Proc. Natl. Acad. Sci. USA
Vol. 93, pp. 15004–15008, December 1996
Mathematics

New infinite families of exact sums of squares formulas, Jacobi elliptic functions, and Ramanujan's tau function

(Jacobi continued fractions/Hankel or Turánian determinants/Fourier series/Lambert series/Schur functions)

STEPHEN C. MILNE

Department of Mathematics, Ohio State University, 231 West 18th Avenue, Columbus, OH 43210

Communicated by Walter Feit, Yale University, New Haven, CT, October 22, 1996 (received for review June 24, 1996)

ABSTRACT **In this paper, we give two infinite families of explicit exact formulas that generalize Jacobi's (1829) 4 and 8 squares identities to $4n^2$ or $4n(n + 1)$ squares, respectively, without using cusp forms. Our 24 squares identity leads to a different formula for Ramanujan's tau function $\tau(n)$, when n is odd. These results arise in the setting of Jacobi elliptic functions, Jacobi continued fractions, Hankel or Turánian determinants, Fourier series, Lambert series, inclusion/exclusion, Laplace expansion formula for determinants, and Schur functions. We have also obtained many additional infinite families of identities in this same setting that are analogous to the η-function identities in appendix I of Macdonald's work [Macdonald, I. G. (1972) *Invent. Math.* 15, 91–143]. A special case of our methods yields a proof of the two conjectured [Kac, V. G. and Wakimoto, M. (1994) in *Progress in Mathematics*, eds. Brylinski, J.-L., Brylinski, R., Guillemin, V. & Kac, V. (Birkhäuser Boston, Boston, MA), Vol. 123, pp. 415–456] identities involving representing a positive integer by sums of $4n^2$ or $4n(n + 1)$ triangular numbers, respectively. Our 16 and 24 squares identities were originally obtained via multiple basic hypergeometric series, Gustafson's C_ℓ nonterminating $_6\phi_5$ summation theorem, and Andrews' basic hypergeometric series proof of Jacobi's 4 and 8 squares identities. We have (elsewhere) applied symmetry and Schur function techniques to this original approach to prove the existence of similar infinite families of sums of squares identities for n^2 or $n(n + 1)$ squares, respectively. Our sums of more than 8 squares identities are not the same as the formulas of Mathews (1895), Glaisher (1907), Ramanujan (1916), Mordell (1917, 1919), Hardy (1918, 1920), Kac and Wakimoto, and many others.**

1. Introduction

In this paper, we announce two infinite families of explicit exact formulas that generalize Jacobi's (1) 4 and 8 squares identities to $4n^2$ or $4n(n + 1)$ squares, respectively, without using cusp forms. Our 24 squares identity leads to a different formula for Ramanujan's (2) tau function $\tau(n)$, when n is odd. These results arise in the setting of Jacobi elliptic functions, Jacobi continued fractions, Hankel or Turánian determinants, Fourier series, Lambert series, inclusion/exclusion, Laplace expansion formula for determinants, and Schur functions. (For this background material, see refs. 1 and 3–16.)

The problem of representing an integer as a sum of squares of integers has had a long and interesting history, which is surveyed in ref. 17 and chapters 6–9 of ref. 18. The review article (19) presents many questions connected with representations of integers as sums of squares. Direct applications of sums of squares to lattice point problems and crystallography can be found in ref. 20. One such example is the computation of the constant Z_N, which occurs in the evaluation of a certain Epstein zeta function, needed in the study of the stability of rare gas crystals and in that of the so-called Madelung constants of ionic salts.

The s squares problem is to count the number $r_s(n)$ of integer solutions $(x_1, \ldots, x_s)$ of the Diophantine equation

$$x_1^2 + \cdots + x_s^2 = n, \qquad [1]$$

in which changing the sign or order of the x_i's gives distinct solutions.

Diophantus (325–409 A.D.) knew that no integer of the form $4n - 1$ is a sum of two squares. Girard conjectured in 1632 that n is a sum of two squares if and only if all prime divisors q of n with $q \equiv 3 \pmod 4$ occur in n to an even power. Fermat in 1641 gave an "irrefutable proof" of this conjecture. Euler gave the first known proof in 1749. Early explicit formulas for $r_2(n)$ were given by Legendre in 1798 and Gauss in 1801. It appears that Diophantus was aware that all positive integers are sums of four integral squares. Bachet conjectured this result in 1621, and Lagrange gave the first proof in 1770.

Jacobi, in his famous *Fundamenta Nova* (1) of 1829, introduced elliptic and theta functions, and used them as tools in the study of Eq. **1**. Motivated by Euler's work on 4 squares, Jacobi knew that the number $r_s(n)$ of integer solutions of Eq. **1** was also determined by

$$\vartheta_3(0, -q)^s := 1 + \sum_{n=1}^{\infty} (-1)^n r_s(n) q^n, \qquad [2]$$

where $\vartheta_3(0, q)$ is the $z = 0$ case of the theta function $\vartheta_3(z, q)$ in ref. 21 given by

$$\vartheta_3(0, q) := \sum_{j=-\infty}^{\infty} q^{j^2}. \qquad [3]$$

Jacobi then used his theory of elliptic and theta functions to derive remarkable identities for the $s = 2, 4, 6, 8$ cases of $\vartheta_3(0, -q)^s$. He immediately obtained elegant explicit formulas for $r_s(n)$, where $s = 2, 4, 6, 8$. We recall Jacobi's identities for $s = 4$ and 8 in the following theorem.

