)$. For such an A, $\|Pu\|^{-2}\langle APu, u\rangle$ is near 1, while $\|(I - P)u\|^{-2}\langle A(I - P)u, u\rangle$ is near 0. Thus one of P and $I - P$ is 0, and $\mathcal{M}$ is a factor. Since $\mathfrak{A}$ is infinite dimensional and $\mathcal{M}$ has a tracial state, $\mathcal{M}$ is a factor of type II_1. We denote by τ, again, the (unique) tracial state on $\mathcal{M}$.

From Lemma 2, there is a 1-separating projection E_0 for $\mathfrak{A}$ in $\mathcal{M}$ such that $\tau(E_0) < a$. Since $\mathcal{M}$ is a factor of type II_1, there is a subprojection E_1 of $I - E_0$ in $\mathcal{M}$ such that $\tau(E_1) = a - \tau(E_0)$. Let E be $E_0 + E_1$. Then E is a 1-separating projection for $\mathfrak{A}$ (from the discussion preceding Lemma 2), and $\tau(E) = a$.

Since u is separating for $\mathcal{M}$, $Eu \neq 0$. Let x be the unit vector $\|Eu\|^{-1}Eu$ and E' be the projection in $\mathcal{M}'$ with range $[\mathcal{M}x]$. From proposition 5.5.5, if $\pi(T) = TE'$ for T in $\mathcal{M}$, then π is a * isomorphism of $\mathcal{M}$ onto the factor $\mathcal{M}E'$ acting on $E'(\mathcal{H})$ with commutant $E'\mathcal{M}'E'$, a factor of type II_1 acting on $E'(\mathcal{H})$. Now, x is generating for $\mathcal{M}E'$ [in $E'(\mathcal{H})$] and

$$[E'\mathcal{M}'E'x] = [E'\mathcal{M}'Eu] = [EE'\mathcal{M}'u] = EE'(\mathcal{H}),$$

the range of $\pi(E)$ $(= EE')$ in $\mathcal{M}E'$. The trace is preserved by π so that EE' has trace a in $\mathcal{M}E'$. Thus $\mathcal{M}E'$ and $E'\mathcal{M}'E'$ have coupling constant a.

As E is separating for $\mathfrak{A}$ and u is separating for $\mathcal{M}$, if $\pi(A)x = 0$, for some A in $\mathfrak{A}$, then $0 = AE'Eu = AEu$, and $A = 0$. Thus, letting $\rho(A)$ be $\langle \pi(A)x, x\rangle$, for A in $\mathfrak{A}$, we have that ρ is a faithful state of $\mathfrak{A}$. Since π is an ultraweak homeomorphism between $\mathcal{M}$ and $\mathcal{M}E'$ (compare remark 7.4.4), $\pi(\mathfrak{A})$ is ultraweakly dense in $\mathcal{M}E'$. Thus x is generating for $\pi(\mathfrak{A})$ in $E'(\mathcal{H})$. It follows, from uniqueness of the GNS representation (proposition 4.5.3), that π restricted to $\mathfrak{A}$ is unitarily equivalent to the GNS representation corresponding to ρ. Hence ρ is a type II_1, faithful, primary state of $\mathfrak{A}$ with coupling a. By choice, $\mathfrak{A}$ acting on $\mathcal{H}$ is the GNS representation corresponding to τ, and π restricted to $\mathfrak{A}$ is the GNS representation corresponding to ρ. But π restricted to $\mathfrak{A}$ extends to the * isomorphism π of the ultraweak closure $\mathcal{M}$ of $\mathfrak{A}$ onto the ultraweak closure $\mathcal{M}E'$ of $\pi(\mathfrak{A})$. Thus τ and ρ are quasi-equivalent (see definition 10.3.1).

Suppose, now, that $\mathfrak{A}$ is a countably generated C*-algebra and π is a faithful, type II_1, primary representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ with generating unit vector x. Let $\mathcal{M}$ be the strong-operator closure of $\pi(\mathfrak{A})$ and F the projection (in $\mathcal{M}$) with range $[\mathcal{M}'x]$. While F may not be separating for $\mathfrak{A}$, from Lemma 2, there is a projection E in $\mathcal{M}$ that is 1-separating for $\pi(\mathfrak{A})$ and has the same trace as F. In this case, there is a partial isometry V in $\mathcal{M}$ with initial projection F and final projection E. Let y be the (unit) vector Vx. Then $[\mathcal{M}y] \supseteq [\mathcal{M}V^*Vx] = [\mathcal{M}x] = \mathcal{H}$. Thus y is generating for $\mathcal{M}$ and for the strong-operator-dense subalgebra $\pi(\mathfrak{A})$ of $\mathcal{M}$. It follows that π is the GNS representation for $\omega_y \circ \pi$.

