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Development of Proportional Reasoning: Where Young Children Go Wrong

Ty W. Boyer¹, Susan C. Levine², and Janelle Huttenlocher²

¹ *Department of Psychological & Brain Sciences, Indiana University*

² *Department of Psychology, The University of Chicago*

Abstract

Previous studies have found that children have difficulty solving proportional reasoning problems involving discrete units until 10- to 12-years of age, but can solve parallel problems involving continuous quantities by 6-years of age. The present studies examine where children go wrong in processing proportions that involve discrete quantities. A computerized proportional equivalence choice task was administered to kindergartners through fourth-graders in Study 1, and to first- and third-graders in Study 2. Both studies involved four between-subjects conditions that were formed by pairing continuous and discrete target proportions with continuous and discrete choice alternatives. In Study 1, target and choice alternatives were presented simultaneously and in Study 2 target and choice alternatives were presented sequentially. In both studies, children performed significantly worse when both the target and choice alternatives were represented with discrete quantities than when either or both of the proportions involved continuous quantities. Taken together, these findings indicate that children go astray on proportional reasoning problems involving discrete units only when a numerical match is possible, suggesting that their difficulty is due to an overextension of numerical equivalence concepts to proportional equivalence problems.

Keywords

Proportional Reasoning; Continuous and Discrete Quantities; Mathematical Development; Intuitive and Explicit Processing

Proportional reasoning involves understanding the multiplicative relationships between rational quantities ($a/b = c/d$), and is a form of reasoning that characterizes important structural relationships in mathematics and science, as well as in every day life (Cramer & Post, 1993; Lesh, Post, & Behr, 1988). As Ahl, Moore, and Dixon (1992) emphasized, “Proportional reasoning is a pervasive activity that transcends topical barriers in adult life.” Proportional information is crucial in dealing with such diverse topics as economic values, relational spatial contrasts, temperatures, densities, concentrations, velocities, chemical compositions, demographic information, and recipe formulation (Karplus, Pulos, & Stage, 1983; Moore, Dixon, & Haines, 1991; Siegler & Vago, 1978; Sophian & Wood, 1997; Spinillo & Bryant, 1999). For example, when baking, one needs to think proportionally about the relative measures of each ingredient (e.g., 2-and- $\frac{1}{2}$ cups flour, $\frac{1}{3}$ cup sugar, and $\frac{1}{4}$ cup butter), and must maintain these proportions whenever deviating from the recipe (e.g., whenever doubling or halving the intended amount). In chemistry, proportionality is central to balancing chemical equations. During election years, candidates strategically allocate their campaign time to particular

geographical locations based on the proportion of the population represented by specific demographic target groups. Understanding of proportionality is also central to mathematics; it is the basis of rational number operations, unit partitioning, and basic algebra and geometry problem solving (Empson, 1999; Fuson & Abrahamson, 2005; Hasemann, 1981; Pitkethly & Hunting, 1996; Saxe, Gearhart, & Seltzer, 1999; Sophian, Garyantes, & Chang, 1997). In fact, this kind of reasoning is viewed as so central to mathematical thinking that the National Council of Teachers of Mathematics (1989) stated that it deserves, “whatever time and effort must be expended to assure its careful development” (cited in Cramer & Post, 1993).

Despite the importance and pervasiveness of proportional reasoning, there is disagreement regarding its developmental time course. One theoretical perspective, originally presented by Piaget and Inhelder (1951/1975; Inhelder & Piaget, 1958), proposes that children are incapable of proportional reasoning until about 11-years of age. According to Piagetian theory, proportional reasoning involves understanding the “relation between relations,” and is a hallmark of formal operations. Piaget and Inhelder’s work, as well as many subsequent studies support this idea (Fujimura, 2001; Schwartz & Moore, 1998). For example, Noeltig (1980) presented 6- to 16-year old children with two proportions, each represented as a set of glasses of orange juice concentrate and a set of glasses of water and participants were asked to choose which proportion would produce a more concentrated orange drink (e.g., three glasses of orange juice to one glass of water vs. one glass of orange juice to three glasses of water). Consistent with Piaget and Inhelder’s perspective, children under 12-years of age failed to select the correct set.

Other studies have shown that 5- to 8- year-olds are unable to reliably predict the outcomes of chance gambles, where outcome probabilities are determined by proportions (e.g., lottery draws and spinner gambles; Brainerd, 1981; Chapman, 1975; Davies, 1965; Falk & Wilkening, 1998). Children’s difficulty with proportional reasoning in the context of conventional fractions is also noted in the mathematics education literature (Carpenter, Fennema, & Romberg, 1993; Pitkethly & Hunting, 1996). For example, Ball (1993) reported that third-grade children systematically misinterpret traditionally notated fractions (e.g., $\frac{3}{4}$), and estimate that fractions with larger denominators are quantitatively greater than fractions with smaller denominators (e.g., $\frac{4}{6} < \frac{4}{8}$).

In contrast to studies indicating that proportional reasoning is a late achievement, some studies report that children as young as 5- to 6-years of age can successfully solve slightly modified proportional reasoning problems (Ginsburg & Rapoport, 1967; Sophian, 2000; Sophian & Wood, 1997; Van Den Brink & Streefland, 1979). One problem modification that has been used to this end has involved framing the proportional equivalence problems in terms of analogy (Farrington-Flint, Canobi, Wood, & Faulkner, 2007). Children become capable of solving simple analogy problems during the preschool years (e.g., Gentner, 1977a e.g., Gentner, 1977b), and researchers have noted that proportional reasoning is a quantitative form of analogical reasoning, in the sense that both conceptual analogies and quantitative proportions require analysis of the relations between relations. For example, understanding that the relation between hands and gloves is analogous to the relation between feet and shoes may involve similar reasoning processes as understanding that the relation between $\frac{4}{5}$ is analogous, or proportionally equivalent to, $\frac{8}{10}$. Working within this framework, Goswami and colleagues (Goswami, 1989; 1995; Singer-Freeman & Goswami, 2001) designed problems to assess young children’s proportional reasoning skills in the context of shape analogies. Their findings show that six- and seven-year-olds understand, for example, that a $\frac{1}{2}$ circle and $\frac{1}{2}$ rectangle pair is analogous to a $\frac{1}{4}$ circle and $\frac{1}{4}$ rectangle pair (Goswami, 1989). Of course, proportions, like other analogies, vary in difficulty depending on the specific terms involved, and Goswami (1989) presented children with $\frac{1}{4}$ -base problems (e.g., $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$), which likely made them easier than other alternatives (e.g., $\frac{2}{3}$ and $\frac{6}{9}$).

Five to six-year-olds also have success on proportional reasoning problems that take the form of probabilistic gambles if a modified response mechanism is used. In these studies, participants are asked to provide a scaled judgment of their satisfaction with a gamble, rather than to predict the outcome of the gamble (Acredolo, O'Connor, Bank, & Horobin, 1989; Schlottmann, 2001). It has been suggested that this type of response may enable young children to use intuitive problem solving strategies that are more likely to be correct than more explicit strategies (Boyer, 2007; Falk & Wilkening, 1998; Reyna & Brainerd, 1994; Schlottmann, 2001).