THEOREM 1.1 (JACOBI).

$$\vartheta_3(0, -q)^4 = 1 - 8\sum_{r=1}^{\infty} (-1)^{r-1} \frac{rq^r}{1 + q^r} = 1 + 8 \sum_{n=1}^{\infty} (-1)^n \left[\sum_{\substack{d|n, d>0 \\ 4\nmid d}} d \right] q^n, \qquad [4]$$

and

$$\vartheta_3(0, -q)^8 = 1 + 16\sum_{r=1}^{\infty} (-1)^r \frac{r^3 q^r}{1 - q^r} = 1 + 16 \sum_{n=1}^{\infty} \left[\sum_{d|n, d>0} (-1)^d d^3 \right] q^n. \qquad [5]$$

The publication costs of this article were defrayed in part by page charge payment. This article must therefore be hereby marked "*advertisement*" in accordance with 18 U.S.C. §1734 solely to indicate this fact.

Consequently, we have

$$r_4(n) = 8 \sum_{\substack{d|n, d>0 \\ 4 \nmid d}} d \text{ and } r_8(n) = 16 \sum_{d|n, d>0} (-1)^{n+d} d^3, \quad [6]$$

respectively.

In general it is true that

$$r_{2s}(n) = \delta_{2s}(n) + e_{2s}(n), \quad [7]$$

where $\delta_{2s}(n)$ is a divisor function and $e_{2s}(n)$ is a function of order substantially lower than that of $\delta_{2s}(n)$. If $2s = 2, 4, 6, 8$, then $e_{2s}(n) = 0$, and Eq. **7** becomes Jacobi's formulas for $r_{2s}(n)$, including Eq. **6**. On the other hand, if $2s > 8$ then $e_{2s}(n)$ is never 0. The function $e_{2s}(n)$ is the coefficient of q^n in a suitable "cusp form." The difficulties of computing Eq. **7**, especially the nondominate term $e_{2s}(n)$, increase rapidly with $2s$. The modular function approach to Eq. **7** and the cusp form $e_{2s}(n)$ is discussed in ref. 13. For $2s > 8$, modular function methods such as those in refs. 22–27, or the more classical elliptic function approach of refs. 28–30, are used to determine general formulas for $\delta_{2s}(n)$ and $e_{2s}(n)$ in Eq. **7**. Explicit, exact examples of Eq. **7** have been worked out for $2 \leq 2s \leq 32$. Similarly, explicit formulas for $r_s(n)$ have been found for (odd) $s < 32$. Alternate, elementary approaches to sums of squares formulas can be found in refs. 31–36.

We next consider classical analogs of Eqs. **4** and **5** corresponding to the $s = 8$ and 12 cases of Eq. **7**.

Glaisher (37, 62–64) used elliptic function methods rather than modular functions to prove the following theorem.

THEOREM 1.2 (GLAISHER).

$$\vartheta_3(0, -q)^{16} = 1 + \frac{32}{17} \sum_{y_1, m_1 \geq 1} (-1)^{m_1} m_1^7 q^{m_1 y_1} \quad [8a]$$

$$- \frac{512}{17} q(q;q)_\infty^8 (q^2;q^2)_\infty^8, \quad [8b]$$

where we have

$$(q;q)_\infty := \prod_{r \geq 1} (1 - q^r). \quad [9]$$

Glaisher took the coefficient of q^n to obtain $r_{16}(n)$. The same formula appears in ref. 13 (equation 7.4.32).

To find $r_{24}(n)$, Ramanujan (ref. 2, entry 7, table VI; see also ref. 13, equation 7.4.37) first proved *Theorem 1.3*.

THEOREM 1.3 (RAMANUJAN). *Let* $(q;q)_\infty$ *be defined by Eq.* **9**. *Then*

$$\vartheta_3(0, -q)^{24} = 1 + \frac{16}{691} \sum_{y_1, m_1 \geq 1} (-1)^{m_1} m_1^{11} q^{m_1 y_1} \quad [10a]$$

$$- \frac{33152}{691} q(q;q)_\infty^{24} - \frac{65536}{691} q^2 (q^2;q^2)_\infty^{24}. \quad [10b]$$

One of the main motivations for this paper was to generalize *Theorem 1.1* to $4n^2$ or $4n(n+1)$ squares, respectively, without using cusp forms such as Eqs. **8b** and **10b** but still using just sums of products of at most n Lambert series similar to either Eq. **4** or Eq. **5**, respectively. This is done in *Theorems 2.1* and *2.2* below. Here, we state the $n = 2$ cases, which determine different formulas for 16 and 24 squares.

