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Nonatomic games on Loeb spaces

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ABSTRACT In the setting of noncooperative game theory, strategic negligibility of individual agents, or diffuseness of information, has been modeled as a nonatomic measure space, typically the unit interval endowed with Lebesgue measure. However, recent work has shown that with uncountable action sets, for example the unit interval, there do not exist pure-strategy Nash equilibria in such nonatomic games. In this brief announcement, we show that there is a perfectly satisfactory existence theory for nonatomic games provided this nonatomicity is formulated on the basis of a particular class of measure spaces, hyperfinite Loeb spaces. We also emphasize other desirable properties of games on hyperfinite Loeb spaces, and present a synthetic treatment, embracing both large games as well as those with incomplete information.

Noncooperative game theory rests fundamentally on the notion of a Nash equilibrium introduced in economic science in ref. 1 and given a definitive formulation in refs. 2 and 3. In this paper, we outline a theory of existence of pure-strategy Nash equilibria based on a class of probability spaces with especially desirable properties. Our work is intended as a contribution to the mathematical theory of games, and, in particular, to the analysis of social science phenomena in which "negligibility" of agents and/or "diffuseness" of information is an essential and substantive issue. In noncooperative game theory, canonical illustrations of ideas based on a nonatomic continuum are given in refs. 4–8, but they also arise naturally in many other contexts, including modern macroeconomics (9, 10). The connection between existence issues in noncooperative game theory and competitive analysis is explicitly laid out in refs. 11–13.

Recent work has presented examples of games without pure-strategy Nash equilibria. These examples all involve the Lebesgue unit interval, and our interpretation of the difficulties revolves around Lusin's theorem. If agents in a "large" society are influenced by aggregate societal responses formalized as a distribution or as an average over a common action set, one cannot proceed without a measurability hypothesis on the function cataloguing the individual responses. The problem then is that in the presence of a topological structure on the space of players' names, as in the Lebesgue unit interval, Lusin's theorem renders measurability as being tantamount to near continuity and thus translates the demand for an equilibrium into a simultaneous demand that the agents behave almost continuously; or alternatively, for almost all players, the response of one player must be related to that of those nearby. This exogenous, but implicit, requirement that the agents should cooperate and act almost continuously goes against the very spirit of the behavioral notion underlying noncooperative game theory; and even though obscured in situations with finite action sets, it is a point of substantive, rather than merely technical, significance. A similar argument underlies nonexistence difficulties in games where incomplete information is modeled on the Lebesgue unit interval. One needs a class of nonatomic probability spaces in which stringent "nearness" properties for names and/or information points are not automatically invoked. It is the contention of this note that hyperfinite nonatomic Loeb spaces (14) furnish just such a class of probability spaces.

An alternative characterization of the work reported here is oriented to researchers uneasy with idealized limit models if they cannot be asymptotically implemented in a large but finite setting. The counterexamples that we report leave open only two logical possibilities: either the very nature of the phenomenon is such that there are no exact results even in the idealized limit setting; or the idealized limit setting is not sharp enough to capture some large finite phenomenon being modeled, necessitating an alternative, more hospitable, limit setting. We present what we consider to be compelling evidence for the viability of such a limit setting, one that yields approximate equilibria not only for the idealization (corollary to theorem 3 in ref. 6; ref. 15), but more importantly, for situations with a large but finite number of players or information samples. The potential of such a limit setting is well-understood in general equilibrium theory (16–18), and we complement this investigation in the direction of noncooperative game theory by a systematic application of the analytical machinery developed in refs. 19 and 20.

Finally, it needs to be made clear that the results that we report are a testimony to the power of probabilistic and measure-theoretic methods as envisaged originally in ref. 21 (p. 14)—we simply work with a class of standard measure spaces with additional properties not shared by general abstract measure spaces. The introduction of Radon measures (22) or the restriction to perfect measures (23) to resolve outstanding difficulties in probability theory represent obvious analogues to our approach in other fields of inquiry.[¶]

Results

Our results are based on nonatomic measure spaces introduced in ref. 14 and now commonly referred to as hyperfinite Loeb measure spaces (see also refs. 18 and 24 and the comprehensive treatment in ref. 16). Throughout this paper, $(T, \mathcal{T}, \nu)$ will denote a hyperfinite internal probability space and $(T, L(\mathcal{T}), \bar{\nu})$ will denote its standardization—the Loeb space. We shall assume that this Loeb space is atomless.