The discrepancy in results among studies showing early versus later success on proportional reasoning problems seem to be explainable by a common thread. In particular, many studies that report later understanding of proportionality tend to present participants with proportions consisting of discrete sets, whereas many studies that report earlier understanding of proportionality tend to present participants with proportions consisting of continuous amounts (Mix, Huttenlocher & Levine, 2002). A few studies have directly examined the impact of discrete versus continuous quantities on proportional reasoning. Spinillo and Bryant (1999), for instance, found that 6-year-olds were more successful in solving a proportional matching task when the stimuli were continuous than when they were discrete (i.e., accuracy was higher when target proportions, which were represented as small round pies, were not "sliced" into discrete units than when they were). Similarly, Jeong, Levine, and Huttenlocher (2007) found that 6-, 8-, and 10-year-olds performed significantly above chance in selecting which of two spinner gambles involves a higher probability for success if the winning and losing proportions on each spinner were represented with continuous sections. None of these age groups, however, selected the more probable spinner if the winning and losing portions were broken into discrete units by demarcating lines. It is possible that proportions represented with continuous amounts are more likely to elicit correct intuitive processes than proportions represented with discrete sets. The question then becomes --what are the explicit processes elicited by discrete quantities that interfere with successful proportional reasoning?

Children's difficulty with discrete unit proportional reasoning problems may be due to over-extension of counting routines to judgments of proportionality (Mix, Levine, & Huttenlocher, 1999; Wynn, 1997). In support of this possibility, Jeong and colleagues (2007) found that even 10-year-olds have particular difficulty on proportional reasoning problems if counting the number of discrete target units produced an outcome inconsistent with the relative proportion of target and non-target units (e.g., they judged 6/10 as more probable than 4/6). Thus, counting and the overextension of numerical counts to proportional problems may impede proportional reasoning. It is also possible that the presence of discrete units interferes with forming a representation of the relative proportion of numerator to denominator amounts, even if children do not try to solve the problem by counting the numerator and denominator units.

In the current studies, participants were given a task that involved selecting a proportion that matched a target juice mixture. Similar to previous studies (Fujimura, 2001; Noelting, 1980; Schwartz and Moore, 1998), proportionality was determined by the relative quantities of juice and water parts. The present task, however, focuses on proportional equivalence between a target and a choice alternative, rather than concentration ordinality, which may serve to reduce task difficulty (Cooper, 1984; Frydman & Bryant, 1988; Mix et al., 2002).

Participants were randomly assigned to one of four continuity conditions, which involved variation in whether the target and choice alternatives were represented with continuous water and juice amounts or with discrete water and juice units. In one condition, both the target proportion and choice alternative proportions were represented with discrete units (hereafter referred to as DD); in another condition, both the target and choice alternatives were represented with continuous amounts (CC); for a third group the target was represented with

discrete units and the choice alternatives with continuous amounts (DC); finally, for a fourth group the target was represented with continuous amounts and the choice alternatives were represented with discrete units (CD). Based on previous findings, we predicted that children would perform relatively well on the continuous target-continuous choice alternatives (CC) problems, and relatively poorly on the discrete-discrete (DD) problems. Children's performance on the mixed conditions, which involved continuous target proportions and discrete choice alternatives (CD), or the reverse (DC), were included to provide information about the nature of children's difficulty. The hypothesis that children's tendency to make numerical matches interferes with their proportional reasoning would gain support if their difficulty is isolated to the DD condition, where both target and choice proportions are represented with discrete units (i.e., $DD < DC = CD = CC$), because the DD condition is the only one in which absolute numerical matches are possible. Alternatively, if the presence of discrete units causes difficulty with forging a proportional representation, we might expect that children will perform poorly anytime discrete units are present in the problem, and the CC condition will be the only condition on which they perform well ($DD = DC = CD < CC$). It is also possible that there will be an incremental effect, such that participants given discrete targets with continuous choice alternatives or the reverse will succeed at a rate that lies between those in the DD and CC conditions (i.e., $DD < DC = CD < CC$). This might be the case because participants given either discrete targets *or* discrete choice alternatives (i.e., DC and CD) are presented with fewer total countable units than those given discrete targets *and* discrete choice alternatives (i.e., DD).

Another major conclusion of previous studies that have examined the development of proportional reasoning is that young children are not able to go beyond the *parts* of the proportion to represent *part-whole* relations (Inhelder & Piaget, 1958; Singer & Resnick, 1992). To examine this issue, we manipulated foil-type to analyze the role of *parts* and *wholes* in children's proportional reasoning. Across trials, the correct choice alternative was a proportional match for the target mixture [i.e., the juice/(juice + water) proportion is preserved]. For half of the trials, the incorrect foil alternative matched the target's absolute juice portion (i.e., a juice *part* foil-type), and for the other half of trials, the foil matched the target's total amount (i.e., a juice + water *whole* foil-type). This was the case in each continuity condition. If younger participants do indeed focus on the *parts* that compose proportions, we might expect an effect of foil-type, such that younger children perform more poorly on trials on which the foil matches the target's "juice" portion (i.e., they are drawn towards forming a *part* match) than on trials for which the foil matches the target's whole juice + water amount.

STUDY 1

Methods

Participants—Participants were 240 students recruited from seven Chicago Public Schools, with 48 participants from kindergarten, first-, second-, third-, and fourth-grades, and approximately the same number of girls and boys tested at each grade level ($M_K = 6$ -years-1-month, 27 girls, 21 boys; $M_1 = 7;1$, 23 girls, 25 boys; $M_2 = 8;0$, 20 girls, 28 boys; $M_3 = 8;10$, 23 girls, 25 boys; $M_4 = 9;10$, 28 girls, 20 boys; $SD = 4$ -months, in all grades). There were 12 participants per grade per continuity condition. All children had written parental consent to participate. We did not expressly collect information on each participating child's ethnicity or socio-economic status (SES), and therefore precise data for these factors are not available. There was, however, substantial diversity within and across the schools sampled, which we can examine at the school level. Based on the statistics reported for each school and the number of children tested at each, we can estimate that approximately 26% of participants were Hispanic, 26% were Asian, 17% were Black, and 31% were White. Using the percentage of students at each school who are eligible for the free or reduced cost lunch program as a metric

of SES, we estimate that about 66% of the children in our sample come from low SES backgrounds ($sd = 27\%$, range = 26% to 96%)². All participants were fluent English speakers.

Procedure—Participating children were individually administered an engaging proportional reasoning task on an IBM T20 laptop computer with a 14.1” screen. Testing was carried out during regular school hours, in familiar rooms adjacent to participants’ classrooms. During task instructions, a picture of a teddy bear appeared on the screen and children were told that his name is “Wally-bear.” The experimenter explained that Wally-bear enjoys drinking all kinds of juice – red, blue, green, yellow, and purple juice – and likes to mix his juice himself. The experimenter then showed participants an example that stressed the importance of maintaining a recipe’s proportion when transforming total amount (See Appendix for the instruction script).

During each trial, a small photo of the character appeared on the upper-left side of the screen, and a mixture of juice + water (target proportion) was shown just below the photo. Two potential matches for the target proportion appeared on the right two-thirds of the screen, one of which was a correct proportional match, and the other of which was a *part* foil or a *whole* foil (as described above). The bottoms of the choice alternatives were vertically aligned with each other, but were not aligned with the target proportion (see Figure 1 for example screenshots of the experiment). With the target and choice proportions on the screen, the experimenter asked, “Which of these two (pointing to the two alternatives) is the right mix for the juice Wally-bear is trying to make? Which of these two would taste like Wally-bear’s juice?” Using the computer’s mouse, the participant registered a selection by clicking a button that appeared below each of the choice alternatives. After the child selected, another target proportion and another two choice alternatives appeared. The “juice” color on each successive trial was different from that on the previous trial. Sixteen self-paced trials were administered in this manner, in one of four predetermined orders (i.e., two pseudo-random orders and their reverses were used). No performance feedback was provided on any trial.