THEOREM 1.4.

$$\vartheta_3(0, -q)^{16} = 1 - \frac{32}{3}(U_1 + U_3 + U_5)$$

$$+ \frac{256}{3}(U_1 U_5 - U_3^2), \quad [11]$$

where

$$U_s \equiv U_s(q) := \sum_{r=1}^{\infty} (-1)^{r-1} \frac{r^s q^r}{1+q^r}$$

$$= \sum_{n=1}^{\infty} \left[\sum_{d|n, d>0} (-1)^{d+n/d} d^s \right] q^n$$

$$= \sum_{y_1, m_1 \geq 1} (-1)^{y_1+m_1} m_1^s q^{m_1 y_1}. \quad [12]$$

Analogous to *Theorem 1.3*, we have *Theorem 1.5*.

THEOREM 1.5.

$$\vartheta_3(0, -q)^{24} = 1 + \frac{16}{9}(17G_3 + 8G_5 + 2G_7)$$

$$+ \frac{512}{9}(G_3 G_7 - G_5^2), \quad [13]$$

where

$$G_s \equiv G_s(q) := \sum_{r=1}^{\infty} (-1)^r \frac{r^s q^r}{1-q^r}$$

$$= \sum_{n=1}^{\infty} \left[\sum_{d|n, d>0} (-1)^d d^s \right] q^n$$

$$= \sum_{y_1, m_1 \geq 1} (-1)^{m_1} m_1^s q^{m_1 y_1}. \quad [14]$$

An analysis of Eq. **10b** depends upon Ramanujan's (2) tau function $\tau(n)$, defined by

$$q(q;q)_\infty^{24} := \sum_{n=1}^{\infty} \tau(n) q^n. \quad [15]$$

For example, $\tau(1) = 1$, $\tau(2) = -24$, $\tau(3) = 252$, $\tau(4) = -1472$, $\tau(5) = 4830$, $\tau(6) = -6048$, and $\tau(7) = -16744$. Ramanujan (ref. 2, equation 103) conjectured, and Mordell (38) proved, that $\tau(n)$ is multiplicative.

In the case where n is an odd integer (in particular an odd prime), equating Eqs. **10a**, **10b**, and **13** yields two formulas for $\tau(n)$ that are different from Dyson's (39) formula. We first obtain *Theorem 1.6*.

THEOREM 1.6. *Let* $\tau(n)$ *be defined by Eq.* **15** *and let* n *be odd. Then*

$$259\tau(n) = \frac{1}{2^3 \cdot 3^2}[17 \cdot 691\sigma_3(n) + 8 \cdot 691\sigma_5(n)$$

$$+ 2 \cdot 691\sigma_7(n) - 9\sigma_{11}(n)]$$

$$- \frac{691 \cdot 2^2}{3^2} \sum_{m=1}^{n-1} [\sigma_3^\dagger(m)\sigma_7^\dagger(n-m) - \sigma_5^\dagger(m)\sigma_5^\dagger(n-m)], \quad [16]$$

where

$$\sigma_r(n) := \sum_{d|n, d>0} d^r \text{ and } \sigma_r^\dagger(n) := \sum_{d|n, d>0} (-1)^d d^r \quad [17]$$

Remark: We can use Eq. **16** to compute $\tau(n)$ in $\leq 6n \ln n$ steps when n is an odd integer. This may also be done in $n^{2+\varepsilon}$ steps by appealing to Euler's infinite-product-representation algorithm (40) applied to $(q;q)_\infty^{24}$ in Eq. **15**.

A different simplification involving a power series formulation of Eq. **13** leads to the following theorem.

THEOREM 1.7. *Let* $\tau(n)$ *be defined by Eq.* **15** *and let* $n \geq 3$ *be odd. Then*

$$259\tau(\mathrm{n}) = \frac{1}{2^3}\sum_{\mathrm{d}|\mathrm{n},\mathrm{d}>0}(-1)^{\mathrm{d}}\mathrm{d}^{11}$$

$$-\frac{691}{2^3\cdot 3^2}\sum_{\mathrm{d}|\mathrm{n},\mathrm{d}>0}(-1)^{\mathrm{d}}\mathrm{d}^3(17+8\mathrm{d}^2+2\mathrm{d}^4) \qquad \textbf{[18a]}$$

$$-\frac{691\cdot 2^2}{3^2}\sum_{\substack{\mathrm{m}_1>\mathrm{m}_2\geq 1\\ \mathrm{m}_1+\mathrm{m}_2\leq \mathrm{n}\\ \gcd(\mathrm{m}_1,\mathrm{m}_2)|\mathrm{n}}}(-1)^{\mathrm{m}_1+\mathrm{m}_2}(\mathrm{m}_1\mathrm{m}_2)^3$$

$$\times(\mathrm{m}_1^2-\mathrm{m}_2^2)^2\sum_{\substack{\mathrm{y}_1,\mathrm{y}_2\geq 1\\ \mathrm{m}_1\mathrm{y}_1+\mathrm{m}_2\mathrm{y}_2=\mathrm{n}}}1. \qquad \textbf{[18b]}$$

Remark: The inner sum in Eq. **18b** counts the number of solutions (y_1, y_2) of the classical linear Diophantine equation $m_1y_1 + m_2y_2 = n$. This relates Eqs. **18a** and **18b** to the combinatorics in sections 4.6 and 4.7 of ref. 15.