For our principal result, we shall work with an ℓ-fold partition $\{T_i\}_{i=1}^{\ell}$, where ℓ is a positive integer and $\bar{\nu}_i$ is a probability measure on T_i given by $\bar{\nu}_i(B) = \bar{\nu}(B)/\bar{\nu}(T_i)$ for any measurable set $B \subseteq T_i$. The action sets are chosen from a Banach space (25), and we use both the weak and the weak* topologies, the second being motivated by sets of probability measures on a compact metric space (26, 27). For the first setting, for each $i = 1, \ldots, \ell$, let A_i be a weakly compact subset of a separable Banach space, $\overline{con}(A_i)$ their closed convex hull,

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and $\mathcal{U}_i^w$ the space of weakly continuous real-valued functions on $(A_i \times \Pi_{j=1}^{\ell} \overline{con}(A_j))$ endowed with the sup-norm topology and the corresponding Borel σ-algebra. For the second setting, we assume the Banach space to be the dual of a separable Banach space, and obtain $\mathcal{U}_i^{w*}$ by phrasing the compactness hypotheses on action sets and continuity hypotheses on the payoffs in terms of weak* topology. We shall work with Bochner integration in the first situation, and with Gel'fand integration in the second (see refs. 19 and 25).

THEOREM 1. *If* $\mathcal{G} = (\mathcal{G}_1, \cdots, \mathcal{G}_\ell)$, $\mathcal{G}_i$ *a measurable map from* T_i *to* $\mathcal{U}_i^w$, *there exists* $g = (g_1, \cdots, g_\ell)$, *where* g_i *is a measurable function from* T_i *to* A_i, *such that for all* $t \in T_i$, *and for all* $a \in A_i$,

$$\mathcal{G}_i(t)\left(g_i(t), \int_{s\in T_1} g_1(s)d\bar{\nu}_1, \cdots, \int_{s\in T_\ell} g_\ell(s)d\bar{\nu}_\ell\right) \geq \mathcal{G}_i(t)\left(a, \int_{s\in T_1} g_1(s)d\bar{\nu}_1, \cdots, \int_{s\in T_\ell} g_\ell(s)d\bar{\nu}_\ell\right).$$

This statement is valid with the space $\mathcal{U}_i^{w*}$ *substituted for* $\mathcal{U}_i^w$, *and Gel'fand integrals for Bochner integrals.*

The interpretation of the theorem is clear. T is the set of players' names with the atomless Loeb measure $\bar{\nu}$ formalizing the fact that each player is strategically negligible. Society is classified into ℓ institutions, with the payoff function of a particular agent depending on the player's action as well as on a statistical measure summarizing the behavior of players in each institution. A nonatomic game $\mathcal{G}$ simply associates in a measurable way a payoff function to each player name, and the function g represents the Nash equilibrium of the game. Such an equilibrium asserts the existence of actions, one action for each player, individually optimal under a particular conception of societal responses, and which engender the particular conception on which their optimality is based.

Both statements in *Theorem 1* are false if the Lebesgue interval is substituted for the Loeb space of players' names. For the second statement, assume $\ell = 1$, and work with an action set constituted by all point measures on the interval $[-1, 1]$, and with the payoff function of $t \in [0, 1]$ given by $u_t(\delta_a, \rho) = h(a, \rho) - |t - |a||$, where δ_a denotes the Dirac measure at a, h a real-valued continuous mapping on $A \times \mathcal{M}([-1, 1])$ taking zero values at the uniform distribution ρ^*, and $\mathcal{M}([-1, 1])$ the space of all probability measures on $[-1, 1]$ endowed with the weak* topology. This game, henceforth $\mathcal{G}_1$, has an equicontinuous set of payoffs, and no Nash equilibrium in pure strategies. Since the induced distribution of a random variable is the Gel'fand integral of the random variable "lifted up" to the space of distributions, and since there is no measurable selection from the correspondence $t \to \{t, -t\}$ that induces ρ^*, the latter cannot be the Gel'fand integral of a Nash equilibrium. Under a further specification of h, we can ensure that for any distribution $\rho \neq \rho^*$, the best response function induces a distribution ρ' such that $d(\rho', \rho^*) < d(\rho, \rho^*)$, d the Prohorov metric generating the weak* topology (27). Nonexistence of equilibrium results from the absence of closure in the relevant space of Gel'fand integrals. The construction of a counterexample for the first assertion of *Theorem 1* is more technical on account of Bochner integration and the strong measurability requirement that underlies it. However, the punch line regarding the absence of closure remains the same.‖

Next, we turn to situations where a player's payoff function depends on the distribution, rather than the mean, of the random variable summarizing societal responses. Towards this end, for each $i = 1, \cdots, \ell$, let A_i be a compact metric space, the unit interval if the reader wishes, and $\mathcal{M}(A_i)$ the compact set of probability measures on A endowed with the weak* topology. Let $\mathcal{U}_i^D$ be the space of real-valued continuous functions on $(A_i \times \Pi_{i=1}^{\ell} \mathcal{M}(A_i))$, endowed with its sup-norm product topology and corresponding Borel σ-algebra. We can now present the following.