Experimental Design—Participants were randomly assigned to one of four continuity conditions (CC, CD, DC, and DD, as described above). The *sole* difference between the continuity conditions was in how the juice and water parts were represented. In the discrete conditions there were lines that demarcated each 1-cm² unit, and in the continuous conditions the juice and water portions formed unitary columns, with visible divisions only occurring at the point where the juice and water parts met (Figure 1 gives example screenshots of each continuity condition). As mentioned above, for each continuity condition, foil-type was varied so as to match the target’s juice *part* on half of the trials and the target’s *whole* extent (e.g., total juice + water quantity) on the other half of the trials (the incorrect alternative in each frame of Figure 1 is a juice *part* foil).

Several other factors were controlled. The target to match scaling direction was balanced across *part* and *whole* foil trials. That is, there were equal numbers of problems that involved scaling up from a target proportion with a smaller numerator/denominator to a proportional match with a larger numerator/denominator (e.g., 2/3 to 6/9), and scaling down from a larger numerator/denominator to a smaller numerator/denominator (e.g., 6/9 to 2/3). In addition, foils were chosen such that the proportional disparity between the target and foil were approximately equivalent across problems (see Table 1). Previous studies have shown that “half” is an especially salient proportion (Jeong, 2005; Spinillo & Bryant, 1991, 1999), so neither the target nor either of the choice alternatives were ever 1/2. Furthermore, all foils were on the opposite side of the half-boundary from the target (i.e., if the target proportion was less than 1/2, then the

²Information regarding the National School Lunch Program, including income eligibility requirements, is available through the United States Department of Agriculture, Food and Nutrition Service (<http://www.fns.usda.gov/cnd/Lunch/>).

foil proportion was greater than $\frac{1}{2}$, or vice versa), which should make the problems easier than had we included foils on the same side of the half boundary as the target proportion. Other factors were varied randomly, as determined by the computer program (i.e., the “juice” color assigned to a particular trial and whether the correct choice appeared on the right or left).

Results

Collapsing across all variables, participants selected the proportional match for the target on 63.2% of all trials. Table 2 summarizes the mean proportion of trials, per school-grade, continuity condition, and foil-type on which participants selected the proportional match. The primary analysis was a $4 \times 5 \times 2 \times 2$ mixed model Analysis of Variance (ANOVA). Continuity condition (CC, CD, DC, or DD), School grade (kindergarten, first-grade, second-grade, third-grade, or fourth-grade), and Sex were between-subjects factors, and Foil-type (*part* foil or *whole* foil) was a within-subjects variable. The dependent measure was the number of times participants selected the proportional match (out of eight trials for each foil-type).

The ANOVA revealed a significant main effect of Continuity condition, $F(3, 200) = 12.88$, $p < .0001$, $\eta^2_p = .16$. Planned pairwise comparisons revealed that participants in the DD condition were significantly less likely to select the proportional match than participants in all other groups (all $ps < .001$), and that there were no significant differences between the performance levels of participants in the other three continuity conditions (all $ps \geq .18$)¹. Follow-up comparisons against chance revealed that performance levels of participants in the CC, CD, and DC conditions all exceeded chance (all $ps \leq .001$), but the performance level of participants in the DD condition did not ($p = .68$).

The main effect of school grade was also significant, $F(4, 200) = 14.49$, $p < .001$, $\eta^2_p = .23$. Planned comparisons showed that fourth-graders were more likely to select the proportional match than all other groups (all $ps \leq .001$). Third-graders and second-graders did not significantly differ from each other ($p = .58$), but both were more likely to select the proportional match than first-graders and kindergartners (all $ps \leq .01$), who in turn did not significantly differ from each other ($p = .74$). On average second-, third-, and fourth-graders selected the proportional match more frequently than would be expected by chance (all $t \geq 4.30$, all $p \leq .001$), but kindergartners and first-graders did not significantly differ from chance performance (both $t \leq 1.00$, both $p \geq .32$). Across grades performance was poorest in the DD condition, and the interaction between school grade and continuity condition was not significant, $F(12, 200) = .51$, $p = .91$, $\eta^2_p = .03$. The main effect of sex was non-significant, $F(1, 200) = .01$, $p = .91$, $\eta^2_p < .01$, and no interactions involving this variable reached statistical significance (all $p \geq .08$).

Finally, the ANOVA revealed a significant main effect of foil-type, $F(1, 200) = 43.42$, $p < .001$, $\eta^2_p = .18$, with participants selecting the proportional match more often when the foil was a *whole* match (juice + water total) than when the foil matched the target’s juice *part* (68.2% vs. 57.4% of trials, respectively). As illustrated in Figure 2, however, foil-type interacted with school grade, $F(4, 200) = 4.24$, $p = .003$, $\eta^2_p = .08$, such that the difference between *whole* and *part* foils decreased with development. Figure 2 also shows that selection of the proportional match when the foil matched the target’s juice *part* did not exceed chance until third-grade whereas it exceeded chance by kindergarten when the foil matched the target’s total juice + water amount (with a Bonferroni control for multiple comparisons).

Individual analyses were conducted to examine whether participants tended to consistently select the proportional match or the foil alternative, and if so, whether this shifted over

¹All pairwise comparisons reported throughout the text were conducted with one-tailed tests.

development. The binomial distribution indicates that in order to exceed chance ($\alpha < .05$, two-tailed) a participant must select a particular choice alternative on at least 13 of the 16 trials. As can be seen in Table 3, the results of the individual analysis largely mirrored the group findings; generally, there was an increase in the number of participants who consistently selected the proportional match with age, and more participants in the CC, CD, and DC conditions (62 children, 34.4%) consistently selected the proportional match across grades than in the DD condition (8 children, 13%). Although relatively few children selected the foil alternative more frequently than chance, more participants in the DD condition (7 children, 12%) did so than in the three other conditions (1 child, 1%).

The amount of time that participants took to complete each trial was also analyzed. On average, participants responded in 8.8-seconds per trial ($SD = 5.2$). Importantly, the time taken to complete the task was negatively correlated with accuracy, $r = -.214$, $p = .001$, indicating that there was not a speed-accuracy trade-off; rather, those who responded more quickly also tended to respond more accurately. A $4 \times 5 \times 2 \times 2$ ANOVA carried out to analyze the effects of condition, school grade, sex, and foil-type on median response time revealed a main effect for continuity condition, $F(3, 200) = 5.08$, $p = .002$, $\eta^2_p = .07$, with participants in the DD condition taking longer to respond than those in all other conditions (all $ps \leq .02$, Sidak adjusted), which did not significantly differ from each other. The analysis also revealed a main effect of grade, $F(4, 200) = 2.97$, $p = .02$, $\eta^2_p = .06$, with second-graders responding faster than kindergartners and first-graders ($p = .03$ and $p = .05$, respectively, Sidak adjusted), but with no other significant pairwise differences (all $p \geq .17$). These findings largely mirror those found in our analyses on accuracy –participants in the DD condition and the youngest participants were slower to respond as well as less accurate. Neither the main effect of sex or foil-type was significant, $F(1, 200) = .03$, $p = .87$, $\eta^2_p < .01$ and $F(1, 200) = 3.04$, $p = .08$, $\eta^2_p = .02$, respectively. The ANOVA, however, revealed an unexpected foil-type \times school grade interaction, $F(4, 200) = 2.96$, $p = .02$, $\eta^2_p = .06$, reflecting faster responding of first-graders when the foil matched the target's juice *part* than when it matched the target's juice + water *whole* ($p = .014$, Sidak adjusted). In addition, an unexpected three-way foil-type \times school grade \times sex interaction emerged, $F(4, 200) = 2.42$, $p = .05$, $\eta^2_p = .05$. Although difficult to interpret, pairwise comparisons indicate that this was due to kindergarten boys, first-grade girls, and third-grade girls responding faster on part foil trials than on whole foil trials (all $p \leq .02$, Sidak adjusted).