In the next section, we present the infinite families of explicit exact formulas that generalize *Theorems 1.1*, *1.4*, and *1.5*.

Our methods yield (elsewhere) many additional infinite families of identities analogous to the η-function identities in appendix I of Macdonald's work (41). A special case of our analysis gives a proof (presented elsewhere) of the two identities conjectured by Kac and Wakimoto (42); these identities involve representing a positive integer by sums of $4n^2$ or $4n(n + 1)$ triangular numbers, respectively. The $n = 1$ case gives the classical identities of Legendre (ref. 43; see also ref. 3, equations ii and iii).

Theorems 1.4 and *1.5* were originally obtained via multiple basic hypergeometric series (44–51) and Gustafson's* C_ℓ nonterminating ${}_6\phi_5$ summation theorem combined with Andrews' (52) basic hypergeometric series proof of Jacobi's 4 and 8 squares identities. We have (elsewhere) applied symmetry and Schur function techniques to this original approach to prove the existence of similar infinite families of sums of squares identities for n^2 or $n(n + 1)$ squares, respectively.

Our sums of more than 8 squares identities are not the same as the formulas of Mathews (31), Glaisher (37, 62–64), Sierpinski (32), Uspensky (33–35), Bulygin (28, 53), Ramanujan (2), Mordell (26, 54), Hardy (23, 24), Bell (55), Estermann (56), Rankin (27, 57), Lomadze (25), Walton (58), Walfisz (59), Ananda-Rau (60), van der Pol (61), Krätzel (29, 30), Gundlach (22), and Kac and Wakimoto (42).

2. The $4n^2$ and $4n(n + 1)$ Squares Identities

To state our identities, we first need the Bernoulli numbers B_n defined by

$$\frac{t}{e^t-1} := \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}, \quad \text{for } |t| < 2\pi. \qquad \textbf{[19]}$$

We also use the notation $I_n := \{1, 2, \ldots, n\}$; $\|S\|$ is the cardinality of the set S, and $\det(M)$ is the determinant of the $n \times n$ matrix M.

The determinant form of the $4n^2$ squares identity is *Theorem 2.1*.

THEOREM 2.1. *Let* n = 1, 2, 3, ... *Then*

$$\vartheta_3(0,-\mathrm{q})^{4\mathrm{n}^2} = 1 + \sum_{\mathrm{p}=1}^{\mathrm{n}}(-1)^{\mathrm{p}}2^{2\mathrm{n}^2+\mathrm{n}}\prod_{\mathrm{r}=1}^{2\mathrm{n}-1}(\mathrm{r}!)^{-1}$$

$$\sum_{\substack{\phi\subset \mathrm{S}\subseteq \mathrm{I}_\mathrm{n}\\ \|\mathrm{S}\|=\mathrm{p}}}\det(\mathrm{M}_{\mathrm{n,S}}), \qquad \textbf{[20]}$$

where $\vartheta_3(0, -\mathrm{q})$ *is determined by Eq.* **3**, *and* $\mathrm{M}_{\mathrm{n,S}}$ *is the* $\mathrm{n}\times\mathrm{n}$ *matrix whose* i*th row is*

$$\mathrm{U}_{2\mathrm{i}-1}, \mathrm{U}_{2(\mathrm{i}+1)-1}, \ldots, \mathrm{U}_{2(\mathrm{i}+\mathrm{n}-1)-1}, \textit{if } \mathrm{i}\in \mathrm{S}$$

$$\textit{and } \mathrm{c}_\mathrm{i}, \mathrm{c}_{\mathrm{i}+1}, \ldots, \mathrm{c}_{\mathrm{i}+\mathrm{n}-1}, \textit{if } \mathrm{i}\notin \mathrm{S}, \qquad \textbf{[21]}$$

where $\mathrm{U}_{2\mathrm{i}-1}$ *is determined by Eq.* **12**, *and* c_i *is defined by*

$$\mathrm{c}_\mathrm{i} := (-1)^{\mathrm{i}-1}\frac{(2^{2\mathrm{i}}-1)}{4\mathrm{i}}\cdot|\mathrm{B}_{2\mathrm{i}}|, \quad \textit{for } \mathrm{i} = 1, 2, 3, \ldots, \qquad \textbf{[22]}$$

with $\mathrm{B}_{2\mathrm{i}}$ *the Bernoulli numbers defined by Eq.* **19**.

We next have *Theorem 2.2*.