COROLLARY 1. *If* $\mathcal{G} = (\mathcal{G}_1, \cdots, \mathcal{G}_\ell)$, $\mathcal{G}_i$ *a measurable map from* T_i *to* $\mathcal{U}_i^D$, *there exists* $(g_1, \cdots, g_\ell)$, *where* g_i *is a measurable function from* T_i *to* A_i, $\bar{\nu}_i g_i^{-1}$ *its distribution on* A_i, *such that for all* $t \in T_i$,

$$\mathcal{G}_i(t)(g_i(t), \bar{\nu}_1 g_1^{-1}, \cdots, \bar{\nu}_\ell g_\ell^{-1}) \geq \mathcal{G}_i(t)(a, \bar{\nu}_1 g_1^{-1}, \cdots, \bar{\nu}_\ell g_\ell^{-1}) \text{ for all } a \in A_i.$$

The interpretation of the corollary is along the same lines as that of *Theorem 1*; externalities or player dependences now involve the entire distribution of societal responses rather than its mean. *Corollary 1* is false if the Lebesgue interval is substituted for the Loeb space of players' names. By "pushing down" the action set in $\mathcal{G}_1$ to the original interval $[-1, 1]$, one can obtain a game $\mathcal{G}_2$, which is a straightforward modification of $\mathcal{G}_1$ and furnishes the counterexample. Finally, *Corollary 1*, along with *Theorem 1*, bears comparison with theorem 2 of ref. 4.

Next, we turn to finite games with incomplete information. A game with incomplete information consists of (*i*) a finite set of ℓ players, each of whom is endowed with a compact metric action space A_i, the unit interval if the reader wishes; (*ii*) an information space constituted by ℓ hyperfinite internal measurable spaces $(Z_i, \mathcal{Z}_i)$, together with $(Z_0, \mathcal{Z}_0)$, Z_0 finite, and μ, an internal probability measure on the product measurable space $(Z, \mathcal{Z}) \equiv (\Pi_{j=0}^{\ell} Z_j, \Pi_{j=0}^{\ell} \mathcal{Z}_j)$ with $(Z, L(\mathcal{Z}), L(\mu))$ its Loeb standardization; and (*iii*) a payoff function $u_i : Z_0 \times Z_i \times \Pi_{j=1}^{\ell} A_j \to \mathbb{R}$, $L(\mathcal{Z}_0 \otimes \mathcal{Z}_i)$-measurable and continuous on $\Pi_{j=1}^{\ell} A_j$. For any $g = (g_1, \cdots, g_\ell)$, g_i a measurable function from Z_i to A_i, we shall denote the expected payoff to the ith player by:

$$U_i(g) = \int_{z\in Z} u_i[z_0, g_1(z_1)), \cdots, g_\ell(z_\ell))]dL(\mu)(z).$$

Let $L(\mu)_i$ denote the marginal of $L(\mu)$ on $(Z_i, L(\mathcal{Z}_i))$, and $(L(\mu)_i|z_0)$ its conditional probability with respect to events in Z_0. The vector $(g; f_i)$ has the usual meaning (2).

COROLLARY 2. *If, for all* $z_0 \in Z_0$, *the information of all the players is independent conditional on* z_0*—i.e.,* $(L(\mu)|z_0) = \Pi_{j=1}^{\ell} (L(\mu)_j|z_0)$*—and is diffuse—i.e.,* $L(\mu)_i$ *is atomless for all* i*—there exists* $g = (g_1, \cdots, g_\ell)$, g_i *a measurable function from* Z_i *to* A_i, *such that for all players* i, $U_i(g) \geq U_i(g; f_i)$, *for all measurable functions* f_i *from* Z_i *to* A_i.

Corollary 2 shows the extent to which (ref. 5; theorem 4 in ref. 6) can be generalized if one is willing to model information with Loeb measure spaces. A variant of *Corollary 2* can be fashioned along the lines of theorem 3 in ref. 29. For a summary of the economic models underlying *Corollary 2* or its variant, see refs. 5 and 6 and the textbook (chapter 6 of ref. 30). Finally, we remind the reader that *Corollary 2* is false for a two-player game** with an identical action set given by $[-1, 1]$, and the Lebesgue unit square substituted for the space of independent information.