Discussion

The current study shows that children have difficulty solving proportional reasoning problems when both the target and the choice proportions are represented with discrete units, but perform significantly better on problems for which the target, the choice alternatives, or both are represented with continuous amounts. That is, performance of participants in the DC and CD conditions did not differ from those in the CC condition, and each of these groups performed above chance and better than those in the DD condition. This pattern of results suggests that proportional reasoning is not limited by the mere presence of discrete, countable entities. Rather, the findings suggest that children's difficulty with proportional reasoning problems involving discrete quantities may stem from an overextension of absolute numerical equivalence strategies to problems that should be solved on the basis of proportional equivalence. Absolute numerical equivalence matching (i.e., matching the target with the foil alternative, which has the same number of units as the target juice part or the target juice + water whole) was only possible when both the target and choice alternatives were represented with discrete units, as in our DD condition (i.e., numerical matching was not possible in the conditions where the target, the choice alternatives, or both were continuous). The difficulty young children have in making proportional matches on DD problems is particularly striking in view of the fact that all foils were on the opposite side of the half-boundary than the target

proportion, in that using an approximate representation of proportion (i.e., more than half, less than half) rather than matching target and foil counts would have led to the correct answer.

As predicted, performance also suffered when the foil matched the target's juice *part*, relative to when the foil matched the target's juice + water *whole*, suggesting that a common absolute juice *part* between the target and foil is a particularly salient similarity feature. As evidenced by an interaction between foil-type and school grade, this was especially the case for younger children. Kindergarten and first-grade students actually performed below chance in choosing the proportional match when the foil matched the target's juice part (See Figure 2). As summarized in Table 2, kindergartners and first-graders in the DD continuity condition selected the proportional match on only 33% and 23% of the trials when the foil alternative matched the target's juice part, compared to 48% and 55% of trials when the foil alternative matched the target's whole (juice + water). This is generally consistent with the previous finding that younger children tend to focus on the *parts* of the problem, rather than analyzing *part-whole* relations (Inhelder & Piaget, 1958; Singer and Resnick, 1992).

These performance data are consistent with our proposal that young children would succeed on problems that they solve through intuitive processes and have greater difficulties with problems that invoke more explicit processes. The response time data are consistent with this intuitive/explicit distinction, in that faster responses were associated with higher performance levels than slower responses (Sloman, 1996; Stanovich, 2004). One potential limitation of this study, however, concerns the simultaneous presentation of target and foil. This may have enabled participants to revise their initial encoding of the target quantity in the DC and CD conditions, because they could go back and forth between the target and the choice alternatives. For example, a participant in the DC condition may have initially represented a $\frac{3}{4}$ discrete target numerically (e.g., as "3 juice parts" or "4 total"), rather than representing it proportionally (e.g., "about $\frac{3}{4}$," "more than half juice," or "3 parts juice to 1 part water"). Upon seeing the choice alternatives represented in a continuous format, however, the child might have reanalyzed the target proportion and re-encoded it in a proportional manner. Thus, the fact that children performed as well in the DC and CD conditions as in the CC condition may have stemmed from the simultaneous availability of target and foil quantities. In order to determine whether this is the case, and to better understand at what point in the problem solving process discrete units disrupt performance, Study 2 adopts the design of Study 1, but modifies the procedure so that the target and choice alternatives are presented sequentially.

STUDY 2

The present study divides the task into encoding and comparison phases with sequential presentation of the target and choice alternatives. The crossing of continuous and discrete targets and choice alternatives with this procedure should provide more definitive information as to whether difficulties with discrete proportions arise at encoding, at comparison, or during both phases. If difficulties arise during encoding, then we should see low and similar performance in the DD and DC conditions, the conditions in which the target proportion consists of discrete units, irrespective of how the choice alternatives are represented. Alternatively, if difficulties arise at the comparison phase, through overextension and matching of the target and foil quantities, then we should see poor performance only in the DD condition, because absolute numeric matches are only possible in this condition. This would replicate the results of Study 1, but would more precisely pinpoint the locus of the difficulty to quantitative comparison because the sequential presentation precludes going back and forth between the target and the choice alternatives. Finally, if difficulties with discrete units arise during encoding *and* comparison, then performance in the DD, DC, and CD conditions should all be lower than performance in the CC condition. In this event, discrete units may disrupt proportional reasoning for one or both of the following reasons; first, children may be unable

to keep from applying counting algorithms in the presence of countable entities and this may decrease their ability to process proportional information; second, the presence of discrete units may decrease the child's ability to perceptually abstract proportional information by breaking up the gestalts of the colored regions.

Methods

Participants—Participants were 144 first- and third-graders recruited from three of the same sites that participated in Study 1 ($M_1 = 7\text{-years-1-month}$, 36 girls, 36 boys; $M_3 = 9;0$, 34 girls, 38 boys; $SD = 3.5\text{-months}$ in each grade). There were 18 participants per grade and continuity condition. Participants recruited for the current study had not participated in the first study. As in the previous study, we did not collect any ethnicity or SES information for each individual child. Again using reported school data and the number of participants recruited from each school, we estimate that 28% of participants were Hispanic, 18% were Asian, 11% were Black, and 43% were White, and that about 46% ($sd = 29\%$) were from low SES backgrounds (based on the percentage of students in each school who qualify for free or reduced-cost lunch).

Procedure—The instructions given to participants prior to the task were identical to those given to participants in Study 1. The procedure was also largely consistent with that of Study 1; however, the targets and choice alternatives in each problem appeared sequentially. On each trial, the participant was first shown the target proportion. This was followed by a solid dark mask that appeared briefly (≈ 100 ms) and then a 5-second blank screen interval. Immediately after this, the two choice alternatives were presented. Participants themselves controlled how long they saw the target proportion; that is, while the target proportion was presented, the participant could click the mouse anywhere on the screen to make it disappear and initiate the mask and the 5-second ISI (with a ceiling of 20-seconds). Problems were presented in random order, as determined by the computer program.

Experimental Design—The experimental design was identical to that of Study 1. Participants were again randomly assigned to one of four continuity conditions (again referred to as CC, CD, DC, and DD). As in Study 1, in each continuity condition, the foil matched the target's juice *part* on half the trials and the target's *whole* (e.g., total juice + water quantity) on the other half.

Results

Collapsing across all variables, participants selected the proportional match for the target on 63.7% of all trials. Table 4 summarizes the data for each continuity condition, school grade, and foil-type. The primary analysis was a $4 \times 2 \times 2 \times 2$ mixed model ANOVA. Continuity Condition (CC, CD, DC, or DD), School grade (first-grade or third-grade), and Sex were between-subjects variables, and Foil-type (*part* or *whole* foil) was a within subjects variable.