We note that y is separating for $\pi(\mathfrak{A})$. Suppose that $\pi(A)y = 0$ for some A in $\mathfrak{A}$. Then

$$0 = \mathcal{M}'\pi(A)y = \pi(A)\mathcal{M}'y = \pi(A)\mathcal{M}'Vx = \pi(A)V\mathcal{M}'x,$$

whence $\pi(A)E = 0$. Thus $\pi(A) = 0$, since E is separating for $\pi(\mathfrak{A})$. As π is faithful, $A = 0$, and $\omega_y \circ \pi$ is a faithful state of $\mathfrak{A}$. Thus π is the GNS representation for the faithful state $\omega_y \circ \pi$ of $\mathfrak{A}$.

Suppose that φ and ψ are unitarily equivalent primary representations of a C*-algebra $\mathfrak{A}$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. There is a unitary transformation U of $\mathcal{H}$ onto $\mathcal{K}$ such that $U\varphi(A)U^{-1} = \psi(A)$ for each A in $\mathfrak{A}$. In this case, $U\varphi(\mathfrak{A})^-U^{-1} = \psi(\mathfrak{A})^-$, where $\varphi(\mathfrak{A})^-$ and $\psi(\mathfrak{A})^-$ are the strong-operator closures of $\varphi(\mathfrak{A})$ in $\mathcal{B}(\mathcal{H})$ and $\psi(\mathfrak{A})$ in $\mathcal{B}(\mathcal{K})$, respectively. In addition, $U\varphi(\mathfrak{A})'U^{-1} = U\psi(\mathfrak{A})'U^{-1}$. Thus φ and ψ have the same coupling when they are cyclic of finite type.

Assume, now, that $\mathfrak{A}$ is a C*-algebra with a unique tracial state and that φ and ψ are faithful, primary, cyclic representations of $\mathfrak{A}$ of finite type with the same coupling on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Since φ and ψ are primary, they are either quasi-equivalent or disjoint [proposition 10.3.12(ii)]. Suppose that they are quasi-equivalent. Then there is a * isomorphism α of $\varphi(\mathfrak{A})^-$ onto $\psi(\mathfrak{A})^-$ such that $\alpha \circ \varphi = \psi$ (from the definition of "quasi-equivalence"). Since $\varphi(\mathfrak{A})^-$, $\varphi(\mathfrak{A})'$ and $\psi(\mathfrak{A})^-$, $\psi(\mathfrak{A})'$ have the same coupling, α is implemented by a unitary transformation (exercise 9.6.30). Thus φ and ψ are unitarily equivalent in this case.

Suppose that φ and ψ are disjoint. We shall derive a contradiction from this assumption. Let π be $\varphi \oplus \psi$. Then π is a faithful representation of $\mathfrak{A}$ of finite type on $\mathcal{H} \oplus \mathcal{K}$. [By definition, $\pi(A)(x, y) = (\varphi(A)x, \psi(A)y)$ when $(x, y) \in \mathcal{H} \oplus \mathcal{K}$.] Since φ and ψ are disjoint, $\pi(\mathfrak{A})^- = \varphi(\mathfrak{A})^- \oplus \psi(\mathfrak{A})^-$ (theorem 10.3.5). With P the orthogonal projection of $\mathcal{H} \oplus \mathcal{K}$ onto $\mathcal{H}$, $I - P$ is the projection on $\mathcal{K}$, and P is in $\pi(\mathfrak{A})^-$. Let x and y be unit vectors in $\mathcal{H}$ and $\mathcal{K}$, respectively, and $\tilde{x}$, $\tilde{y}$ be the unit vectors $(x, 0)$ and $(0, y)$ in $\mathcal{H} \oplus \mathcal{K}$. With τ the center-valued trace on $\pi(\mathfrak{A})^-$ (theorem 8.2.8), the functionals $A \to \langle \tau(\pi(A))\tilde{x}, \tilde{x}\rangle$ and $A \to \langle \tau(\pi(A))\tilde{y}, \tilde{y}\rangle$ $(A \in \mathfrak{A})$ are tracial states of $\mathfrak{A}$. By assumption, these states are equal. Since they coincide on $\pi(\mathfrak{A})$, and τ is ultraweakly continuous, they coincide on the weak (and strong)-operator closure of the unit ball in $\pi(\mathfrak{A})$. By the Kaplansky Density Theorem (theorem 5.3.5), P is in this closure. Moreover, $\tau(P) = P$. Thus $1 = \langle P\tilde{x}, \tilde{x}\rangle = \langle P\tilde{y}, \tilde{y}\rangle = 0$, a contradiction. It follows that φ and ψ are not disjoint, that they are quasi-equivalent, and that they are unitarily equivalent. ■