THEOREM 2.2. *Let* n = 1, 2, 3, ... *Then*

$$\vartheta_3(0,-\mathrm{q})^{4\mathrm{n}(\mathrm{n}+1)} = 1 + \sum_{\mathrm{p}=1}^{\mathrm{n}}(-1)^{\mathrm{n}-\mathrm{p}}2^{2\mathrm{n}^2+3\mathrm{n}}\prod_{\mathrm{r}=1}^{2\mathrm{n}}(\mathrm{r}!)^{-1}$$

$$\sum_{\substack{\phi\subset \mathrm{S}\subseteq \mathrm{I}_\mathrm{n}\\ \|\mathrm{S}\|=\mathrm{p}}}\det(\mathrm{M}_{\mathrm{n,S}}), \qquad \textbf{[23]}$$

where $\vartheta_3(0, -\mathrm{q})$ *is determined by Eq.* **3**, *and* $\mathrm{M}_{\mathrm{n,S}}$ *is the* $\mathrm{n}\times\mathrm{n}$ *matrix whose* i*th row is*

$$\mathrm{G}_{2\mathrm{i}+1}, \mathrm{G}_{2(\mathrm{i}+1)+1}, \ldots, \mathrm{G}_{2(\mathrm{i}+\mathrm{n}-1)+1}, \textit{if } \mathrm{i}\in \mathrm{S}$$

$$\textit{and } \mathrm{a}_\mathrm{i}, \mathrm{a}_{\mathrm{i}+1}, \ldots, \mathrm{a}_{\mathrm{i}+\mathrm{n}-1}, \textit{if } \mathrm{i}\notin \mathrm{S}, \qquad \textbf{[24]}$$

where $\mathrm{G}_{2\mathrm{i}+1}$ *and* $\mathrm{a}_\mathrm{i} := \mathrm{c}_{\mathrm{i}+1}$ *are determined by Eqs.* **14** *and* **22**, *respectively*.

We next use Schur functions $s_\lambda(x_1, \ldots, x_p)$ to rewrite *Theorems 2.1* and *2.2*. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots)$ be a partition of nonnegative integers in decreasing order, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \ldots$, such that only finitely many of the λ_i are nonzero. The length $\ell(\lambda)$ is the number of nonzero parts of λ.

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_p)$ of length $\leq p$,

$$s_\lambda(x) \equiv s_\lambda(x_1,\ldots,x_p) := \frac{\det(x_i^{\lambda_j+p-j})}{\det(x_i^{p-j})} \qquad \textbf{[25]}$$

is the Schur function (12) corresponding to the partition λ. [Here, $\det(a_{ij})$ denotes the determinant of a $p \times p$ matrix with (i, j)th entry a_{ij}]. The Schur function $s_\lambda(x)$ is a symmetric polynomial in $x_1, \ldots, x_p$ with nonnegative integer coefficients. We typically have $1 \leq p \leq n$.

We use Schur functions in Eq. **25** corresponding to the partitions λ and ν, with

$$\lambda_r := \ell_{p-r+1} - \ell_1 + r - p$$

$$\text{and } \nu_r := j_{p-r+1} - j_1 + r - p,$$

$$\text{for } r = 1, 2, \ldots, p, \qquad \textbf{[26]}$$

where the ℓ_r and j_r are elements of the sets S and T, with

$$S := \{\ell_1 < \ell_2 < \cdots < \ell_p\}$$

$$\text{and} \quad S^c := \{\ell_{p+1} < \cdots < \ell_n\}, \qquad \textbf{[27]}$$

$$T := \{j_i < j_2 < \cdots < j_p\}$$

$$\text{and} \quad T^c := \{j_{p+1} < \cdots < j_n\}, \qquad \textbf{[28]}$$

where $S^c := I_n - S$ is the compliment of the set S. We also have

$$\Sigma(S) := \ell_1 + \ell_2 + \cdots + \ell_p$$

$$\text{and} \quad \Sigma(T) := j_1 + j_2 + \cdots + j_p. \qquad \textbf{[29]}$$

Keeping in mind Eqs. **25–29**, symmetry and skew-symmetry arguments, row and column operations, and the Laplace

*Gustafson, R. A., Ramanujan International Symposium on Analysis, Dec. 26–28, 1987, Pune, India, pp. 187–224.

expansion formula (9) for a determinant, we now rewrite *Theorem 2.1* as *Theorem 2.3*.

THEOREM 2.3. *Let* n = 1, 2, 3, ... *Then*

$$\vartheta_3(0,-q)^{4n^2} = 1 + \sum_{p=1}^{n} (-1)^p 2^{2n^2+n} \prod_{r=1}^{2n-1} (r!)^{-1} \times \sum_{\substack{y_1,\ldots,y_p\geq 1\\ m_1>m_2>\cdots>m_p\geq 1}} (-1)^{y_1+\cdots+y_p}$$

$$\cdot (-1)^{m_1+\cdots+m_p} q^{m_1y_1+\cdots+m_py_p} \prod_{1\leq r<s\leq p} (m_r^2-m_s^2)^2$$

$$\cdot (m_1m_2\cdots m_p) \sum_{\substack{\phi\subset S,T\subseteq I_n\\ \|S\|=\|T\|=p}} (-1)^{\Sigma(S)+\Sigma(T))}\cdot \det(D_{n-p,S^c,T^c})$$