The analysis revealed a significant main effect of continuity condition, $F(3, 128) = 3.49, p = .02, \eta^2_p = .08$. Planned pairwise comparisons revealed that participants in the DD continuity condition were significantly less likely to select the proportional match than those in the CC, CD, and DC conditions (all $t_s \geq 1.70, p_s < .05$). There were no significant differences in performance between the CC, CD, and DC conditions (all $p_s \geq .41$). Comparisons against chance revealed that participants in the CC, CD, and DC conditions selected the proportional match at a rate that exceeded chance (all $t_s \geq 3.71, p_s \leq .001$), whereas those in the DD condition did not ($t = 1.26, p = .22$). The analysis also revealed a main effect of school grade, $F(1, 128) = 21.84, p \leq .001, \eta^2_p = .15$, with third-graders selecting the proportional match more frequently than first-graders (71.3% vs. 56.1% of trials). On average, both first-graders and third-graders selected the proportional match more frequently than expected by chance, $t(71) = 2.81, p = .006$ and $t(71) = 7.49, p < .001$, respectively. The main effect of sex was not

significant, $F(1, 128) = 1.60, p = .21, \eta^2_p = .01$; however, there was an unexpected statistically significant sex \times continuity condition interaction, $F(3, 128) = 5.11, p = .002, \eta^2_p = .11$, which was due to girls, but not boys, performing better in the CC, CD, and DC conditions than the DD condition (all $t_s \geq 3.22, p_s \leq .002$, Sidak adjusted). Girls selected the proportional match on 75%, 72%, 71%, and 46% of trials for the CC, CD, DC, and DD conditions, respectively, and boys selected the proportional match on 60%, 65%, 58%, and 64% of trials for the CC, CD, DC, and DD conditions, respectively (this also resulted in a significant difference between boys and girls in the CC and DD conditions, both $p_s \leq .035$, Sidak adjusted). Finally, the ANOVA revealed a main effect of foil-type, $F(1, 128) = 4.12, p = .04, \eta^2_p = .03$, with better performance when the foil alternative matched the target's juice + water *whole* than when it matched the target's colored juice *part* (65.9% vs. 61.4%).

As in Study 1, individual analyses mirrored the findings in the group analyses. The number of participants who consistently selected the proportional match increased with age (13% of 1st-graders and 50% of 3rd-graders exceeded chance), and more participants in the CC, CD, and DC conditions (39%, 33%, and 33%, respectively) consistently selected the proportional match than in the DD condition (19%). Conversely, 3 of 36 participants (8%) in the DD condition consistently selected the absolute foil alternative, whereas only 1 of 108 participants (1%) in the other three conditions combined did so.

On average, participants took 9.1-seconds to encode the target proportion ($SD = 5.6$), and took 6.9-seconds to choose once the choice alternatives appeared ($SD = 5.6$). As in Study 1, accuracy and timing were negatively correlated, which was true both for encoding time and accuracy, $r = -.21, p .012$, and for choice time and accuracy, $r = -.16, p .064$. A $4 \times 2 \times 2 \times 2 \times 2$ mixed model ANOVA was conducted to analyze the effects of continuity condition (CC, CD, DC, and DD), school grade (1st- and 3rd-grades), sex, foil-type (part foils and whole foils), and process (encoding and choice), with median times as the dependent variable. There was a main effect of continuity condition, $F(3, 128) = 3.75, p = .01, \eta^2_p = .08$. Participants in the CC condition ($M = 5.8$) were significantly faster than those in the DD condition ($M = 7.8, p = .007$, Sidak adjusted), but neither significantly differed from those in the CD and DC conditions ($M_{CD} = 6.9$ and $M_{DC} = 6.8$). As in Study 1, these results show problem solving times that are generally consistent with performance, rather than reflecting a speed-accuracy trade-off. Although not of central theoretical interest, there was also a main effect of process, $F(1, 128) = 80.74, p < .001, \eta^2_p = .39$, with participants taking significantly longer to encode the target proportion than to select a choice alternative. The main effects for foil-type, school grade, and sex were not significant ($F \leq 1.20, p \geq .28$). Finally, there was a significant three-way process \times condition \times school grade interaction, $F(3, 128) = 3.57, p = .02, \eta^2_p = .08$. Pairwise comparisons suggest this was due to first-graders in the CC condition encoding the target proportion faster than first-graders in the DD and DC conditions ($p_s \leq .04$, Sidak adjusted), with no other significant pairwise differences.

Discussion

Consistent with Study 1, the results of Study 2 revealed that as early as first-grade (i.e., at about 6- or 7-years of age) children can successfully solve proportional equivalence problems when the target proportion, the choice alternative proportions, or both are represented with continuous quantities. They failed to solve otherwise parallel problems, however, when the target and choice proportions were *both* represented with discrete quantities. The finding that performance levels in the DC and CD conditions were significantly better than in the DD condition and not significantly different from performance in the CC condition indicates that the mere presence of discrete units does not disrupt proportional reasoning. The combination of above chance performance in the DC condition and chance performance in the DD condition is particularly telling. This pattern suggests that children can correctly encode the target's

proportion when a quantity is made up of discrete units (as indicated by performance in the DC condition), but that when numerical unit information is available at both target encoding and choice alternative comparison (as in the DD condition), the possibility of a numerical match, particularly a match to the highly salient juice parts, detracts from making a proportional match.

The significant interaction between continuity condition and sex was unexpected. Consistent with the performance patterns of Study 1, girls performed more poorly in the DD condition than in the other three conditions, but in contrast to the findings in Study 1, boys' performance did not differ significantly across conditions. Furthermore, girls exhibited stronger performance than boys in the CC condition, and boys exhibited stronger performance than girls in the DD condition. This crossover interaction suggests that sequential presentation may decrease the propensity of boys to make a numerical match in the DD condition more than it does for girls, and raises the possibility that there may be a sex-based divergence between intuitive and more explicit proportional reasoning capacities. However, because the sex \times condition interaction was unique to Study 2 (this effect did not approach significance in Study 1), it must be interpreted with caution.

General Discussion

The current studies extend our understanding of the development of proportional reasoning, and illuminate some of the reasons why children have difficulty solving proportional equivalence problems involving discrete quantities compared to problems involving continuous amounts (Jeong et al., 2007; Sophian, 2000; Spinillo & Bryant, 1999). Specifically, our continuity condition manipulation revealed that children are able to solve proportional equivalence problems when the target, the choice alternatives, or both are represented by continuous amounts several years before they are able to solve parallel problems in which both the target and the choice proportions are represented by discrete units. This pattern of findings suggests that children's difficulties stem, at least partly, from their propensity to compare quantities on the basis of the number of elements in the target quantity rather than on the basis of proportional relations. In Studies 1 and 2, performance in the DD condition, which is the only condition where absolute numeric matches were possible, was significantly worse than in the other conditions, and did not rise above chance level. That participants in both studies had more success in solving proportional problems when the target was represented with discrete units and the choice alternatives were represented with continuous amounts (the DC condition) or vice versa (the CD condition), suggests that they are not simply misrepresenting proportional information whenever discrete, countable units are involved. Rather, the finding that performance was at chance only in the discrete-target, discrete-choice alternatives condition indicates that children's difficulties reflects a tendency to match the number of units in the target and choice alternatives when this is possible.

In both studies there was also an effect of foil-type, such that performance was lower when the foil matched the target's colored juice part than when it matched its juice + water whole. This indicates that a common absolute *part* quantity between the target and foil is particularly attractive. This effect did not interact with continuity condition, which suggests that it is robust, and occurs even when continuous quantities are involved and children are representing proportional information. As mentioned, this finding is consistent with previous results showing that young children have a tendency to focus on the *parts* of a proportion (Inhelder & Piaget, 1958; Singer & Resnick, 1992). This does not necessarily mean that children are unable to code part-whole relationships, but rather, that parts may be more salient to them than wholes. It should be noted that the types of foils used in the current study were not exhaustive. For example, the foils could have involved a match to the target proportion's water *part*. We reasoned that the colored juice part of each proportion, which varied from trial to trial, was

more perceptually salient than the water portion of each proportion, which remained constant in color, and therefore the juice part was more likely to provide participants with an attractive, albeit erroneous, choice alternative. We expect that equivalent target and foil alternative water parts may have drawn some participants to the foil, because such a foil would provide another sort of numerical match, but this effect may have been less dramatic than we see for juice part foils. Further, our current findings do not provide information as to whether the potency of the numerical foil would vary depending on whether it was shown in the lower or upper portion of the stimulus column, or whether the water and juice parts were randomly interspersed (e.g., Jeong et al., 2007), as the juice part was always shown cohesively in the lower part of each proportion column.