‖For details concerning $\mathcal{G}_1$, see example 3 of ref. 28. The second counterexample involves an uncountable action set in the Hilbert space ℓ_2, and the details will be presented elsewhere.

**Details to be presented elsewhere. The possibility of an example has also been indicated in footnote 18 of ref. 6.

Next, we turn to the translation of these results for a large but finite setting of players (*Theorem 1* and *Corollary 1*) and of information samples (*Corollary 2*). The fact that this can be done for hyperfinite models goes back to the pioneering work in ref. 31 (see refs. 16 and 18 for details). The concern with asymptotic implementability arises on two counts: an insistence that an ideal model ought to capture the asymptotic nature of the large finite phenomenon being modeled and that assumptions on the idealized limit case ought to be translated into assumptions on the primitives of the finite case. We illustrate these points by presenting an asymptotic version only of *Corollary 1*. We draw on a conventional notion for putting some control on the extent to which the characteristics of the players in the class of finite games are allowed to vary. A sequence of measurable mappings from a measure space to a topological space is tight if for any $\varepsilon > 0$, there exists a compact set containing more than $(1 - \varepsilon)$ of the mass of the measure induced by each mapping. The tightness hypothesis guarantees that the standard part map (14, 16–18, 24) is well-defined and Loeb measurable, thereby furnishing the measurability hypothesis in *Corollary 1*. We can now present, in the notation of *Corollary 1*, our next result.

COROLLARY 3. *For each* $n \geq 1$, *let* T^n *be a finite set, and let* $G^n = (G^n_1, \cdots, G^n_\ell)$, G^n_i *a mapping from* T^n_i *into* $\mathcal{U}^D_i$, $\{T^n_1, \cdots, T^n_\ell\}$ *a partition of* T^n, *be a finite game. Assume that the sequence of finite games is tight and that there is a positive number* c *such that for all* $1 \leq i \leq \ell$, *and sufficiently large* n, $|T^n_i|/n > c$. *Then for any* $\varepsilon > 0$, *there exists a positive integer* N *such that for all* $n \geq N$, *there exists* $g^n_i : T^n_i \to A_i$ *such that for all* $t \in T^n_i$, *for all* $a \in A_i$,

$$G^n_i(t)\left(g^n_i(t), \frac{1}{|T^n_1|}\sum_{s\in T^n_1}\delta_{g^n_1}(s), \cdots, \frac{1}{|T^n_\ell|}\sum_{s\in T^n_\ell}\delta_{g^n_\ell}(s)\right)$$
$$\geq G^n_i(g)\left(a, \frac{1}{|T^n_1|}\sum_{s\in T^n_1}\delta_{g^n_1}(s), \cdots, \frac{1}{|T^n_\ell|}\sum_{s\in T^n_\ell}\delta_{g^n_\ell}(s)\right) - \varepsilon.$$

We emphasize that *Corollary 3* furnishes ε-equilibria for a large finite game rather than for an idealized limit game; see corollary to theorem 3 in ref. 6 for this distinction.

It is noted in ref. 32 that Lebesgue spaces are not homogeneous in the sense that two identically distributed random variables on the Lebesgue unit interval are not necessarily connected by an automorphism. In our context, this translates to the statement that the equilibrium profile is not necessarily insensitive to a permutation of the players' names. Consider a game $\mathcal{G}_3$ manufactured from $\mathcal{G}_2$ by a transformation which can be informally described as "expanding and shrinking" of the set of players' names such that the distribution of their characteristics remains the same. Formally, let the payoff function $v_t: A \times \mathcal{M}([-1, 1]) \to \mathbb{R}$ of any player $t \in [0, 1]$ be given by:

$$v_t(\cdot, \cdot) = \begin{cases} u_{2t}(\cdot, \cdot), & \text{if } 0 \leq t \leq (1/2) \\ u_{2-2t}(\cdot, \cdot), & \text{if } (1/2) < t \leq 1, \end{cases}$$

where u_t is the payoff function of player t in $\mathcal{G}_2$. The point is that even though the two games are identical in some essential macroscopic sense, $\mathcal{G}_2$ has no Nash equilibria and $\mathcal{G}_3$ does! Simply check that one such Nash equilibrium is given by $g : [0, 1] \to [-1, 1]$, where:

$$g(t) = \begin{cases} 2t, & \text{if } 0 \leq t \leq (1/2) \\ 2t - 2, & \text{if } (1/2) < t \leq 1. \end{cases}$$