$$\cdot (m_1m_2\cdots m_p)^{2\ell_1+2j_1-4} s_\lambda(m_1^2,\ldots,m_p^2) s_\nu(m_1^2,\ldots,m_p^2), \quad [30]$$

where $\vartheta_3(0,-q)$ *is determined by Eq.* **3**; *the sets* S, S^c, T, *and* T^c *are given by Eqs.* **27** *and* **28**; $\Sigma(S)$ *and* $\Sigma(T)$ *are given by Eq.* **29**; *the* $(n-p)\times(n-p)$ *matrix* $D_{n-p,S^c,T^c} := [c_{(\ell_{p+r}+j_{p+s}-1)}]_{1\leq r,s\leq n-p}$, *where the* c_i *are determined by Eq.* **22**, *with the* B_{2i} *in Eq.* **19**; *and* s_λ *and* s_ν *are the Schur functions in Eq.* **25**, *with the partitions* λ *and* ν *given by Eq.* **26**.

We next rewrite *Theorem 2.2* as *Theorem 2.4*.

THEOREM 2.4. *Let* n = 1, 2, 3, ... *Then*

$$\vartheta_3(0,-q)^{4n(n+1)} = 1 + \sum_{p=1}^{n} (-1)^{n-p} 2^{2n^2+3n} \prod_{r=1}^{2n} (r!)^{-1} \times \sum_{\substack{y_1,\ldots,y_p\geq 1\\ m_1>m_2>\cdots>m_p\geq 1}} (-1)^{m_1+\cdots+m_p}$$

$$\cdot q^{m_1y_1+\cdots+m_py_p} (m_1m_2\cdots m_p)^3 \prod_{1\leq r<s\leq p} (m_r^2-m_s^2)^2$$

$$\cdot \sum_{\substack{\phi\subset S,T\subseteq I_n\\ \|S\|=\|T\|=p}} (-1)^{\Sigma(S)+\Sigma(T)}\cdot \det(D_{n-p,S^c,T^c})$$

$$\cdot (m_1m_2\cdots m_p)^{2\ell_1+2j_1-4} s_\lambda(m_1^2,\ldots,m_p^2) s_\nu(m_1^2,\ldots,m_p^2), \quad [31]$$

where the same assumptions hold as in Theorem 2.3, *except that the* $(n-p)\times(n-p)$ *matrix* $D_{n-p,S^c,T^c} := [a_{(\ell_{p+r}+j_{p+s}-1)}]_{1\leq r,s\leq n-p}$, *where the* $a_i := c_{i+1}$ *are determined by Eq.* **22**.

We close this section with some comments about the above theorems. To prove *Theorem 2.1*, we first compare the Fourier and Taylor series expansions of the Jacobi elliptic function $f_1(u,k) := sc(u,k)dn(u,k)$, where k is the modulus. An analysis similar to that in refs. 3, 4, and 16 leads to the relation $U_{2m-1}(-q) = c_m + d_m$, for $m = 1, 2, 3, \ldots$, where $U_{2m-1}(-q)$ and c_m are defined by Eqs. **12** and **22**, respectively, and d_m is given by $d_m = [(-1)^m z^{2m}/2^{2m+1}]\cdot(sd/c)_m(k^2)$, where $z := {}_2F_1(1/2, 1/2; 1; k^2) = 2\boldsymbol{K}(k)/\pi \equiv 2\boldsymbol{K}/\pi$, with $\boldsymbol{K}(k) \equiv \boldsymbol{K}$ the complete elliptic integral of the first kind in ref. 21, and $(sd/c)_m(k^2)$ is the coefficient of $u^{2m-1}/(2m-1)!$ in the Taylor series expansion of $f_1(u,k)$ about $u = 0$.

An inclusion/exclusion argument then reduces the $q \leftrightarrow -q$ case of Eq. **20** to finding suitable product formulas for the $n \times n$ Hankel determinants $\det(d_{i+j-1})$ and $\det(c_{i+j-1})$. Row and column operations immediately imply that

$$\det(d_{i+j-1}) = (z^{2n^2}(-1)^n/2^{2n^2+n})\det[(sd/c)_{i+j-1}(k^2)]. \quad [32]$$

From theorem 7.9 of ref. 4, we have $z = \vartheta_3(0,q)^2$, where $q = \exp[-\pi\boldsymbol{K}(\sqrt{1-k^2})/\boldsymbol{K}(k)]$. Setting $z = \vartheta_3(0,q)^2$ in Eq. **32** and then taking $q \leftrightarrow -q$ produces the $\vartheta_3(0,-q)^{4n^2}$ in Eq. **20**. The proof of *Theorem 2.1* is complete once we show that

$$\det[(sd/c)_{i+j-1}(k^2)] = \prod_{r=1}^{2n-1} (r!)$$

and

$$\det(c_{i+j-1}) = 2^{-(2n^2+n)}\cdot \prod_{r=1}^{2n-1} (r!). \quad [33]$$

By a classical result of Heilermann (7, 8), more recently presented in ref. 10 (theorem 7.14), Hankel determinants whose entries are the coefficients in a formal power series L can be expressed as a certain product of the "numerator" coefficients of the associated Jacobi continued fraction J corresponding to L, provided that J exists. Modular transformations, followed by row and column operations, reduce the evaluation of $\det[(sd/c)_{i+j-1}(k^2)]$ in Eq. **33** to applying Heilermann's formula to Rogers' (14) J-fraction expansion of the Laplace transform of $sd(u,k)cn(u,k)$. The evaluation of $\det(c_{i+j-1})$ can be done similarly, starting with $sc(u,k)$ and the relation $sc(u,0) = \tan(u)$.