Also of note is our use of a rather minimalistic approach to discretizing the presented elements; that is, although demarcated in a way that makes them countable, our discrete representational format does not actually involve independent entities, with inter-item distances. In the present studies, our intent was to make the continuous and discrete representational formats as comparable as possible, and therefore, we elected to use the described, perhaps less dramatic, demarcated columns for discrete trials. Basically, we conceptualized our continuous stimuli as existing in an undifferentiated cylinder, and our discrete stimuli as existing in something along the lines of a graduated cylinder. It is possible that independent discrete units, which participants might be even more likely to explicitly count, would heighten the effects we obtained with demarcated columns, and would lead to even worse proportional reasoning performance.

Our results show that children are able to reason about proportionality at earlier ages than Piaget and Inhelder's (1951/1975; Inhelder & Piaget, 1958) classic studies indicated. That is, if the problem involves at least one proportion consisting of continuous amounts, which effectively prevents children from making matches on the basis of number, they demonstrate an ability to reason about proportional equivalence earlier – by about first-grade (i.e., between 6- and 8-years of age). At the same time, however, our results are consistent with Piaget and Inhelder's findings that children have marked difficulty with problems represented entirely with discrete units for which there is the possibility of a numerical match to the target. That is, participants in the DD condition were at chance levels of performance, and in fact tended to select the absolute numeric match slightly more often than the proportional match until fourth-grade (i.e., about 10-years of age). If continuous amounts are conceptualized as eliciting intuitive problem solving processes, and problems with discrete targets and choice alternatives are conceptualized as eliciting more explicit problem solving processes, our results can be viewed as consistent with the Piagetian perspective. That is, Piaget and Inhelder (1975/1951) acknowledged the possibility of intuitive problem solving by young children, but did not consider this to reflect true understanding of probability and proportionality. Thus, differing views about the developmental course of proportional understanding may hinge on what constitutes evidence of understanding. Current views, which characterize the development of mathematical understanding as moving from partial to more complete, and from more to less contextually dependent, rather than as nonexistent until a point in development until a certain kind of problem is solved correctly, are helpful in making sense of seemingly disparate findings (e.g., Mix, 2002; Mix, Huttenlocher & Levine, 2002).

Related to the issue of graded and contextually dependent development, the individual analyses we conducted revealed that children tend to use a variety of strategies to solve the problems. Relatively few participants, particularly younger participants, selected either option (i.e., the proportional match or the absolute numerical foil) at a rate that exceeded chance, which suggests that most children were not consistently using a specific strategy across problems. As suggested by Siegler (1996; 2005), in the context of studying children's solutions to numerical calculations, this variability may be an important engine for learning and development. A

microgenetic study that probes children's explanations for their solutions to proportional reasoning problems could provide more information about their strategies, and thus shed light on whether variable solution strategies are related to learning trajectories for this kind of problem. There may, however, be an inherent difficulty in doing this; specifically, intuitive reasoning processes are by definition less open to scrutiny than more explicit processes, and although children may be able to verbalize the explicit processes they engage in (e.g., counting the discrete units – anecdotally, a strategy used by numerous participants in the current study), they may have difficulty verbally explaining the intuitive strategies they use (Siegler, 2000). Furthermore, requesting participants to verbalize how they solved a given problem may actually affect the sort of processes they use to solve that problem or subsequent problems.

Another contribution of the present studies to the cognitive developmental literature is the identification of a perceptual manipulation – continuous vs. discrete representation – that drives engagement of intuitive and more explicit problem solving processes. Others have noted the role of perceptual manipulations on children's numerical processing skills. For example, Mix (1999; 2008) demonstrated that 3- to 4-year-olds are more likely to select a choice alternative that is numerically equivalent to a target if the target and choice arrays have similar surface features. Mix (2008) suggests that this occurs through alignment processes, and that highly similar item sets invite more thorough comparison processes than less similar item sets, resulting in more frequent numerical matching. This raises the possibility that young participants in our study may have been particularly drawn to making numerical target-foil matches in the DD condition by the surface feature similarity between the two (e.g., 3 yellow target units could be matched with 3 yellow foil units). It is possible that if there were variation between the color of the target and choice alternatives' juice parts the incorrect numerical match would have been less attractive and proportional matching would have been even better.

Our finding of early intuitive proportional reasoning in the context of continuous amounts is also consistent with results showing sensitivity to proportional relations in young children and infants. For example, several studies show that very young children are able to encode the length of a target object relative to a comparison standard at a point in development when they are unable to encode the length of the target object in the absence of a comparison standard, a finding that implicates proportional reasoning strategies (e.g., Baillargeon, 1991; Duffy, Huttenlocher, & Levine, 2005; Huttenlocher, Duffy, & Levine, 2002). Further, McCrink and Wynn (2007) report that 6-month-old infants are sensitive to the ratio captured by arrays of two categories of units (i.e., relatively larger pacman figures and smaller pellet objects), as long as the ratio is sufficiently large (i.e., a 2:1 ratio). This finding may seem somewhat inconsistent with our results, as the units incorporated into each of McCrink and Wynn's ratios were discrete; however, the number of objects in the sets that were presented was quite high (i.e., ranging from 12 to 60 total units presented at once). In light of our findings, it is possible that infants' sensitivity to the ratios may actually be linked to their *inability* to exactly enumerate the set of objects shown, and the engagement of an approximate, analogue magnitude system (Feigenson, Dehaene, & Spelke, 2004). This raises the somewhat counter-intuitive hypothesis that given a task like the one used in the present studies, children younger than those tested here, who have not yet developed mature counting skills, might be able to make proportional matches irrespective of the stimulus format (i.e., continuous vs. discrete), perhaps out-performing older children who erroneously make numeric matches on discrete-discrete problems. Conversely, elementary school age children may perform better on proportional reasoning tasks if the number of units to be counted is increased to a point that eliminates counting and matching as a feasible problem solving strategy. In this sense, understanding of number, and the ease or ability to count sets of items, is the very thing that negatively impacts elementary school aged children's success in solving proportional equivalence problems. A similar argument is made by Thompson and Opfer (in press), who suggest that the beneficial development of a linear numerical representation is associated with

the cost of losing access to a fractional power function representation. In their studies, higher accuracy in a number line estimation task was associated with lower accuracy in a fraction line estimation task. These findings are similar to the present results, and suggest that an interesting future direction may be to more specifically analyze the relation between performance on the sort of proportional reasoning task used here and performance on problems assessing arithmetical skills.