Remark 5: The fact that each faithful, type II_1, primary, cyclic representation of a countably generated C*-algebra is the GNS representation for some faithful state of the C*-algebra, noted in Theorem 4 does not remain valid under "beckoning" weakening of the hypotheses. For example, replacing "type II_1" by "finite-type" or dropping the assumption that the C*-algebra is countably generated do not yield valid statements. To see this, we note that no irreducible representation of a C*-algebra (other than the 1-dimensional $\mathbb{C}$) is the GNS representation for a faithful state. Although each irreducible representation π is cyclic with each unit vector x in the representation space as a generating vector, so that π is the GNS representation for each of the states $\omega_x \circ \pi$, a state ρ has an irreducible GNS representation if and only if the null space of ρ is $\mathcal{L}_\rho + \mathcal{L}_\rho^*$, where $\mathcal{L}_\rho$ is the left kernel of ρ (theorem 10.2.8). If ρ is faithful, $\mathcal{L}_\rho$ must be (0). Thus ρ is faithful and has an irreducible GNS representation if and only if its null space is (0), in which case, the C*-algebra is $\mathbb{C}$. In particular, $\mathcal{B}(\mathcal{H})$, acting on $\mathcal{H}$, in its identity representation, acts irreducibly and is countably generated when $\mathcal{H}$ is finite dimensional. Hence the identity representation of $\mathcal{B}(\mathcal{H})$ on $\mathcal{H}$ is the GNS representation of $\mathcal{B}(\mathcal{H})$ for no faithful state when $\mathcal{H}$ has dimension 2 or more.

To illustrate the need for the hypothesis that our C*-algebra is countably generated, we examine a factor $\mathcal{M}$ of type II_1 acting on a Hilbert space $\mathcal{H}$ with coupling a in $(0, 1)$. If x is a generating vector for $\mathcal{M}$, then $[\mathcal{M}'x]$ is the range of a projection in $\mathcal{M}$ of trace a (< 1). Hence x is not generating for $\mathcal{M}'$ and therefore, not separating for $\mathcal{M}$ (corollary 5.5.12). Thus the identity representation of $\mathcal{M}$ on $\mathcal{H}$ is a faithful, type II_1, primary, cyclic representation of $\mathcal{M}$ that is the GNS representation for no faithful state of $\mathcal{M}$. Of course, $\mathcal{M}$ is not countably generated as a C*-algebra in this case. ■

THEOREM 6. (*i*) *If $\mathfrak{A}$ has a unique tracial state, then each representation of finite type of $\mathfrak{A}$ is primary.*

(*ii*) *If each cyclic representation of $\mathfrak{A}$ of finite type is primary, then $\mathfrak{A}$ has at most one tracial state.*

Proof: (*i*) Let π be a representation of finite type of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ and τ be the center-valued trace on $\pi(\mathfrak{A})^-$. As argued in the last paragraph of the proof of Theorem 4, $\langle\tau(T)x, x\rangle = \langle\tau(T)y, y\rangle$ for all T in $\pi(\mathfrak{A})^-$ and all unit vectors x and y in $\mathcal{H}$. In particular, if P is a non-zero central projection in $\pi(\mathfrak{A})^-$ and x is a unit vector in its range, then

$$1 = \langle Px, x\rangle = \langle\tau(P)x, x\rangle = \langle\tau(P)y, y\rangle = \langle Py, y\rangle$$

for each unit vector y in $\mathcal{H}$. Thus $P = I$, $\pi(\mathfrak{A})^-$ is a factor, and π is primary.