The proof of *Theorem 2.2* is similar to *Theorem 2.1*, except that we start with $sc^2(u,k)dn^2(u,k)$.

Our proofs of the Kac and Wakimoto conjectures do not require inclusion/exclusion, and the analysis involving Schur functions is simpler than in those in Eqs. **30** and **31**.

We have (elsewhere) written down the $n = 3$ cases of *Theorems 2.3* and *2.4* which yield explicit formulas for 36 and 48 squares, respectively.

This work was partially supported by National Security Agency Grant MDA 904-93-H-3032.

1. Jacobi, C. G. J. (1829) *Fundamenta Nova Theoriae Functionum Ellipticarum*, Regiomonti, Sumptibus fratrum Bornträger; reprinted in Jacobi, C. G. J. (1881–1891) *Gesammelte Werke* (Reimer, Berlin), Vol. 1, pp. 49–239 [reprinted (1969) by Chelsea, New York; now distributed by Am. Mathematical Soc., Providence, RI].
2. Ramanujan, S. (1916) *Trans. Cambridge Philos. Soc.* **22,** 159–184.
3. Berndt, B. C. (1991) *Ramanujan's Notebooks*, Part III (Springer, New York), p. 139.
4. Berndt, B. C. (1993) in *CRM Proceedings & Lecture Notes*, ed. Murty, M. R. (Am. Mathematical Soc., Providence, RI), Vol. 1, pp. 1–63.
5. Goulden, I. P. & Jackson, D. M. (1983) *Combinatorial Enumeration* (Wiley, New York).
6. Gradshteyn, I. S. and Ryzhik, I. M. (1980) *Table of Integrals, Series, and Products* (Academic, San Diego), 4th Ed.; trans. Scripta Technica, Inc., ed. Jeffrey, A.
7. Heilermann, J. B. H. (1845) *De Transformatione Serierum in Fractiones Continuas*, Dr. Phil. dissertation (Royal Academy of Münster).
8. Heilermann, J. B. H. (1846) *J. Reine Angew. Math.* **33,** 174–188.
9. Jacobson, N. (1974) *Basic Algebra I* (Freeman, San Francisco), pp. 396–397.
10. Jones, W. B. & Thron, W. J. (1980) in *Encyclopedia of Mathematics and Its Applications*, ed. Rota, G.-C. (Addison–Wesley, London), Vol. 11 (now distributed by Cambridge Univ. Press, Cambridge, U.K.), pp. 244–246.
11. Lawden, D. F. (1989) *Applied Mathematical Sciences* (Springer, New York), Vol. 80.
12. Macdonald, I. G. (1995) *Symmetric Functions and Hall Polynomials* (Oxford Univ. Press, Oxford), 2nd Ed.
13. Rankin, R. A. (1977) *Modular Forms and Functions* (Cambridge Univ. Press, Cambridge, U.K.), pp. 241–244.
14. Rogers, L. J. (1907) *Proc. London Math. Soc.* (Ser. 2) **4,** 72–89.
15. Stanley, R. P. (1986) *Enumerative Combinatorics* (Wadsworth & Brooks Cole, Belmont, CA), Vol. 1.
16. Zucker, I. J. (1979) *SIAM J. Math. Anal.* **10,** 192–206.
17. Grosswald, E. (1985) *Representations of Integers as Sums of Squares* (Springer, New York).