The current results have implications for efforts to increase children's understanding of fractions and proportions (e.g., Ball, 1993; Empson, 1999; Pitkethly & Hunting, 1996; Sophian et al., 1997; Streefland, 1993). The main effect of school grade, coupled with a non-significant interaction between school grade and continuity condition, shows that there are significant improvements in proportional reasoning with age, even when conditions are favorable for representing proportional information (i.e., when at least one of the proportions in the problem is represented with continuous quantities and foils are on the opposite side of the half boundary). This may reflect the impact of school curricula, as fraction and rational number operations are increasingly emphasized at higher elementary grade levels. In the particular schools attended by children in our studies, fractions and rational number operations are introduced in 3rd-grade and are given increasing emphasis in 4th grade and in subsequent grade levels. Furthermore, the fact that only the third- and fourth-graders in Study 1 and the third-graders in Study 2 performed significantly above chance when the foil was a part match in the DD condition may reflect the knowledge they are gaining in school. It would be very interesting to further examine the effect of instruction on fraction problem solving on proportional reasoning processes, perhaps by tying this instructional effort to the intuitive proportional reasoning that young children in our studies demonstrate. This sort of instructional strategy could make use of children's success on proportional problems involving continuous amounts to scaffold their performance on proportional problems involving discrete sets. Drawing parallels between the two kinds of problems may increase the likelihood that children will apply correct intuitive processes, rather than erroneous counting strategies, to proportional problems involving discrete sets. Relating instruction to children's intuitive knowledge is an approach that is widely advocated by educational researchers (e.g., Fischbein, 1987), and by those specifically interested in improving children's proportional reasoning and fraction understanding (e.g., Ahl et al., 1992; Fuson & Abrahamson, 2005; Pitkethly & Hunting, 1996).

In conclusion, the present findings indicate that children's proportional reasoning abilities vary as a function of the structure of the representations they are given. When absolute numerical matches are not possible, even six- and seven-year-olds demonstrate proportional reasoning abilities. Conversely, when absolute numerical matches are possible, even eight- and nine-year-olds have difficulty reasoning proportionally. Our results suggest that young children go wrong in reasoning about proportions when the knowledge they have acquired about counting to compare set sizes gets in the way of their intuitive, relative visual comparison, proportional reasoning processes.

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Appendix Task instructions script

The participant is shown a photo of a teddy bear on the computer screen, and is told:

“His name is Wally-bear, and Wally-bear really, really likes drinking juice.”

The participant is prompted to click a button on the screen, and upon doing so, the character photo changes so that he appears to be holding a glass of orange juice:

“See, he has a glass of orange juice. Wally-bear loves orange juice, but he also loves all other kinds of juice – he loves red juice, blue juice, green juice, yellow juice, and purple juice – all kinds of juice. Wally-bear loves juice so much that sometimes he likes to mix his juice himself, but he needs to be really careful when he’s mixing his juice, so he has just enough juice and just enough water, so that it tastes just right! Let me show you.

The participant is again prompted to click the screen button, and a column depicting two (or three) orange “juice” units, and two (or three) light blue “water” units appeared below the character (always resulting in a one-to-one, juice to water ratio). Note, the program randomly determined whether the participant was shown two or three initial juice and water units. Also,

as in the task itself, in the conditions where the target was continuous, the initial column's juice and water parts are continuous, with visible division only occurring where the two meet.

“See, when he was mixing this glass of orange juice [pointing to the glass the character appears to be holding], Wally needed just the right amount of orange [gesturing to the orange portion of the column], and he needed just the right amount of water [gesturing to the light blue portion of the column], so that when he mixed them up [gesturing with a circular motion around the entire column], it would taste just right. Let me show you what would happen if Wally wanted more (less) juice.”

The participant represses the button, and a second column, also composed of juice and water parts appears. The second column is proportionally equivalent to the initial column, but is more (or less) in absolute amount:

“See, if Wally wanted more (less) juice, he would need to mix more (less) orange [pointing to the orange portion of the secondary column, and gesturing back and forth between the initial and secondary orange amounts], *and* he would need more (less) water [pointing to the light blue portion of the secondary column, and gesturing back and forth between the initial and secondary water amounts], so that when he mixes his juice [gesturing with a circular motion around the entire secondary column], it would still taste just right [gesturing with a circular motion around the entire initial column], just like it is supposed to [with a point to the glass of juice the character appears to be holding].”

Finally, the participant is prompted to click the button again, the juice mix columns disappear, and the experimenter says:

“Now I have a question for you. You see, Wally knows how to mix his juice, and he knows what the different kinds of juice are supposed to taste like, but sometimes he gets a little confused, and he doesn't know how much juice and how much water he needs to mix for his different kinds of juice. Do you think you could help him by picking out the right mixes for his different kinds of juice? Do you think you can pick out the right mixes so that Wally's juice tastes just right?”

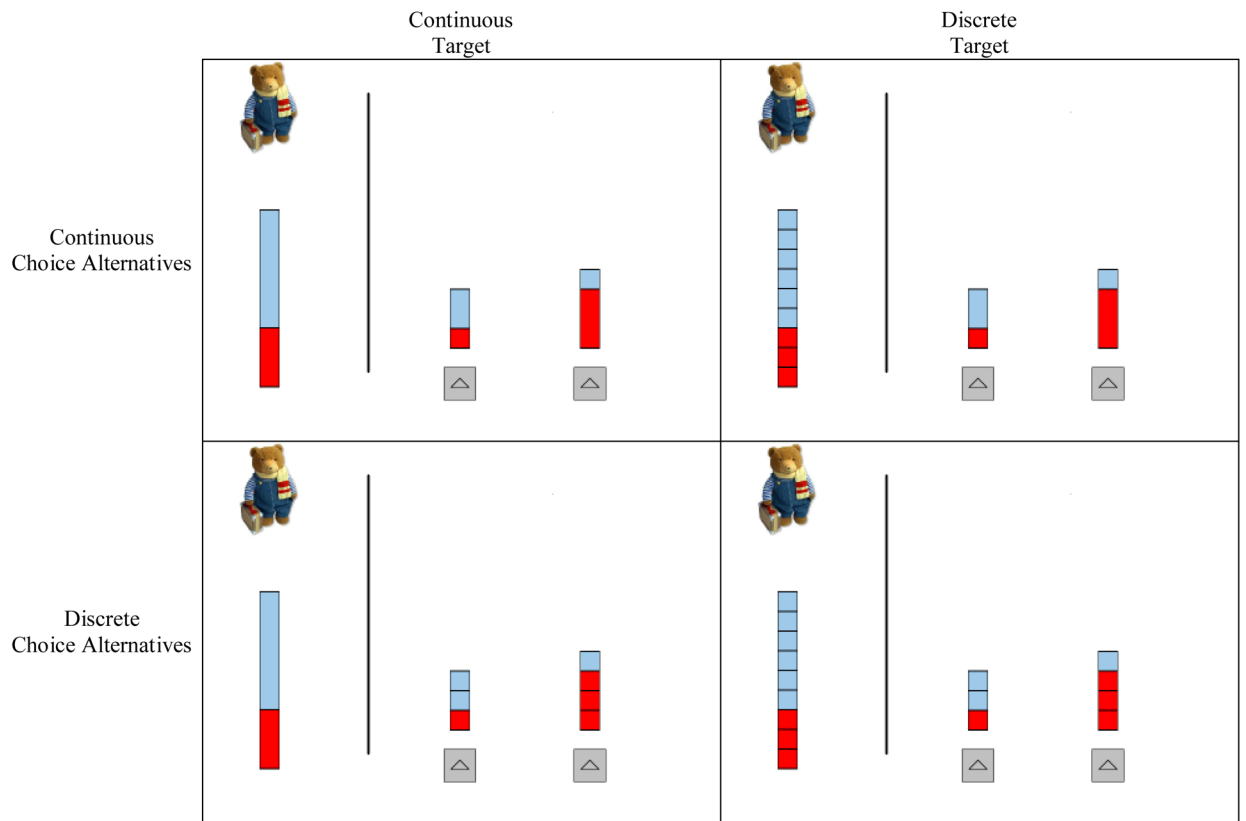


Figure 1. Example screenshots from each of the four between-subjects continuity conditions formed through 2×2 combination of continuous targets vs. discrete targets \times continuous choice alternatives vs. discrete choice alternatives.