(*ii*) Let τ_1 and τ_2 be tracial states of $\mathfrak{A}$, π_1 and π_2 the GNS representations of $\mathfrak{A}$ on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, for π_1 and π_2, with cyclic unit vectors u_1 and u_2 such that $\tau_j(T) = \langle\pi_j(T)u_j, u_j\rangle$ $(j = 1, 2)$. From Proposition 3, u_j is a separating trace vector for $\pi_j(\mathfrak{A})^-$, and $\pi_j(\mathfrak{A})^-$ is of finite type $(j = 1, 2)$.

By assumption π_j is primary, and $\pi_j(\mathfrak{A})^-$ is a finite factor $(j = 1, 2)$. Thus the sum $\pi_1(\mathfrak{A})^- \oplus \pi_2(\mathfrak{A})^-$ is a finite von Neumann algebra that is not a factor. In addition, (u_1, u_2) is a generating vector for $\pi_1(\mathfrak{A})^- \oplus \pi_2(\mathfrak{A})^-$ (acting on $\mathcal{H}_1 \oplus \mathcal{H}_2$). If π_1 and π_2 are disjoint, then $(\pi_1 \oplus \pi_2)(\mathfrak{A})^- = \pi_1(\mathfrak{A})^- \oplus \pi_2(\mathfrak{A})^-$ (theorem 10.3.5), whence (u_1, u_2) is a cyclic vector for $(\pi_1 \oplus \pi_2)(\mathfrak{A})^-$. In this case, $\pi_1 \oplus \pi_2$ is a cyclic representation of finite type of $\mathfrak{A}$ that is not primary, contrary to assumption. Thus π_1 and π_2 are not disjoint.

Since π_1 and π_2 are primary and not disjoint, they are quasi-equivalent representations [proposition 10.3.12(ii)]. In particular, there is a normal state ω of $\pi_1(\mathfrak{A})^-$ such that $\omega \circ \pi_1 = \omega_{u_2} \circ \pi_2$ (proposition 10.3.13). Thus $\omega \circ \pi_1 = \tau_2$, and ω is a tracial state of $\pi_1(\mathfrak{A})^-$. Since $\pi_1(\mathfrak{A})^-$ is a finite factor, it has a unique tracial state (compare theorem 8.2.8), whence ω is the restriction of ω_{u_1} to $\pi_1(\mathfrak{A})^-$, and $\tau_1 = \omega_{u_1} \circ \pi_1 = \omega \circ \pi_1 = \omega_{u_2} \circ \pi_2 = \tau_2$. Hence $\mathfrak{A}$ has at most one tracial state. ■

Dedicated to Erling Størmer on the occasion of his sixtieth birthday.

1. Connes, A. (1976) *Ann. Math.* **104,** 73–115.
2. Evans, D. & Kawahigashi, Y. (1998) *Quantum Symmetries on Operator Algebras* (Oxford, New York).
3. Glimm, J. (1960) *Trans. Am. Math. Soc.* **95,** 318–340.
4. Gårding, L. & Wightman, A. (1954) *Proc. Natl. Acad. Sci. USA* **40,** 318–340.
5. Haag, R., Hugenholtz, N. & Winnink, M. (1967) *Commun. Math. Phys.* **5,** 215–236.
6. Jones, V. F. R. (1983) *Invent. Math.* **72,** 1–25.
7. Kadison, R. & Ringrose, J. (1997, 1998) *Fundamentals of the Theory of Operator Algebras, Vols. 1–4* (Am. Math. Soc., Providence, RI).
8. Kaplansky, I. (1951) *Ann. Math.* **53,** 235–249.
9. Murray, F. & von Neumann, J. (1936) *Ann. Math.* **37,** 116–229.
10. Murray, F. & von Neumann, J. (1943) *Ann. Math.* **44,** 716–808.
11. Powers, R. T. (1967) *Ann. Math.* **86,** 138–178.
12. Powers, R. T. (1975) *Duke Math. J.* **49,** 151–156.
13. Powers, R. T. & Størmer, E. (1970) *Commun. Math. Phys.* **16,** 1–33.