18. Dickson, L. E. (1966) *History of the Theory of Numbers* (Chelsea, New York), Vol. 2.
19. Taussky, O. (1970) *Am. Math. Monthly* **77,** 805–830.
20. Glasser, M. L. & Zucker, I. J. (1980) *Theoretical Chemistry: Advances and Perspectives*, eds. Eyring, H. & Henderson, D. (Academic, New York), Vol. 5, pp. 67–139.
21. Whittaker, E. T. & Watson, G. N. (1927) *A Course of Modern Analysis* (Cambridge Univ. Press, Cambridge, U.K.), 4th Ed., pp. 464, 498.
22. Gundlach, K.-B. (1978) *Glasgow Math. J.* **19,** 173–197.
23. Hardy, G. H. (1918) *Proc. Natl. Acad. Sci. USA* **4,** 189–193.
24. Hardy, G. H. (1920) *Trans. Am. Math. Soc.* **21,** 255–284.
25. Lomadze, G. A. (1948) *Akad. Nauk Gruzin. SSR Trudy Tbiliss. Mat. Inst. Razmadze* **16,** 231–275 (in Russian; Georgian summary).
26. Mordell, L. J. (1917) *Q. J. Pure Appl. Math. Oxford* **48,** 93–104.
27. Rankin, R. A. (1962) *Acta Arith.* **7,** 399–407.
28. Bulygin, V. (1914) *Bull. Acad. Imp. Sci. St. Petersbourg Ser. VI* **8,** 389–404; announced by Boulyguine, B. (1914) *Comptes Rendus Paris* **158,** 328–330.
29. Krätzel, E. (1961) *Wiss. Z. Friedrich–Schiller–Univ. Jena Math. Naturwiss. Reihe* **10,** 33–37.
30. Krätzel, E. (1962) *Wiss. Z. Friedrich–Schiller–Univ. Jena Math. Naturwiss. Reihe* **11,** 115–120.
31. Mathews, G. B. (1895) *Proc. London Math. Soc.* **27,** 55–60.
32. Sierpinski, W. (1907) *Wiad. Mat. Warszawa* **11,** 225–231 (in Polish).
33. Uspensky, J. V. (1913) *Commun. Soc. Math. Kharkow* (Ser. 2) **14,** 31–64 (in Russian).
34. Uspensky, J. V. (1925) *Bull. Acad. Sci. USSR Leningrad* (Ser. 6) **19,** 647–662 (in French).
35. Uspensky, J. V. (1928) *Trans. Am. Math. Soc.* **30,** 385–404.
36. Uspensky, J. V. & Heaslet, M. A. (1939) *Elementary Number Theory* (McGraw–Hill, New York).
37. Glaisher, J. W. L. (1907) *Q. J. Pure Appl. Math. Oxford* **38,** 1–62.
38. Mordell, L. J. (1917) *Proc. Cambridge Philos. Soc.* **19,** 117–124.
39. Dyson, F. J. (1972) *Bull. Am. Math. Soc.* **78,** 635–653.
40. Andrews, G. E. (1986) *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics and Computer Algebra*, National Science Foundation Conference Board of the Mathematical Sciences Regional Conference Series, Vol. 66, p. 104.
41. Macdonald, I. G. (1972) *Invent. Math.* **15,** 91–143.
42. Kac, V. G. & Wakimoto, M. (1994) in *Progress in Mathematics*, eds. Brylinski, J.-L., Brylinski, R., Guillemin, V. & Kac, V. (Birkhäuser Boston, Boston, MA), Vol. 123, pp. 415–456.
43. Legendre, A. M. (1828) *Traité des Fonctions Elliptiques et des Intégrales Euleriennes* (Huzard-Courcier, Paris), Vol. III, pp. 133–134.
44. Lilly, G. M. & Milne, S. C. (1993) *Constr. Approx.* **9,** 473–500.
45. Milne, S. C. (1985) *Adv. Math.* **57,** 34–70.
46. Milne, S. C. (1987) *J. Math. Anal. Appl.* **122,** 223–256.
47. Milne, S. C. (1992) *Discrete Math.* **99,** 199–246.
48. Milne, S. C. (1994) *SIAM J. Math. Anal.* **25,** 571–595.
49. Milne, S. C. (1995) *Adv. Math.*, in press.
50. Milne, S. C. & Lilly, G. M. (1992) *Bull. Am. Math. Soc. (N.S.)* **26,** 258–263.
51. Milne, S. C. & Lilly, G. M. (1995) *Discrete Math.* **139,** 319–346.
52. Andrews, G. E. (1974) *SIAM Rev.* **16,** 441–484.
53. Bulygin, V. (B. Boulyguine) (1915) *Comptes Rendus Paris* **161,** 28–30 (in French).
54. Mordell, L. J. (1919) *Trans. Cambridge Philos. Soc.* **22,** 361–372.
55. Bell, E. T. (1919) *Bull. Am. Math. Soc.* **26,** 19–25.
56. Estermann, T. (1936) *Acta Arith.* **2,** 47–79.
57. Rankin, R. A. (1945) *Proc. Cambridge Philos. Soc.* **41,** 1–11.
58. Walton, J. B. (1949) *Duke Math. J.* **16,** 479–491.
59. Walfisz, A. Z. (1952) *Uspehi Mat. Nauk (N.S.)* (Ser. 6) **52,** 91–178 (in Russian); English transl., (1956) *Am. Math. Soc. Transl.* (Ser. 2) **3,** 163–248.
60. Ananda-Rau, K. (1954) *J. Madras Univ. Sect. B* **24,** 61–89.
61. van der Pol, B. (1954) *Ned. Akad. Wetensch. Indag. Math.* **16,** 349–361.
62. Glaisher, J. W. L. (1907) *Q. J. Pure Appl. Math. Oxford* **38,** 178–236.
63. Glaisher, J. W. L. (1907) *Q. J. Pure Appl. Math. Oxford* **38,** 210.
64. Glaisher, J. W. L. (1907) *Q. J. Pure Appl. Math. Oxford* **38,** 289–351.