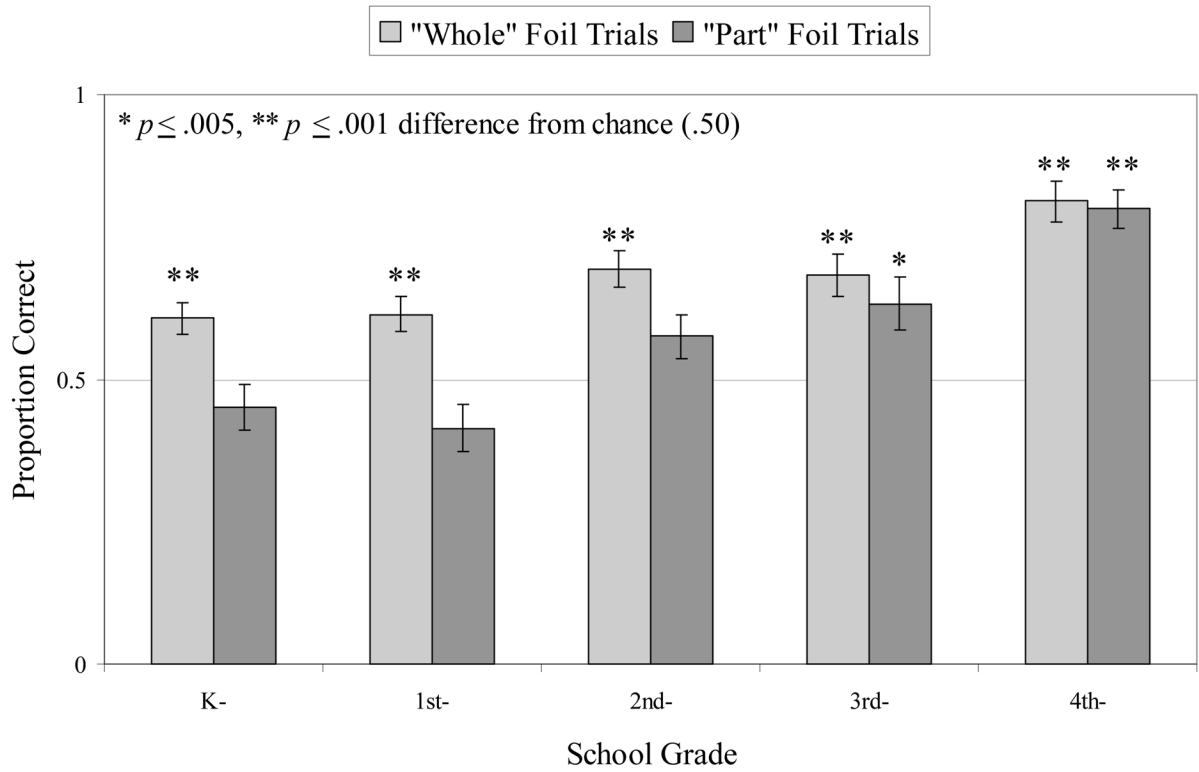


Figure 2.
Study 1: Mean proportion correct for school grade \times foil-type. Error bars represent the standard error of the mean.

Table 1

Proportions that were used across the 16 experimental trials.

Trial	Target	Match	Foil	Proportional Disparity
<i>Part</i> Foils	1	4/10	4/5	.40
	2	2/8	2/3	.42
	3	3/9	3/4	.42
	4	4/12	4/6	.33
	5	2/6	2/3	.33
	6	4/6	4/12	.33
	7	2/3	2/6	.33
	8	4/5	8/10	.40
<i>Whole</i> Foils	9	1/5	3/5	.40
	10	1/3	2/3	.33
	11	2/6	4/6	.33
	12	4/6	2/6	.33
	13	8/12	4/12	.33
	14	9/12	3/4	.42
	15	8/12	3/12	.42
	16	6/10	3/5	.40

Note: The fractions displayed represent juice units/(juice + water units). "Part" foils always matched the target proportion's juice units or amount, and "whole" foils always matched the target proportion's total (juice + water) units or amount. Proportional disparity = $|\text{target} - \text{foil}|$.

Table 2
 Study 1: Mean proportion of trials that participants selected the proportional match, per continuity condition, grade, and foil-type.

School Grade	Continuity Condition					Total
	CC	CD	DC	DD	DD	
Kindergarten						
Whole Foils	0.67	0.61	0.67	0.48	0.61	0.61
Part Foils	0.57	0.39	0.51	0.33	0.45	0.45
First-Grade						
Whole Foils	0.68	0.60	0.63	0.55	0.61	0.61
Part Foils	0.53	0.46	0.44	0.23	0.41	0.41
Second-Grade						
Whole Foils	0.69	0.66	0.81	0.61	0.69	0.69
Part Foils	0.57	0.59	0.71	0.43	0.58	0.58
Third-Grade						
Whole Foils	0.77	0.69	0.77	0.50	0.68	0.68
Part Foils	0.73	0.72	0.66	0.43	0.63	0.63
Fourth-Grade						
Whole Foils	0.84	0.91	0.84	0.66	0.81	0.81
Part Foils	0.84	0.81	0.89	0.66	0.80	0.80
Total	0.69	0.64	0.69	0.49	0.63	0.63

Table 3

Study 1: Number (proportion) of participants who chose the proportional match or the numeric match foil more frequently than chance, per grade and continuity condition.

Proportional Match	Grade	Continuity Condition					TOTAL
		CC	CD	DC	DD	TOTAL	
Proportional Match	Kindergarten	2 (.17)	1 (.08)	2 (.17)	-	5 (.10)	
	1 st _Grade	2 (.17)	1 (.08)	2 (.17)	1 (.08)	6 (.13)	
	2 nd _Grade	3 (.25)	3 (.25)	6 (.50)	1 (.08)	13 (.27)	
	3 rd _Grade	6 (.50)	6 (.50)	4 (.33)	1 (.08)	17 (.35)	
	4 th _Grade	8 (.67)	8 (.67)	8 (.67)	5 (.42)	29 (.60)	
TOTAL		21 (.35)	19 (.32)	22 (.37)	8 (.13)	70 (.29)	
Numeric Match	Kindergarten	-	-	-	3 (.25)	3 (.06)	
	1 st _Grade	-	1 (.08)	-	1 (.08)	2 (.04)	
	2 nd _Grade	-	-	-	-	0 (.00)	
	3 rd _Grade	-	-	-	2 (.17)	2 (.04)	
	4 th _Grade	-	-	-	1 (.08)	1 (.02)	
TOTAL		0 (.00)	1 (.02)	0 (.00)	7 (.12)	8 (.03)	

Note: The significant difference criterion was selecting either alternative on 13/16 trials ($\alpha < .05$, two-tailed).

Table 4
 Study 2: Mean proportion of trials that participants selected the proportional match, per continuity condition, grade, and foil-type.

School Grade	Continuity Condition					Total
	CC	CD	DC	DD		
First-Grade						
Whole Foils	0.63	0.68	0.53	0.46	0.58	
Part Foils	0.54	0.58	0.56	0.51	0.55	
Third-Grade						
Whole Foils	0.81	0.80	0.72	0.64	0.74	
Part Foils	0.72	0.67	0.75	0.58	0.68	
Total	0.68	0.68	0.64	0.55	0.64	