Our final result responds to this rather anomalous state of affairs. When we work with a subclass of atomless Loeb spaces, hyperfinite Loeb counting spaces, we are always guaranteed of the existence of a suitable isomorphism between two random variables with the same distribution (33); the specific microspecification of two situations is of no consequence if they are "identical" from the macroscopic game-theoretic point of view. We can use this observation to prove the following illustrative result related to *Corollary 1.*

PROPOSITION 1. *Let* $\mathcal{G} = (\mathcal{G}_1, \cdots, \mathcal{G}_\ell)$ *and* $\mathcal{F} = (\mathcal{F}_1, \cdots, \mathcal{F}_\ell)$, $\mathcal{G}_i$ *and* $\mathcal{F}_i$, *measurable mappings from* T_i *to* $\mathcal{U}^D_i$ *such that for all* i, $\bar{\nu}_i\mathcal{G}_i^{-1} = \bar{\nu}_i\mathcal{F}_i^{-1}$. *Then for each* i, *there exists automorphisms* $\phi_i : (T_i, \bar{\nu}_i) \to (T_i, \bar{\nu}_i)$ *such that* $\mathcal{G}_i(t) = \mathcal{F}_i(\phi_i(t))$ *for almost all* $t \in T_i$. *Let* $f = (f_1, \cdots, f_\ell)$, $f_i : T_i \to A_i$ *be a Nash equilibrium of the atomless game* $\mathcal{F}$, *and define* $g_i : T_i \to A_i$ *such that* $g_i(t) = f_i(\phi_i(t))$ *for all* $t \in T_i$. *Then* g *is a Nash equilibrium of the atomless game* $\mathcal{G}$ *and every Nash equilibrium of* $\mathcal{G}$ *is obtained in this way.*

Ideas of Proof

The proof of *Theorem 1* is based on the theory of Bochner and Gel′fand integration of a correspondence on an atomless Loeb space; see ref. 19 for results and also for counterexamples based on Lebesgue unit intervals. The theory is analogous to that developed in ref. 34 for correspondences with values in an Euclidean space. The proof of *Theorem 1* simply appeals to the Fan-Glicksberg fixed point theorem (35, 36) along the lines of ref. 37. *Corollary 1* follows from *Theorem 1*, because of the connection, already observed earlier, between the Gel′fand integral and the induced distribution of a random variable. *Corollary 2* follows from *Corollary 1*, because the setting of the incomplete information game is one where the expected payoff of a player depends only on his type and on the distribution of the opponents' actions. *Corollary 3* follows from *Corollary 1* as a routine consequence of the transfer property of the nonstandard extension (see refs. 16, 18, and 31).

Since *Corollaries 1* and *2* involve distributions rather than integrals, we can also furnish direct proofs based on the notion of a distribution of a correspondence. This is simply the collection of the distributions of all random variables selected from the correspondence and is a well-behaved object when the domain space is an atomless Loeb space; see ref. 20 for the results, as well as for counterexamples based on Lebesgue intervals.

In passing, we can ask what is the special structure of Loeb spaces that delivers existence, even though existence fails in models based on the unit interval? We simply note that Loeb spaces, as used here, can be constructed from sequences and as such are particularly amenable to capturing phenomena valid for a large but finite asymptotic setting. On the other hand, by requiring "almost" continuity on the primitive data of the hyperfinite game, one can reduce it to a game on the Lebesgue interval, but one may not be able to reduce any of its equilibria, guaranteed by *Theorem 1* above, in a similar way. The results in refs. 19 and 20 can be used to prove existence theorems for the ideal case along conventional measure-theoretic lines. These results are essential. We cannot *restrict* results that are valid for an abstract measure space to a Loeb setting (the counterexamples show that there are no such results!); or *transfer* asymptotic results to a nonstandard universe and then *push down* to a Loeb setting (our asymptotic results are new). Of course, once the asymptotic results have been stated and proved, future work may furnish direct proofs without any of the special measure-theoretic tools utilized here (see section 6 of ref. 16). For earlier work along one or both of these coordinates, see refs. 16, 18, 31, and 38.

Concluding Remarks

First, the counterexamples constituting the negative aspect of this research are robust in that they do not hinge on any measure-theoretic intricacies. Indeed, the mappings associating payoff functions to the space of players' names or of information are even equicontinuous, and the action sets can be reset in a variety of ways formally described in ref. 28.

Second, Loeb measure spaces are standard measure spaces in the sense that any result valid for an abstract measure space applies to them. They can be used without familiarity with the nonstandard construction but simply as probability spaces with additional properties, much as Lebesgue measure spaces are used without regard to their particular construction or to the Dedekind set-theoretic basis of the set of real numbers. As such, the models reported here can be applied by researchers interested in questions involving *negligibility* and *diffuseness*.

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